Assortment Optimization under a Mixture of Mallows Model

Assortment optimization is a key decision problem that commonly arises in several application contexts. A concrete setting is that of a retailing where the decision maker needs to decide on the subset of products to offer from possibly a large universe of substitutable products to maximize the expected revenue or profit from each arrive customer with random preferences and substitution behavior. One of the key ingredients for the assortment optimization problem is a model to capture the customer preferences that accounts for product substitution behavior, whereby customers substitute to an available product (say, a dark blue shirt) when her most preferred product (say, a black one) is not offered. We assume that the substitution behavior is captured by a choice model which makes the demand for each offered product a function of the entire offer set.

The nonparametric rank-based choice model is a general model to capture preferences. In its more general form, it models customer preferences as a distribution over all possible rankings/preference lists of the products [3, 5]. Each preference list specifies a rank ordering of the products, with lower ranked products being more preferred. The model accommodates distributions with exponentially large support sizes and, therefore, can capture complex substitution patterns. However, available data are usually not sufficient to identify such a complex model. Therefore, ‘sparsity’ is used as a model selection criterion to pick a model from the set of models consistent with the data. Specifically, it is assumed that the distribution has a support size $K$, for some $K$ that is polynomial in the number of products. Sparsity results in data-driven model selection [3], obviating the need for imposing arbitrary parametric structures.

Despite their generality, however, sparse rank-based models cannot account for “noise” or any deviations from the $K$ ranked-lists in the support. This limits their modeling flexibility, resulting in unrealistic predictions and inability to model individual-level observations. Specifically, because $K \ll n!$ (where $n$ is the number of products), the model specifies that there is a zero chance that a customer uses a ranking that is even slightly different from any of the $K$ rankings in the support and a zero chance of observing certain choices. However, choices may be observed in real (holdout) data that are not consistent with any of the $K$ rankings, making the model predictions unrealistic. In addition, a natural way to interpret sparse choice models is to assume that the population consists of $K$ types of customers, with each type described by one of the ranked lists. When this interpretation is applied to individual-level observations, it implies that all the choice observations of each individual must be consistent with at least one of the $K$ rankings, which again may not be the case in real data.

In order to address these issues, we generalize the sparse rank-based models by “smoothing” them using the Mallows kernel. Specifically, we suppose that the choice model is a mixture of $K$ Mallows models and focus on assortment optimization under
this generalized class. The Mallows distribution was introduced in the mid-1950’s [4] and is the most popular member of the so-called distance-based ranking models, which are characterized by a modal ranking $\omega$ and a concentration parameter $\theta$. The probability that a ranking $\sigma$ is sampled falls exponentially as $e^{-\theta \cdot d(\sigma, \omega)}$, where $d(\cdot, \cdot)$ is the Kendall-Tau distance between $\sigma$ and $\omega$ (which is equal to the number of disagreements in pairwise comparisons between $\omega$ and $\sigma$). Intuitively, the Mallows model assumes that consumer preferences are concentrated around a central permutation, with the likelihood of large deviations being low.

Our contribution. Our main contribution in this paper is to give a near-optimal algorithm for a large class of constrained assortment optimization for the mixture of Mallows model under a reasonable assumption. We address the two key computational challenges that arise in solving our problem: (a) efficiently computing the choice probabilities and hence, the expected revenue/profit, for a given offer set $S$ and (b) finding a near-optimal assortment. We also present a compact mixed integer program (MIP) that leads to a practical approach for the constrained assortment optimization problem under a general mixture of Mallows model. We describe the main contributions below.

Exact computation of choice probabilities. We present an efficient procedure to compute the choice probabilities $P(a|S)$ exactly under a general mixture of Mallows model. Because the mixture of Mallows distribution has an exponential support size, computing the choice probabilities for a fixed offer set $S$ requires marginalizing the distribution by summing it over an exponential number of rankings, and therefore, is a non-trivial computational task. We exploit the structural symmetries in the Mallows distribution to derive an efficiently computable closed-form expression for the choice probabilities for a given offer set under the mixture of Mallows distribution. In particular, we first consider a single Mallows distribution and show that the choice probabilities under the Mallows distribution can be expressed as a discrete convolution. Using fast Fourier transform, the choice probabilities can be computed in $O(n^2 \cdot \log n)$ time where $n$ is the number of products. Therefore, we obtain a procedure with running time $O(K \cdot n^2 \cdot \log n)$ to compute the choice probabilities for a fixed offer set under a mixture of $K$ Mallows.

Near-optimal algorithm for constrained assortment optimization problem. We present a polynomial time approximation scheme (PTAS) for a large class of constrained assortment optimization for the mixture of Mallows model including cardinality constraints, knapsack constraints, and matroid constraints. Our PTAS holds under the assumption that the no-purchase option is ranked last in the modal rankings for all Mallows segments in the mixture or more generally, the customers are indifferent between the products that are less preferred than no-purchase in a modal ranking and these products are same for modal rankings of all Mallows segments in the mixture. Such an assumption is quite reasonable in practice. Moreover, it is important to note that it is not possible to
design a near-optimal algorithm for a general mixture of Mallows distribution because [1] show that the assortment optimization under a sparse rank-based model with $K > n$ is hard to approximate within a factor better than $O(1/n^{1-\epsilon})$ for any $\epsilon > 0$.

Under the above assumption, for any $\epsilon > 0$, our algorithm computes an assortment with expected revenue at least $(1-\epsilon)$ times the optimal in running time that is polynomial in $n$ and $K$ but depends exponentially on $1/\epsilon$. The PTAS is based on establishing a surprising sparsity property about near-optimal assortments, namely, that there exist a near-optimal assortment of size $O(1/\epsilon)$. Therefore, enumerating over all such subsets gives a $(1-\epsilon)$-approximation to the constrained assortment optimization problem. To the best of our knowledge, this is the first provably near-optimal algorithm for the assortment optimization under Mallows or the mixture of Mallows model in such generality.

**A mixed integer programming formulation.** We present a compact mixed integer linear program (MIP) with $O(K \cdot n^3)$ variables, $O(n)$ binary variables and $O(K \cdot n^3)$ constraints for the constrained assortment optimization under a general mixture of Mallows model with $K$ segments. The compact formulation is based on an alternative efficient procedure to compute the choice probabilities for a fixed offer set exactly. In particular, we exploit the repeated insertion method (RIM) introduced by [2] for sampling rankings according to the Mallows distribution and show that the choice probabilities for the Mallows model can be expressed as the unique solution to a system of linear equations that can be solved in $O(n^3)$ time. This gives us an alternate procedure to efficiently compute the choice probabilities for a fixed offer set exactly. While this is less efficient than using fast Fourier transform ($O(n^3)$ versus $O(n^2 \cdot \log n)$), it allows us to formulate a compact MIP for the constrained assortment optimization problem under a general mixture of Mallows model. Our MIP formulation holds for general mixture of Mallows model and does not require any assumption on the rank of no-purchase in the modal rankings.

**References**


