Learning in Repeated Auctions with Budgets: Regret Minimization and Equilibrium

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Abstract
In display advertising markets, advertisers often purchase ad placements through bidding in repeated auctions based on realized viewer information. We study how budget-constrained advertisers may compete in such sequential auctions in the presence of uncertainty about future bidding opportunities as well as competitors’ heterogenous preferences, budgets, and strategies. We formulate this problem as a sequential game of incomplete information, where bidders know neither their own valuation distribution, nor the budgets and valuation distributions of their competitors. We introduce a family of dynamic bidding strategies we refer to as adaptive pacing strategies, in which advertisers adjust their bids throughout the campaign according to the sample path of expenditures they exhibit. We analyze the performance of these strategies under different assumptions on competitors’ behavior. Under arbitrary competitors’ bids, we establish through matching lower and upper bounds the asymptotic optimality of this class of strategies as the number of auctions grows large. When all the bidders adopt these strategies, the dynamics converge to a tractable and meaningful steady state. Moreover, we characterize a regime (well motivated in the context of display advertising markets) under which these strategies constitute an approximate Nash equilibrium in dynamic strategies: The benefit of unilaterally deviating to other strategies, including ones with access to complete information, becomes negligible as the number of auctions and competitors grows large. This establishes a connection between regret minimization and market stability, by which advertisers can essentially follow equilibrium bidding strategies that also ensure the best performance that can be guaranteed off-equilibrium.

Keywords: Sequential auctions, online advertising, online learning, stochastic optimization, stochastic approximation, incomplete information, regret analysis, dynamic games

1 Introduction
Display ad spending has grown dramatically in recent years, reaching over $30 billion in the United States in 2016. More than two thirds of the spending is generated through algorithms, which

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are used by advertisers to acquire ad placements in spot markets called ad exchanges (eMarketer, 2016). At a high level, ad exchanges operate as online platforms where advertisers compete in sequential auctions to purchase online ad placements from publishers, as these become available, in real-time. Each time a user visits a website, the publisher posts the available ad slot in the exchange, together with some descriptive user information (typically based on browsing history). Based on this information, advertisers compete in an auction for placing their ad. Advertisers participate in these auctions with the objective of fulfilling marketing campaigns that are subject to budget constraints that limit their total expenditure throughout the horizon. While we refer the reader to Muthukrishnan (2009) for a more detailed description of ad exchanges, we next highlight the main characteristics of the competitive landscape formed in these online markets.

**Complexity.** Competition in ad exchange markets features repeated interactions between budget constrained competitors. Throughout its campaign, each advertiser typically bids in numerous (order of millions) auctions in which it competes with many (order of thousands) other advertisers over ad placements that become available. While there are many competing advertisers in the market, the number of advertisers that actively bid in a given auction may be small. Finally, these numerous competing interactions are *coupled* by budgets that limit the total expenditures of advertisers throughout the horizon. Given the frequency of opportunities and the time scale on which decisions are made, bidding is fully automated and governed by algorithms.

**Heterogeneity.** Advertisers that compete in ad exchange platforms may differ one from the other in various aspects. Each advertiser may value differently the opportunity to advertise to a given user, may have different expectations regarding future opportunities, and may be constrained by a different budget. In addition, advertisers may have various levels of strategic and technical sophistication: While some firms devote considerable resources to develop complex data-driven algorithms that are designed to identify and respond to strategies of competitors in real-time to maximize their campaign utility, other, less resourceful firms, may adopt simple bidding strategies that may be independent of idiosyncratic user information (e.g., “bid $x$ until running out of budget”), or even appear as arbitrary.

**Uncertainty.** Another distinctive feature is that advertisers interact in a highly uncertain environment. First, just before each auction advertisers evaluate in real-time the current opportunity (in terms of, e.g., likelihood of purchase) in case of winning the auction based on the information about the visiting user. However, advertisers naturally do not hold such information about future opportunities that may be realized throughout the campaign. Second, advertisers typically know
very little about the extent of competition they face in ad exchange markets. Advertisers typically know neither the total number of competing advertisers, nor the number of advertisers that actively bid in each auction. In particular, advertisers typically do not know the aforementioned varying characteristics of their competitors, including their valuations, their budgets, their strategies as well as the level of their strategic and technical sophistication.

**Research questions.** The main questions we address in this paper are: (i) How should budget-constrained advertisers compete in repeated auctions under uncertainty? (ii) What type of performance can be *guaranteed* without any assumptions on the competitors’ bidding behavior? (iii) Can a single class of strategies constitute an equilibrium (when adopted by all advertisers) while achieving the best performance that can be guaranteed off-equilibrium? Due to their practical relevance, these questions draw significant attention from online advertisers and advertising platforms.

1.1 **Main contribution**

At a high level, the main contribution of the paper lies in introducing a new family of practical and intuitive bidding strategies that dynamically adapt to uncertainties and competition throughout the campaign based only on observed expenditures, and analyzing the performance of these strategies *off-equilibrium* and *in-equilibrium*. In more detail, our contribution is along the following lines.

**Formulation and solution concepts.** We formulate the budget-constrained competition between advertisers in ad exchange markets as a sequential game of incomplete information, where bidders know neither their own valuation distribution, nor the budgets and valuation distributions of their competitors. To evaluate the performance of a given strategy without any assumptions on competitors’ behavior, we quantify the portion it can *guarantee* out of the performance of the best (feasible) dynamic sequence of bids one could have selected with the benefit of hindsight. The proposed performance metric, referred to as $\gamma$-competitiveness, extends the one of no-regret strategies (also called Hannan consistent strategies, and commonly used in unconstrained repeated games settings), to a more stringent benchmark that allows for dynamic sequences of actions subject to global constraints. By showing that the maximal portion that can be guaranteed is, in general, smaller than one, we establish the impossibility of a “no-regret” notion relative to a dynamic benchmark: No admissible strategy can guarantee long-run average performance that asymptotically converges to the one achieved by the best sequence of bids.
A new class of adaptive bidding strategies. We introduce a class of practical bidding strategies that adapt the bidding behavior of advertisers throughout the campaign, based only on the sample path of expenditures they exhibit. We refer to this class as adaptive pacing strategies, as these strategies dynamically adjust the “pace” at which the advertiser depletes its budget. The depletion pace is adapted by updating a multiplier that captures the shadow price associated with the budget constraint, and effectively determines the extent at which the advertiser bids below its true values (“shades” bids). This multiplier is dynamically updated according to an intuitive primal-dual learning scheme that extends single-agent learning ideas. While the proposed class of strategies has appealing practical features, it is worthwhile noting that a key purpose in its design is to facilitate formalizing the type of performance that can be achieved and to highlight key trade-offs at play in a simple and intuitive manner.

Performance and stability. We analyze the performance of this class of strategies under different assumptions on competitors’ bidding behavior. Through matching lower and upper bounds on the γ-competitiveness we establish the asymptotic optimality of this class of strategies, i.e., no other strategy can asymptotically guarantee a larger portion out of the performance attainable with the benefit of hindsight. When adopted by all the bidders, we show that the resulting dynamics converge to a tractable and meaningful steady state, and characterize a regime (that is well motivated in the context of ad exchanges) under which these strategies constitute an approximate Nash equilibrium: The benefit of unilaterally deviating to fully informed dynamic strategies becomes negligible as the number of auctions and competitors grows large. This establishes a connection between regret minimization under uncertainty and a strong notion of market stability in our setting: Advertisers can follow bidding strategies that constitute an equilibrium when adopted simultaneously and, at the same time, ensure the best performance that can be guaranteed against arbitrary bids. The latter connection may have implications on the design of online markets, for example, in predicting the impact of advertisers’ learning strategies on publishers’ revenues and advertisers’ utilities.

1.2 Related work

Our study lies in the intersection of literature streams in Marketing, Economics, Operations, and Computer Science studying the broad application area of online advertising as well as methodological aspects in games of incomplete information, auctions, and stochastic approximation.

Sequential games of incomplete information. A rich literature stream has been studying conditions under which various observation-based dynamics may or may not converge to an equi-
librium. For extensive reviews of related work, which excludes budget considerations and mostly focuses on discrete action sets and complete information, see books by Weibull (1995), Hofbauer and Sandholm (1998), Fudenberg and Levine (1998), Cesa-Bianchi and Lugosi (2006), as well as Hart and Mass-Colell (2013). Exceptions that consider incomplete information include Hart and Mass-Colell (2001), which suggest a stimulus-based procedure that, when adopted by all players, induces an empirical distribution of play that converges to a correlated equilibria in the fully informed one-shot game, as well as Germano and Lugosi (2007) that modify a regret-testing procedure introduced by Foster and Young (2006) to achieve convergence to an $\varepsilon$-Nash equilibrium.

While convergence to equilibrium has appealing features, it generally does not imply any guarantees on the performance along the strategy’s decision path. One main notion that has been used to evaluate performance along the decision path is Hannan consistency (Hannan 1957), also referred to as universal consistency, or equivalently, no-regret. A strategy is Hannan consistent if under arbitrary competitors’ actions it guarantees payoff at least as high as the one achieved by the single best action that could have been selected with the benefit of hindsight. Hannan consistency was used as a solution concept in many sequential settings; Examples include Foster and Vohra (1993), Fudenberg and Levine (1995), and Freund and Schapire (1999), among many others. An important connection between Hannan consistency and convergence to equilibrium was established by Hart and Mass-Colell (2000), which provided a no-regret strategy that, when adopted by all players, induces an empirical distribution of play that converges asymptotically to a correlated equilibrium. Stoltz and Lugosi (2007) extended the former to compact and convex action sets.

In the current paper we advance similar ideas by establishing a connection between regret-minimizing and equilibrium in a setting with budget constraints that limit the sequence of actions players may take. Since in such setting repeating any single action may be infeasible as well as generate poor performance, we replace the static benchmark that appears in Hannan consistency with the best dynamic and feasible sequence of actions. We show that in general, no learning strategy can guarantee “no-regret” relative to this stringent benchmark, and characterize the maximal portion that may be guaranteed out of it. We provide a class of strategies that achieve this maximal portion, and show that these strategies constitute an $\varepsilon$-Nash equilibrium among the class of fully informed dynamic strategies. The latter solution concept is particularly ambitious under incomplete information as it involves present as well as look-ahead considerations, and is in general hard to maintain even in more information abundant settings (see, e.g., Brafman and Tennenholtz 2004 and Ashlagi et al. 2012).
Learning and equilibrium in sequential auctions. Our work relates to other papers that study learning in various settings of repeated auctions. Iyer et al. (2014) adapt a mean field approximation to study repeated auctions in which bidders learn about their own private value over time. Weed et al. (2016) consider a sequential common value auction setting, where bidders do not know the associated intrinsic value, and adapts multi-arm-bandit policies to analyze the exploration-exploitation tradeoff faced by bidders. Bayesian learning approaches were analyzed, e.g., by Hon-Snir et al. (1998) in the context of sequential first price auctions, as well as by Han et al. (2011) under monitoring and entry costs. All these studies, however, address settings significantly different than ours, and in particular, do not consider budget constraints. In our setting, items are sold via repeated second-price auctions and bidders observe their private values at the time of bidding (but are uncertain about future opportunities). While bidding truthfully is weakly dominant in a static second-price auction, budget-constrained advertisers need to shade their bids to account for the (endogenous) option value of future opportunities. To perform well, a bidder needs to learn this option value, which depends directly on its budget, together with unknown factors such as its value distribution as well as the budgets, value distributions, and strategic behavior of its competitors.

The aforementioned wide array of advertisers’ sophistication levels rises the question of whether advertisers should aim to learn an equilibrium solution concept. The question of whether bidders “find” and “stick” with equilibrium bidding strategies in reality was extensively studied; see survey of results in Chapter 7 of Kagel et al. (1995). In the context of display ad auctions, Nekipelov et al. (2015) recently suggested a regret-based approach for estimating players values from their bids under more lenient behavior assumption relative to equilibrium-based approaches, and demonstrated based on data of ad auction bids that their approach leads to estimation results that are roughly on par with econometric analysis that is based on equilibrium-based structural models (see, e.g., Athey and Nekipelov 2010). In another recent paper, Nisan and Noti (2016) conduct experiments where (human) bidders compete in sequential auctions that simulate ad auction platforms. In their repeated experiment, regret-based estimation was shown to achieve accuracy that is roughly on par with equilibrium-based econometric methods. While concerned with settings that are different then ours (fixed values, no budget constrains), these findings emphasize the necessity of a twofold performance analysis for candidate bidding strategies: in-equilibrium, when adapted by all advertisers, together with off-equilibrium guarantees against arbitrary competitors’ bids.

Ad exchange markets. When advertisers’ types are known, Balseiro et al. (2015) introduce a fluid mean-field approximation to study the outcome of the strategic interaction between budget-
constrained advertisers bidding in ad exchanges. While they show that stationary strategies that
shade bids using a constant multiplier constitute an approximate Nash equilibrium, these strategies
are based on complete information on the value distributions and the budget constraints of all the
advertisers. The current paper focuses on the practical challenge of bidding in these repeated
auctions, in particular, when such broad information on the entire market is not available to
the bidder, and when competitors do not necessarily follow an equilibrium strategy. While we
do not impose a fluid mean-field approximation in the current paper, under such approximation
our proposed strategies could be viewed as ones that converge under incomplete information to
an equilibrium in fluid-based strategies. To some extent, the convergence of simple and intuitive
dynamics to a fluid mean-field equilibrium provides support to the latter as a solution concept.

More broadly, our study contributes to a growing stream of literature that studies display
ad allocation in general and ad exchanges in particular. Most of this literature focuses on the
publishers’ perspective, and treat the advertisers’ competition as exogenous; See, e.g., McAfee
et al. (2009), Alaei et al. (2009), Yang et al. (2012) and Balseiro et al. (2014). Focusing on the
advertiser’s perspective, Ghosh et al. (2009) study the design of a bidding agent for a campaign, and
Jiang et al. (2014) provide some numerical analysis of a bidding procedure based on approximating
scheme that is similar to ours; However, both papers take the perspective of a single agent that
bids in the presence of an exogenous and stationary market. An additional line of research studies
online advertising while focusing on guaranteed contracts, mostly from scheduling and revenue
management perspectives; See, e.g., Roels and Fridgeirsdottir (2009), Araman and Fridgeirsdottir

**Stochastic approximation methods.** The bidding strategies we provide are based on approx-
imating a Lagrangian dual objective through a stochastic approximation scheme that essentially
constructs and follows sub-gradient estimates. The study of the broad class of stochastic approxi-
mination methods originates with the work of Robbins and Monro (1951) and Kiefer and Wolfowitz
(1952), and since its inception has been widely studied and applied to diverse problems in a variety
of fields, including Economics, Statistics, Operations Research, Engineering and Computer Science;
for a survey of reviews and applications see Benveniste et al. (1990), Kushner and Yin (2003), and
Lai (2003), as well as a review by Araman and Caldentey (2011) for more recent applications in the
broad area of revenue management. This literature stream focuses on single-agent learning methods
of adjusting to uncertain environment that is typically exogenous and stationary, with only few
exceptions that are relevant in our context. Besbes et al. (2015) provide a general non-stationary
stochastic approximation framework that allows temporal changes in the underlying environment, but these changes are exogenous and the decision maker operates as a monopolist. Rosen (1965) demonstrates a method for finding an equilibrium in concave games using a gradient-based method that is applied in a centralized manner under complete information on the payoff functions; See also related results for two players with discrete action sets in Singh et al. (2000), Bowling and Veloso (2001), and Shamma and Arslan (2005). In another related work, Nedic and Ozdaglar (2009) study the convergence of distributed gradient-based methods adopted by cooperative agents to minimize a centralized objective, where some information is exchanged between agents.

The current paper contributes to this literature by extending single-agent learning ideas in two key aspects. First, we apply and analyze our procedure under incomplete information in a complex, competitive setting of practical importance. When the learning strategies are adopted independently by all advertisers the resulting environment is endogenous and naturally non-stationary. Second, our scheme uses expenditure observations to construct sub-gradient estimates period-by-period, each time for a different component of the dual objective that we approximate. This non-standard approach allows advertisers to learn efficiently throughout the campaign, rather than from one campaign to another.

Budget pacing strategies. In this paper we propose a class of dynamic strategies that adjust the bidding behaviour of advertisers through controlling the pace at which they deplete their budget. The practice of budget pacing has received increasing attention in the online advertising industry and this notion has become a common term among companies in the industry, which often recommend advertisers to pace their budget or provide budget pacing services using a variety of heuristics.\footnote{A short and insightful tutorial on the budget pacing algorithm Facebook offers to advertisers is available at \url{https://developers.facebook.com/docs/marketing-api/pacing}; For further examples see discussions in online blogs powered by Twitter and ExactDrive as well as the leading ad exchange platform DoubleClick.} Few budget management approaches have been recently considered in the literature; see, e.g., Karande et al. (2013), Charles et al. (2013), and Balseiro et al. (2017), as well as Zhou et al. (2008) in the context of sponsored search in an adversarial setting. However, to the best of our knowledge, this paper is the first to propose a class of practical budget pacing strategies with proved performance guarantees (and concrete notions of optimality) under competition and incomplete information.

Temporal spacing of repeated exposures. Notably, the notion of budget pacing may appear akin to the one of temporal spacing of repeated exposures that has been discussed in the Marketing and Psychology literature in various contexts of advertising; See a review by Noel and Vallen (2009) as well as more recent work by Sahni (2015). It is worthwhile to highlight the conceptual difference
between these two notions. Temporal spacing of exposures refers to spreading exposures throughout
time to maximize the impact of the campaign, and motivated by the *phycological* “spacing effect,”
by which repeated exposures are more effective when spaced. On the other hand, the budget pacing
notion discussed in the current paper is motivated by the *operational* value that lies in being able to
exploit attractive bidding opportunities that may appear throughout the entire campaign horizon.

## 2 Incomplete information model

We model a sequential game of incomplete information with budget constraints, where *K* risk-
neutral heterogeneous advertisers repeatedly bid to place ads through second-price auctions. At
each time period *t* = 1, ..., *T* there is an available ad slot that is auctioned. The auctioneer (i.e.,
the ad exchange platform) first shares with the advertisers some information about a visiting user,
and then runs a second-price auction with a reserve price of zero to determine which ad to show to
the user.\(^2\) The information provided by the auctioneer heterogeneously affects the value advertisers
perceive for the impression based on their targeting criteria.

The subjective values advertisers assign to the impression they bid for at time *t* are denoted by
the random vector \(\{v_{k,t}\}_{k=1}^{K}\), that is assumed to be independently distributed across impressions
and advertisers, with a support over \([0, \bar{v}]^K \subset \mathbb{R}_+^K\). We denote the marginal cumulative distribution
function of \(v_{k,t}\) by \(F_k\), and assume that \(F_k\) is absolutely continuous with bounded density \(f_k\). At
the beginning of the sequence of auctions advertiser *k* has a budget \(B_k\), which limits the total
payments that can be made by the advertiser throughout the campaign. We denote by \(\rho_k := B_k/T\)
the target expenditure (per impression) rate of advertiser *k*; we assume \(0 < \rho_k \leq \bar{v}\) for each
advertiser \(k \in \{1, \ldots, K\}\), else the problem is trivial as the advertiser never runs out of budget.\(^3\)
Each advertiser *k* is therefore characterized by a type \(\theta_k := (F_k, \rho_k)\).

At each period *t* each advertiser privately observes a valuation \(v_{k,t}\), and then posts a bid \(b_{k,t}\).
Given the bidding profile \(\{b_{k,t}\}_{k=1}^{K} \in \mathbb{R}_+^K\), we denote by \(d_{k,t}\) the highest competing bid faced by
advertiser *k*:

\[
d_{k,t} := \max_{i : i \neq k} \{b_{i,t}\}.
\]

We assume that advertisers have a quasilinear utility function given by the difference between the
sum of the valuations generated by the impressions won and the expenditures corresponding to the

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\(^2\) The simplifying assumption of a zero reserve price can be easily adjusted to accommodate positive reserve prices.

\(^3\) Corresponding, for example, to a case when advertisers have periodic (daily or weekly) budgets, we focus on the
case of *synchronous campaigns* where all campaigns start and finish simultaneously. This model captures some of the
key market features, while providing the tractability required to transparently highlight key problem characteristics.
second-price rule. We denote by $z_{k,t}$ the expenditure of advertiser $k$ at time $t$:

$$z_{k,t} := 1\{d_{k,t} \leq b_{k,t}\}d_{k,t},$$

and the corresponding net utility by $u_{k,t} := 1\{d_{k,t} \leq b_{k,t}\}(v_{k,t} - d_{k,t})$. After the bidding takes place, the auctioneer allocates the ad placement to the highest bidder, and each advertiser $k$ privately observes its own expenditure $z_{k,t}$ and net utility $u_{k,t}$.

**Information structure and admissible budget-feasible bidding strategies.** We assume that in the beginning of the horizon advertisers have no information on their participation probabilities and the valuation distributions, as well as their competitors’, that is, the distributions $\{F_k : k = 1 \ldots, K\}$ are unknown. Each advertiser $k \in \{1, \ldots, K\}$ knows its own target expenditure rate $\rho_k$, as well as the length of the campaign $T$, but has no information about the budgets or the target expenditure rates of its competitors.\(^4\) In particular, each advertiser does not know the number of competitors in the market, their types, or even its own type.

We next formalize the class of (potentially randomized) admissible and budget feasible bidding strategies. Let $y$ be a random variable defined over a probability space $(\mathcal{Y}, \mathcal{Y}, \mathbb{P}_y)$. We denote by $\mathcal{H}_{k,t}$ the history available at time $t$ to advertiser $k$, defined by

$$\mathcal{H}_{k,t} := \sigma\left(\left\{v_{k,\tau}, b_{k,\tau}, z_{k,\tau}, u_{k,\tau}\right\}_{\tau=1}^{t-1}, v_{k,t}, y\right)$$

for any $t \geq 2$, with $\mathcal{H}_{k,1} := \sigma(v_{k,1}, y)$. A bidding strategy $\beta$ for advertiser $k$ is a sequence of bids $\{b_{k,t}^\beta : t = 1, \ldots, T\}$. We say that a bidding strategy $\beta$ is admissible, or non-anticipating, if it depends only on available histories, that is, at each period $t$ the bid $b_{k,t}^\beta$ is measurable with respect to the filtration $\mathcal{H}_{k,t}$. We say that $\beta$ is budget-feasible if it generates expenditures that are constrained by the available budget, that is, $\frac{1}{T} \sum_{t=1}^{T} 1\{d_{k,t} \leq b_{k,t}^\beta\}d_{k,t} \leq \rho_k$ for any realized vector of highest competitors’ bids $d_k = \{d_{k,t}\}_{t=1}^{T} \in \mathbb{R}_+^T$. We denote by $\mathcal{B}_k$ the class of admissible budget-feasible strategies for advertiser $k$ with target expenditure rate $\rho_k$, and by $\mathcal{B} = \mathcal{B}_1 \times \ldots \times \mathcal{B}_K$ the product space of strategies for all advertisers.

**Off-equilibrium performance.** To evaluate off-equilibrium performance of a candidate strategy $\beta \in \mathcal{B}_k$ without any assumptions on the primitives and strategies that govern competitors’ decisions, we define the sample-path performance given vectors of realized values $v_k = \{v_{k,t}\}_{t=1}^{T} \in [0, \bar{v}]^T$ and

\(^4\)In practice advertisers may infer the length of their campaign (in terms of total number of impressions) prior to its beginning of their campaign using user traffic statistics commonly provided by publishers and analytics companies.
realized competing bids $d_k = \{d_{k,t}\}_{t=1}^T \in \mathbb{R}_+^T$ as follows:
\[
\pi^\beta_k (v_k; d_k) := \mathbb{E}^\beta \left[ \sum_{t=1}^T 1\{d_{k,t} \leq b_{k,t}^\beta\} (v_{k,t} - d_{k,t}) \right],
\]
where the expectation is taken with respect to any randomness embedded in the strategy $\beta$.

We quantify the minimal loss a strategy can guarantee relative to any sequence of bids one could have made with the benefit of hindsight. Given sequences of realized valuations $v_k$ and highest competing bids $d_k$, we denote by $\pi^H_k (v_k; d_k)$ the best performance advertiser $k$ could have achieved with the benefit of hindsight:
\[
\pi^H_k (v_k; d_k) := \max_{x_k \in \{0,1\}^T} \sum_{t=1}^T x_{k,t} (v_{k,t} - d_{k,t})
\]
\[\text{s.t. } \frac{1}{T} \sum_{t=1}^T x_{k,t} d_{k,t} \leq \rho_k,\]
where the binary variable $x_{k,t}$ indicates whether the auction at time $t$ is won by advertiser $k$. The solution of (1) is the best dynamic response to $v_k$ and $d_k$, capturing the best sequence of bids that “could have been made” against a realized competitive environment. While the identification of the best sequence of bids in hindsight does not take into account the potential competitive reaction to that sequence, this benchmark is tractable and has appealing practical features. For example, the best performance in hindsight can be computed using only historical data and requires no behavioral assumptions on competitors (Talluri and van Ryzin, 2004, Ch. 11.3).

Non-anticipating strategies might not be able to achieve or approach $\pi^H_k$. For some $\gamma \in [1, \infty)$, a bidding strategy $\beta \in B_k$ is said to be asymptotic $\gamma$-competitive (Borodin and El-Yaniv, 1998) if:
\[
\limsup_{T \to \infty} \sup_{\rho_k T \in [0, \infty]} \sup_{d_k \in \mathbb{R}_+^T} \frac{1}{T} \left( \pi^H_k (v_k; d_k) - \gamma \pi^\beta_k (v_k; d_k) \right) \leq 0,
\]
An asymptotic $\gamma$-competitive bidding strategy (asymptotically) guarantees a portion of at least $1/\gamma$ out of the performance of any dynamic sequence of bids that could have been selected in hindsight.\(^5\)

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5While in the definition of $\pi^\beta_k (v_k; d_k)$, the vectors $v_k$ in $[0, \bar{v}]^T$ and $d_k$ in $\mathbb{R}_+^T$ are fixed in advance, we also allow the components of these vectors to be selected dynamically according to the realized path of $\beta$, that is, $v_{k,t}$ and $d_{k,t}$ could be measurable with respect to the filtration $\sigma \{v_{k,\tau}, b_{k,\tau}, z_{k,\tau}, u_{k,\tau}\}_{\tau=1}^{t-1}$. This allows nonoblivious or adaptive adversaries, and in particular, valuations and competing bids that are affected by the player’s strategy. For further details see Chapter 7 of Cesa-Bianchi and Lugosi (2006).
Equilibrium solution concept. Given a strategy profile $\beta = (\beta_k)_{k=1}^K \in \mathcal{B}$, we denote by $\Pi_k^\beta$ the total expected payoff for advertiser $k$:

$$
\Pi_k^\beta := \mathbb{E}_\nu \left[ \sum_{t=1}^T \mathbb{1}\{d_{k,t}^\beta \leq b_{k,t}^\beta\}(v_{k,t} - d_{k,t}^\beta) \right],
$$

where the expectation is taken with respect to any randomness embedded in the strategies $\beta$ and the random values $\{v_{k,t}\}_{k,t} \in [0, \bar{v}]^{K \times T}$ of all the advertisers, and where we denote the highest competing bid induced by competitors’ strategies $\beta_{-k} = (\beta_i)_{i \neq k}$ by $d_{k,t}^{\beta_{-k}} = \max_{i : i \neq k} \{b_{i,t}^\beta\}$. We say that a strategy profile $\beta \in \mathcal{B}$ constitutes an $\varepsilon$-Nash equilibrium in dynamic strategies if each player’s incentive to unilaterally deviate to another strategy is at most $\varepsilon$, that is, if for all $k$:

$$
\frac{1}{T} \left( \sup_{\beta \in \mathcal{B}_{CI}^k} \Pi_k^{\beta,\beta_{-k}} - \Pi_k^{\beta} \right) \leq \varepsilon,
$$

where $\mathcal{B}_{CI}^k \supseteq \mathcal{B}_k$ denotes the class of strategies with complete information on market primitives. While a more precise description of this class will be advanced in §5 we note that this class also includes strategies with access to complete information on the types $(\theta_i)_{i=1}^K$, as well as the bids made in past periods by all advertisers. Naturally, the solution concept in (3) takes into account the dynamic response of the competitive environment to the actions of an advertiser.

3 Off-equilibrium performance guarantees

In this section we study the performance that can be guaranteed by bidding strategies without any assumptions on the primitives (budgets, valuations) and strategies that govern competitors’ bids. We obtain a lower bound on the minimal loss one must incur relative to the best dynamic response in hindsight. We then introduce a class of adaptive pacing strategies that react to realized valuations and expenditures throughout the campaign, and obtain an upper bound for the performance of these strategies that asymptotically matches the aforementioned lower bound.

3.1 Lower bound on the achievable guaranteed performance

Before investigating the type of performance an advertiser bidding for impressions under a budget constraint could aspire to, we first formalize the type of performance that cannot be achieved. The following result bounds the minimal loss one must incur relative to the best dynamic (and feasible) sequence of bids that could have been selected in hindsight.
Theorem 3.1. (Lower bound on the achievable guaranteed performance) For any target expenditure rate $\rho_k \in (0, \bar{v}]$ and a number $\gamma < \bar{v}/\rho_k$ there exists some constant $C > 0$ such that for any bidding strategy $\beta \in B_k$:

$$\limsup_{T \to \infty} \sup_{B_k = \rho_k T} \frac{1}{T} \left( \pi^{\text{H}}_k(v_k; d_k) - \gamma \pi^{\beta}_k(v_k; d_k) \right) \geq C.$$ 

Theorem 3.1 establishes that no admissible strategy can guarantee asymptotic $\gamma$-competitiveness for any $\gamma < \bar{v}/\rho_k$, and therefore cannot guarantee a portion larger than $1 : (\bar{v}/\rho_k)$ of the total net utility achieved by a dynamic response that is taken with the benefit of hindsight. This implies that any bidding strategy may perform poorly relative to such dynamic responses when the target expenditure rate $\rho_k$ is small relative to the maximal valuation $\bar{v}$, but better guaranteed performance is achievable as $\rho_k$ gets closer to $\bar{v}$ (when $\rho_k \geq \bar{v}$ the advertiser can bid thoughtfully to guarantee optimal performance and 1-competitiveness). Since $\rho_k$ may be smaller than $\bar{v}$, Theorem 3.1 shows that in general the performance of any learning bidding strategy cannot be guaranteed to converge asymptotically to $\pi^{\text{H}}_k(v_k; d_k)$, and thus establishes the impossibility of a “no-regret” notion relative to dynamic sequences of bids.

We next describe the main ideas in the proof of the Theorem. To analyze the loss relative to the best dynamic response in hindsight we first observe, by adapting Yao’s principle (Yao, 1977, Borodin and El-Yaniv, 1998) to our setting, that in order to bound the worst-case loss of any strategy (deterministic or not) relative to the best response in hindsight, it suffices to analyze the expected loss of deterministic strategies relative to the same benchmark, where the sequence of valuations is drawn from a certain distribution (see Lemma A.1). We construct an instance that essentially follows the next structure. We assume for simplicity that the advertiser knows in advance that the highest competing bids will be fixed at $d$ along the campaign, and that the sequence of values will be randomly selected from the set $\{v^1, v^2\}$, where

$$v^1 = \left( v_{\text{low}}^{\tau \text{ auctions}}, \ldots, v_{\text{low}}^{\tau \text{ auctions}}, d, \ldots, d \right)_{T - \tau \text{ auctions}}$$

$$v^2 = \left( v_{\text{low}}^{\tau \text{ auctions}}, \ldots, v_{\text{low}}^{\tau \text{ auctions}}, v_{\text{high}}, \ldots, v_{\text{high}} \right)_{T - \tau \text{ auctions}}$$

---

6When $\beta$ is deterministic the lower bound in Theorem 3.1 is held in every sample path. In particular, it holds also when valuations and competitors’ bids are allowed to be arbitrarily correlated across impressions and advertisers.
for some $\bar{v} \geq v_{\text{high}} > v_{\text{low}} > d > 0$. We establish that under such structure one may restrict analysis to strategies that determine before the beginning of the campaign how many auctions to win at different stages of the campaign (see Lemma [A.2]). This presents the advertiser with the following tradeoff: while early auctions introduce a return per unit of budget that is certain but low, later auctions introduce a return per unit of budget that may be higher but may also decrease to zero. Since the number of auctions in the first stage, denoted by $\tau$, may be designed to grow with $T$, this tradeoff may drive loss that does not diminish to zero asymptotically. The proof of the theorem presents a more general construction that follows the ideas illustrated above, and tunes the structural parameters to maximize the worst-case loss that must be incurred by any strategy relative to the best dynamic response in hindsight.

3.2 Asymptotic optimal bidding strategy

We introduce an adaptive pacing strategy that adjust bids based on observations. In what follows we denote by $P_{[a,b]}(x) = \min\{\max\{x,a\},b\}$ the Euclidean projection operator on the interval $[a,b]$.

Adaptive pacing strategy (A). Input: a number $\epsilon_k > 0$.

1. Select an initial multiplier $\mu_1$ in $[0, \bar{\mu}_k]$, and set the remaining budget to $\tilde{B}_{k,1} = B_k = \rho_k T$.

2. For each $t = 1, \ldots, T$:
   
   (a) Observe the realization of the random valuation $v_{k,t}$, and post a bid:
   $\
b_{k,t}^A = \min \left\{ \frac{v_{k,t}}{1 + \mu_{k,t}}, \tilde{B}_{k,t} \right\}.$

   (b) Observe the expenditure $z_{k,t}$. Update the multiplier by
   $\
   \mu_{k,t+1} = P_{[0,\bar{\mu}_k]} \left( \mu_{k,t} - \epsilon_k \left( \rho_k - z_{k,t} \right) \right)
   $ and the remaining budget by $\tilde{B}_{k,t+1} = \tilde{B}_{k,t} - z_{k,t}$.

The adaptive pacing strategy dynamically adjusts the pace at which the advertiser depletes its budget by updating a multiplier that determines the extent at which the advertiser bids below its true values (shades bids). The strategy consists of a sequential approximation scheme taking place in the dual space, designed to approximate the best solution in hindsight defined in (1). The objective of the Lagrangian dual of (1) is $\sum_{t=1}^{T} x_{k,t}(v_{k,t} - (1 + \mu)d_{k,t}) + \mu \rho_k$, where $\mu$ is a nonnegative multiplier capturing the shadow price associated with the budget constraint. For a fixed $\mu$, the dual objective is maximized by winning all items with $v_{k,t} \geq (1 + \mu)d_{k,t}$, which is obtained by bidding $b_{k,t} = v_{k,t}/(1 + \mu)$, since advertiser $k$ wins whenever $b_{k,t} \geq d_{k,t}$. Therefore, one has:
\[
\pi_k^H(v_k; d_k) \leq \inf_{\mu \geq 0} \sum_{t=1}^{T} (v_{k,t} - (1 + \mu)d_{k,t})^+ + \mu \rho_k, \tag{4}
\]
where we denote by \(y^+ = \max(y, 0)\) the positive part of a number \(y \in \mathbb{R}\). The tightest upper bound can be obtained by solving the minimization problem in the right hand side of (4); Since the latter cannot be solved without prior information on all the values and competing bids throughout the campaign, the proposed adaptive pacing strategy approximates (4) by estimating a multiplier \(\mu_{k,t}\) and bidding \(b_{k,t}^H = v_{k,t}/(1 + \mu_{k,t})\) at each around, as long as the remaining budget suffices.\(^7\)

Essentially, the dual approximation scheme consists of estimating a direction of improvement and following that direction. More precisely, the sequence of multipliers follows a subgradient descent scheme, using the noisy point access each advertiser has to \(\partial^- \psi_{k,t}(\mu_{k,t})\), the left derivative of the \(t\)-period component of dual objective in (4). Namely, at each period \(t = 1, 2\ldots\) one has:

\[
\partial^- \psi_{k,t}(\mu_{k,t}) = \rho_k - d_{k,t}1\{v_{k,t} \geq (1 + \mu_{k,t})d_{k,t}\} = \rho_k - d_{k,t}1\{b_{k,t}^H \geq d_{k,t}\} = \rho_k - z_{k,t}.
\]

At each period \(t\) the advertiser compares its expenditure \(z_{k,t}\) to the target expenditure rate \(\rho_k\) that is “affordable” given the initial budget. Whenever the advertiser’s expenditure exceeds the target expenditure rate, \(\mu_{k,t}\) is increased by \(\epsilon_k(z_{k,t} - \rho_k)\), implying an increase in the shadow price associated with the budget constraint (that just became more binding). On the other hand, if the advertiser’s expenditure is lower than the target expenditure rate (including periods in which the expenditure is zero) the multiplier \(\mu_{k,t}\) is decreased by \(\epsilon_k(\rho_k - z_{k,t})\), implying a decrease in that shadow price (the budget constraint becomes less binding). Then, given the valuation at that period, the advertiser shades its value using the current estimated multiplier.

At a high level, the multiplier \(\mu_{k,t}\) captures the “belief” advertiser \(k\) has at period \(t\) regarding the shadow price associated with its budget constraint. Notably, the “correct” shadow price, reflecting the “correct” value of future opportunities, depends not only on the budget (which is known to the advertiser), but also on the value distribution of the advertiser, as well as the value distributions, budgets, and strategies of its competitors. Rather than aiming at estimating the latter unknown factors, the proposed class of strategies directly learns the shadow price – a single parameter that captures the impact of these factors on the advertiser’s optimal bidding strategy.

\(^7\)The heuristic of bidding \(b_{k,t}^H = v_{k,t}/(1 + \mu^H)\) until the budget is depleted, with \(\mu^H\) optimal for (4), can be shown to be asymptotically optimal for the hindsight problem \(^1\) as \(T\) grows large (see, e.g., Talluri and van Ryzin 1998).
**Assumption 3.2. (Appropriate selection of step size)** The number $\epsilon_k$ satisfies $0 < \epsilon_k \leq 1/\bar{v}$, $\lim_{T \to \infty} \epsilon_k = 0$ and $\lim_{T \to \infty} T \epsilon_k = \infty$.

Even though Assumption 3.2 places some restrictions on the step size selection, it still defines a broad class of step sizes; for example, $\epsilon_k = c/T$ satisfies Assumption 3.2 for any constants $0 < c \leq 1/\bar{v}$ and $0 < \gamma < 1$.

**Theorem 3.3. (Asymptotic optimality of the adaptive pacing strategy)** Let $A$ be the adaptive pacing strategy with a step size $\epsilon_k$ satisfying Assumption 3.2 and $\bar{\mu}_k \geq \bar{v}/\rho_k - 1$. Then:

$$\limsup_{T \to \infty} \sup_{B_k = \rho_k T \in \mathbb{R}_+^T} \frac{1}{T} \left( \pi_k^A(v_k, d_k) - \frac{\bar{v}}{\rho_k} \pi_k^A(v_k, d_k) \right) = 0.$$  

Theorem 3.3 establishes that the adaptive pacing strategy is an asymptotic $\frac{\bar{v}}{\rho_k}$-competitive online algorithm [Borodin and El-Yaniv 1998], guaranteeing at least $1 : (\bar{v}/\rho_k)$ out of the total net utility achieved by any dynamic response that could have been guaranteed with the benefit of hindsight.

Given Theorem 3.1, this also establishes the asymptotic optimality of the adaptive pacing strategy, as no other admissible bidding strategy can guarantee a larger fraction of the best performance in hindsight. Theorems 3.1 and 3.3 together characterize the best achievable guaranteed performance: a fraction of $1 : (\rho_k/\bar{v})$ out of the performance of any dynamic response that can be selected in hindsight. This portion depends on the target expenditure rate $\rho_k$ of the advertiser: when the latter is low, the best guaranteed performance diminishes, and when it is high, and closer to $\bar{v}$, the best guaranteed performance approaches the one that can be obtained with the benefit of hindsight.

In the proof of the Theorem, we first analyze the time at which the budget is depleted under the adaptive bidding strategy, and show that this time is “close” to $T$. We then express the loss incurred by that strategy relative to the best response in hindsight in terms of the value of lost auctions, and bound the potential value of these auctions. Together, we establish that there exist some positive constants $C_1, C_2$ and $C_3$, independent of $T$, such that for any $d_k \in \mathbb{R}_+^T$ and $v_k \in [0, \bar{v}]^T$,

---

8 We restrict the formal definition of the adaptive pacing strategy to stationary step sizes $\epsilon_k$ only to simplify and shorten analysis. Our results can be adapted to allow time-varying step sizes through a more general form of Assumption 3.2. Nevertheless, we note that the optimality notions we consider are defined for the general class of admissible budget-feasible strategies and achieved within the subclass of strategies with stationary step sizes.

9 We note that the upper bound in Theorem 3.3 is extremely robust with respect to the underlining formulation. For example, it holds also when campaigns can start and finish in different times and under arbitrary sequences of valuations and competing bids, including ones that are correlated across impressions or advertisers, as well as dynamic sequences that are based on knowledge about the bidding strategy and its realized path.
\[
\pi^w_k(v_k; d_k) - \frac{\bar{v}}{\rho_k} \pi^2_k(v_k, d_k) \leq C_1 + \frac{C_2}{\epsilon_k} + C_3 T \epsilon_k .
\]

This allows one to analyze the performance of various step size selections, and in particular, establishes asymptotical optimality whenever Assumption 3.2 holds.

4 Simultaneous learning and convergence

Based on the intuitive nature of adaptive pacing strategies and their asymptotic optimality in the presence of arbitrary competitors’ actions (that was establish in §3), in this section we turn to study the dynamics that emerge when all advertisers follow these strategies simultaneously. We establish that the induced competitive dynamics converge to a tractable and meaningful steady state.

4.1 A steady state candidate

Under simultaneous adoption of the adaptive pacing strategy, each advertiser sequentially follows the sub-gradient \( \partial_{-} \psi_{k,t}(\mu_{k,t}) = \rho_k - z_{k,t} \) to approximate \( \pi^w_k(v_k; d_k) \) using the dual Lagrangian problem in (4). Therefore, if the dynamics that result from simultaneous adoption of the adaptive pacing strategies indeed converge to a steady state, one may expect this steady state to consist of multipliers under which each advertiser’s expected expenditure equals its target expenditure (whenever its budget is binding). Denote the expected expenditure per auction of advertiser \( k \) when advertisers shade bids according to a profile of multipliers \( \mu \in \mathbb{R}^K \) by

\[
G_k(\mu) := \mathbb{E}_v[1\{(1 + \mu_k)d_{k,1} \leq v_{k,1}\}d_{k,1}],
\]

where \( d_{k,1} = \max_{i: i \neq k} \left\{ \frac{v_{i,1}}{(1 + \mu_i)} \right\} \). Following the above intuition, we consider the candidate steady state vector \( \mu^* \), defined by the complementary conditions:

\[
\mu^*_k \geq 0 \quad \perp \quad G_k(\mu^*) \leq \rho_k, \quad \forall k = 1, \ldots, K, \tag{5}
\]

where \( \perp \) indicates a complementarity condition between the multiplier and the expenditure, that is, at least one condition must hold with equality. Intuitively, in a steady state each advertiser shades bids if its expected expenditure is equal to its target rate. Conversely, if an advertiser’s target rate exceeds its expected expenditure, then it bids truthfully by setting the multiplier equal to zero.\(^{10}\)

We next provide sufficient conditions for the uniqueness of the vector \( \mu^* \). The vector function

\(^{10}\)Some features of the vector \( \mu^* \) are studied in Balseiro et al. (2015) under a fluid mean-field approximation.
\( \mathbf{G} : \mathbb{R}^K \to \mathbb{R}^K \) is said to be \( \lambda \)-strongly monotone in a set \( \mathcal{U} \subset \mathbb{R}^K \) if 
\[ (\mathbf{\mu} - \mathbf{\mu}')^T (\mathbf{G}(\mathbf{\mu}') - \mathbf{G}(\mathbf{\mu})) \geq \lambda \|\mathbf{\mu} - \mathbf{\mu}'\|_2^2 \]
for all \( \mathbf{\mu}, \mathbf{\mu}' \in \mathcal{U} \). We refer to \( \lambda \) as the monotonicity constant.

**Assumption 4.1. (Stability)**

1. There exists a set \( \mathcal{U} = \prod_{k=1}^K [0, \bar{\mu}_k] \) and a monotonicity constant \( \lambda > 0 \), independent of \( K \), such that the expected expenditure function \( \mathbf{G}(\mathbf{\mu}) \) is \( \lambda \)-strongly monotone over \( \mathcal{U} \).

2. The target expenditure rate satisfy \( \rho_k \geq \bar{\nu}/\bar{\mu}_k \) for every bidder \( k \).

The first part of the assumption requires that the expenditure function is strongly monotone on a set of feasible multipliers.\(^{11}\) The second part imposes that budgets are not too low and guarantees that \( \mathbf{\mu}^* \in \mathcal{U} \). In Appendix \( \text{C} \) (see Proposition \( \text{C.1} \)) we show that Assumption 4.1 guarantees the existence of a unique vector of multipliers \( \mathbf{\mu}^* \). As we further discuss in §6, uniqueness of \( \mathbf{\mu}^* \) has practical benefits that may motivate the market designer to place restrictions on target expenditure rates of advertisers to guarantee that Assumption 4.1 holds.

### 4.2 Convergence of multipliers

To analyze the sample path of multipliers resulting from simultaneous adoption of the adaptive pacing strategy, we consider some conditions on the step sizes selected by the different advertisers.

**Assumption 4.2. (Joint selection of step sizes)** Let \( \bar{\epsilon} = \max_{k \in \{1, \ldots, K\}} \epsilon_k \) and \( \underline{\epsilon} = \min_{k \in \{1, \ldots, K\}} \epsilon_k \), and let \( \lambda \) be the monotonicity constant that appears in Assumption 4.1. The profile of step sizes \( (\epsilon_k)_{k=1}^K \) satisfies:

\begin{align*}
(i) & \quad \bar{\epsilon} \leq \min \{1/\bar{\nu}, 1/(2\lambda)\} \\
(ii) & \quad \lim_{T \to \infty} \frac{T\bar{\epsilon}^2}{\underline{\epsilon}} = 0 \\
(iii) & \quad \lim_{T \to \infty} \frac{\epsilon_k^2}{\bar{\epsilon}} = 0.
\end{align*}

Assumption 4.2 details conditions on the \textit{joint} selections of step sizes (extending the individual conditions that appear in Assumption 3.2), essentially requiring the step sizes selected by the different advertisers to be “reasonably close” one to the another. Nevertheless, the class of step sizes defined by Assumption 4.2 is quite general and allows flexibility in the individual step size

\(^{11}\)The first part of Assumption 4.1 is similar to other common conditions in the literature that guarantee steady state uniqueness in concave (unconstrained) games (see, e.g., Rosen [1965]), as well as convergence of stochastic approximation methods in multi-dimensional action spaces (see, e.g., §1.10 in Part 2 of Benveniste et al. [1990]). In Appendix \( \text{C} \) we show that this part of the assumption is implied by the diagonal strict concavity condition defined in Rosen [1965], prove that it holds in symmetric settings, and demonstrate its validity in simple cases.
selection. For example, step sizes $\epsilon_k = c_k T^{-\gamma_k}$ satisfy Assumption 4.2 for any constants $c_k > 0$ and $0 < \gamma_k < 1$ such that $\max_k \gamma_k \leq 2 \min_k \gamma_k$ and $2 \max_k \gamma_k \leq 1 + \min_k \gamma_k$. Moreover, when steps sizes are equal across advertisers and satisfy $\epsilon \leq 1/(2\lambda)$, Assumption 4.2 coincides with Assumption 3.2.

The following result states that Assumption 4.2 guarantees that the multipliers selected by advertisers under simultaneous adoption of the adaptive pacing strategy converge to the vector $\mu^*$. In what follows we use the standard norm notation $\|x\|_p := \left(\sum_{k=1}^K |x_k|^p\right)^{1/p}$ for a vector $x \in \mathbb{R}^K$.

**Theorem 4.3. (Convergence of multipliers)** Suppose that Assumption 4.1 holds, and let $\mu^*$ be the profile multipliers defined by (5). Assume that all the advertisers follow adaptive pacing strategies with step sizes that together satisfy Assumption 4.2. Then:

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^T \mathbb{E}\left[\|\mu_t - \mu^*\|_2^2\right] = 0.$$ 

At a high level, Theorem 4.3 adapts standard stochastic approximation techniques (see, e.g., Nemirovski et al. 2009) to accommodate the expenditure feedback form and possibility of different step sizes across agents. In the proof of the Theorem we establish that under Assumption 4.1 there exist positive constants $C_1$ and $C_2$, independent of $T$, such that:

$$\mathbb{E}\left[\|\mu_t - \mu^*\|_2^2 1\left\{\bar{B}_{k,t+1} \geq \bar{v} \forall k\right\}\right] \leq C_1 \frac{\bar{\epsilon}}{\xi} (1 - 2\lambda \xi)^t - 1 + C_2 \frac{\bar{\epsilon}^2}{\xi},$$

for any $t \in \{1, \ldots, T\}$, where the dependence of $C_1$ and $C_2$ on the number of advertisers is of linear order. This implies that insofar as the respective primal solutions are feasible under the remaining budgets, the vector of multipliers $\mu_t$ selected by advertisers under the adaptive pacing strategy converges in $L_2$ to the vector $\mu^*$. We then argue that the adaptive pacing strategies do not run out of budget too early, and establish the result using the conditions in Assumption 4.2.

As we discuss in the following subsection, due to the primal feasibility constraints, convergence in the dual space does not necessarily guarantee convergence in performance, or any performance guarantees. However, the convergence of multipliers established in Theorem 4.3 has a stand-alone practical importance as it provides the market designer an ability to predict bidding behaviour of sophisticated advertisers (assuming it has access to the advertisers’ types) through the simple characterization in (5). This implication is further discussed in §6.
4.3 Convergence in performance

We next study long-run average performance achieved when all advertisers follow the adaptive pacing strategy. Recall that Theorem 4.3 implies that the vector of multipliers $\mathbf{\mu}_t$ selected by advertisers under the adaptive pacing strategy converges to the vector $\mathbf{\mu}^\ast$. However, since at every period the remaining budget may potentially constrain bids, the convergence of multipliers to $\mathbf{\mu}^\ast$ does not guarantee performance that is close to the one achieved under the vector $\mathbf{\mu}^\ast$. Nevertheless, establishing such a performance guarantee is a key step towards equilibrium analysis. As a natural candidate for performance convergence we consider the dual performance that is obtained under the fixed vector $\mathbf{\mu}^\ast$. Given a vector of multipliers $\mathbf{\mu}$, we denote:

$$
\Psi_k(\mathbf{\mu}) := T \left( \mathbb{E}_v \left[ (v_{k,1} - (1 + \mu_k)d_{k,1})^+ \right] + \mu_k \rho_k \right),
$$

with $d_{k,1}$ as defined in §4.1. Recalling (4), $\Psi_k(\mathbf{\mu}^\ast)$ is the dual performance of advertiser $k$ when at each period $t$ each advertiser $i$ bids $b_{i,t} = v_{i,t} / (1 + \mu_i^\ast)$.

**Theorem 4.4. (Convergence in performance)** Suppose that Assumption 4.1 holds, and let $\mathbf{\mu}^\ast$ be the profile of multipliers defined by (5). Let $\mathbf{A} = (A_k)_{k=1}^K$ be a profile of adaptive pacing strategies, with step sizes that together satisfy Assumption 4.2. Then, for each $k \in \{1, \ldots, K\}$:

$$
\limsup_{T \to \infty \atop B_k = \rho_k T} \frac{1}{T} (\Psi_k(\mathbf{\mu}^\ast) - \Pi_k^\mathbf{A}) \leq 0.
$$

In the proof we establish that, when all advertisers follow the adaptive pacing strategy, deviations from target expenditure rates are controlled in a manner such that advertisers very often deplete their budgets “close” to the end of the campaign. Therefore the potential loss along the sample path due to running out of budgets is relatively small compared to the cumulative payoff. We establish that there exists some positive constants $C_1$ through $C_4$, independent of $T$, such that:

$$
\Psi_k(\mathbf{\mu}^\ast) - \Pi_k^\mathbf{A} \leq C_1 + C_2 \bar{\epsilon}^{1/2} + C_3 \bar{\epsilon} T^{1/2} + C_4 \frac{1}{\xi}.
$$

We note that the dependence of the payoff gap on the number of advertisers is of order $K$. This allows performance analysis under various selections of step sizes, and in particular, establishes convergence in performance whenever that sequence of step sizes satisfies Assumption 4.2.
5 Approximate Nash equilibrium in dynamic strategies

In this section we study conditions under which adaptive pacing strategies constitute an $\varepsilon$-Nash equilibrium. First, we note that while under our formulation a profile of adaptive pacing strategies converges in multipliers and performance to a steady state, such a strategy profile does not necessarily constitute an approximate Nash equilibrium. We then detail one extension of our formulation (well-motivated in the context of display ad auctions) where in each period each advertiser participates in one among multiple auctions that take place in parallel. We show that under such structure adaptive pacing strategies constitute an $\varepsilon$-Nash equilibrium in dynamic strategies.

At a high level, an advertiser could potentially gain by deviating from adaptive pacing strategies if this deviation: (i) causes competitors to deplete their budgets earlier to reduce competition in later auctions; or (ii) induces competitors to learn “wrong” multipliers so as to reduce competition before their budgets are depleted. Under Assumption 4.2, the self-correcting nature of the adaptive pacing strategies prevents competitors from depleting budgets too early and thus the first avenue is not effective. However, when the same set of advertisers frequently interact, an advertiser can cause competitors to excessively shade bids by driving them to learn multipliers that are higher than the steady state ones. This may allow a deviating advertiser to reduce competition and profit throughout the campaign.

In practice, however, the number of advertisers actively bidding in an ad exchange is large and, due to high frequency of auctions as well as sophisticated targeting technologies, advertisers often participate only in a fraction of all auctions that take place (Celis et al., 2014). As a result, each advertiser typically competes with a different set of bidders in every auction (Balseiro et al., 2014). This limits the potential impact a single advertiser may have on the expenditure trajectory of any other advertiser and, therefore, the benefit from unilaterally deviating to the strategies described above should be expected to be small.

To demonstrate this intuition, we next detail one extension of our basic framework, where at each period each advertiser participates in one auction among multiple auctions taking place in parallel. (This captures, for example, the case of different publishers simultaneously auctioning one ad slot each or a single publisher auctioning multiple ad slots via parallel auctions.) Each advertiser is “matched” with one of the auctioned ad slots based on its targeting criteria and the attributes associated to the impressions auctioned in each slot. We establish that in markets with these characteristics adaptive pacing strategies constitute an $\varepsilon$-Nash equilibrium.\footnote{While one could consider multiple ways to extend this simple matching model, the main purpose in its design is to...}

...
Parallel auctions model. At each time period $t = 1, \ldots, T$ we now assume that there are $M$ ad slots sold simultaneously in parallel auctions. We assume that each advertiser participates in one of the $M$ auctions independently at random. We denote by $m_{k,t} \in \{1, \ldots, M\}$ the auction advertiser $k$ participates in at time $t$; At each period $t$ the random variable $m_{k,t}$ is independently drawn from a distribution $\alpha_k = \{\alpha_{k,m}\}_{m=1}^M$, where $\alpha_{k,m}$ denotes the probability that advertiser $k$ participates in auction $m$. The type of advertiser $k$ is therefore given by $\theta_k = (F_k, \rho_k, \alpha_k)$.\(^{13}\) The definition of $d_{k,t}$, the highest competing bid faced by advertiser $k$, is adjusted to be
\[
d_{k,t} := \max \left\{ \max \left\{ b_{i,t} : i \neq k \right\}, 0 \right\}.
\]

Notably, all the results we established thus far hold also when advertisers compete in $M$ parallel auctions as discussed above. We next establish that under this extension adaptive pacing strategies constitute an $\varepsilon$-Nash equilibrium in dynamic strategies, in the sense that no advertiser can benefit from unilaterally deviating to any other strategy when the number of time periods and players is large, even when considering strategies with complete information.

We denote by $B_{CI_k} \supseteq B_k$ the space of non-anticipating and budget-feasible strategies with complete information. In particular, strategies in $B_{CI_k}$ may be based on knowledge of all the types $(\theta_i)_{i=1}^K$, as well as the bids made in past periods by all the competitors. To analyze unilateral deviations from adaptive pacing strategies we consider conditions on the likelihood that different advertisers compete in the same auction at a given time period. Let $a_{k,i} = P\{m_{k,t} = m_{i,t}\} = \sum_{m=1}^M \alpha_{k,m} \alpha_{i,m}$ denote the probability that advertisers $k$ and $i$ compete in the same auction at a given time period, and let $a_k = (a_{k,i})_{i \neq k} \in [0, 1]^{K-1}$. We refer to these as the matching probabilities of advertiser $k$.

Assumption 5.1. (Matching probabilities) The matching probabilities $(a_k)_{k=1}^K$ and profile of step sizes $(\epsilon_k)_{k=1}^K$ satisfy:
\[
\begin{align*}
(i) & \quad \lim_{M,K \to \infty} K^{1/2} \max_{k=1,\ldots,K} \| a_k \|_2 < \infty, \\
(ii) & \quad \lim_{M,K,T \to \infty} \bar{\epsilon} / \epsilon \max_{k=1,\ldots,K} \| a_k \|_2^2 = 0.
\end{align*}
\]

The first condition on the matching probabilities in Assumption 5.1 imposes that each advertiser demonstrate simple conditions (that are well-grounded in practice) under which adaptive pacing strategies constitute an $\varepsilon$-Nash equilibrium in dynamic strategies. Nevertheless, we note that by applying similar analysis adaptive pacing strategies can also be shown to constitute $\varepsilon$-Nash equilibrium in dynamic strategies under other models that have been considered in the literature, such as those in Iyer et al. (2014) and Balseiro et al. (2014).\(^{13}\) Our results could generalize to more sophisticated matching models. For example, our model and analysis can be easily extended to allow for the distribution of values to be auction dependent by having a distribution $F_{k,m}$ for each advertiser $k$ and auction $m$.

\[22\]
interacts with a small number of different competitors per time period, even when the total number of advertisers and auctions per period is large. The second condition ties the matching probabilities with the spectrum of steps sizes selected by advertisers, essentially stating that advertisers can interact with more competitors per time period when the learning rates of the adaptive pacing strategies are similar one to the other. The class of matching probabilities and step sizes defined by Assumption 5.1 includes many practical settings. For example, when each advertiser participates in each auction with the same probability, one has $\|a_k\|_2 \approx K^{1/2}/M$ since $\alpha_{i,m} = 1/M$ for all advertiser $i$ and auction $m$. When the expected number of bidders per auction $\kappa := K/M$ is fixed (implying that the number of auctions is proportional to the number of players), we have $\|a_k\|_2 \approx \kappa/K^{1/2}$ and the first condition is satisfied. The second condition states that in such case the difference between the learning rates of different advertisers should be of order $o(K/\kappa^2)$.

**Theorem 5.2. (ε-Nash equilibrium in dynamic strategies)** Suppose that Assumption 4.1 and 5.1 hold. Let $A$ be a profile of adaptive pacing strategies with step sizes that together satisfy Assumption 4.2. Then:

$$\limsup_{T,K,M \to \infty} \sup_{k \in \{1, \ldots, K\}} \sup_{\beta \in B_1} \frac{1}{T} \left( \Pi_{k}^{\beta, A_k} - \Pi_{k}^{A_k} \right) \leq 0.$$

Theorem 5.2 establishes that adaptive pacing strategies (with step sizes and matching probabilities that together satisfy Assumptions 4.2 and 5.1) constitute an ε-Nash equilibrium among the class of dynamic strategies with access to complete information on market primitives.

We next describe the key ideas of the proof. Recall that Theorem 4.4 shows that the performance of advertiser $k$ when all players (including itself) implement the adaptive pacing strategies is lower bounded by $\Psi_k(\mu^*)$. We prove the result by upper bounding the benefit of unilaterally deviating to an arbitrary strategy in terms of $\Psi_k(\mu^*)$.

Under Assumption 5.1 each advertiser interacts with a small number of different competitors per time period and the impact of a single advertiser on any other advertiser is limited. Thus, competitors learn the stable multipliers $\mu^*$ regardless of the actions of the deviating advertiser and an advertiser cannot induce competitors to learn a “wrong” multiplier so as to reduce competition. We show that the benefit of deviating to any other strategy is small when competitors’ multipliers are “close” to $\mu^*$. We bound the performance of an arbitrary budget-feasible strategy using a Lagrangian relaxation in which we add the budget constraint to the objective with $\mu_k^*$ as the Lagrange multiplier. This yields a bound on the performance of any admissible strategy, including ones with
full information on market primitives. Thus, we obtain that there exist constants $C_1$ through $C_5$, independent of $T$, $K$ and $M$, such that for any profile of advertisers satisfying Assumption 4.1 and any strategy $\beta \in \mathcal{B}_k^{C_1}$:

$$
\Pi_k^{\beta} - \Psi_k(\mu^*) \leq C_1 + C_2 \|a_k\|_2 K^{1/2} \frac{\epsilon_1^{1/2}}{\xi^{3/2}} + C_3 \|a_k\|_2 K^{1/2} \frac{\epsilon T}{\xi^{1/2}} + C_4 \frac{1}{\xi} + C_5 \|a_k\|_2^2 \frac{\epsilon T}{\xi}.
$$

for each advertiser $k$. This allows one to evaluate unilateral deviations under different of step sizes, and to establish the result under the conditions in Assumptions 4.2 and 5.1.

The vector function $\Psi(\cdot)$ defined in (6) has the following practical meaning. Given information on the advertisers’ types, as well as the vector $\mu^*$ characterized in (5), the vector $\Psi(\mu^*)$ provides a simple approximation to the payoff advertisers may expect in equilibrium, and may be used to forecast the revenue of the auction platform itself. We further discuss this implication in §6.

6 Concluding remarks

Summary of results. In this paper we studied how budget-constrained advertisers may bid in repeated auctions under uncertainties about future opportunities and competition. We formulated the problem as a repeated game of incomplete information, where bidders know neither their own valuation distribution, nor the budgets and valuation distributions of their competitors. We introduced a family of adaptive pacing strategies, in which advertisers adjust the pace at which they spend their budget throughout the campaign according to their expenditures. Under arbitrary competitors’ bids, we established through matching lower and upper bounds that the proposed strategies asymptotically achieve the maximal portion out of the best performance attainable with the benefit of hindsight. When adopted by all the bidders, these strategies induce dynamics that converge to a tractable and meaningful steady state and, under conditions motivated in the context of display ad auctions, constitute an approximate Nash equilibrium in large markets, among fully informed strategies. As dynamic pacing strategies minimize regret relative to the best sequence of bids, these results establish a connection between (individual) regret minimization and a strong notion of stability in ad exchange markets. This connection, which is similar in spirit to the ones established in repeated games of incomplete information without global constrains (e.g., in Hart and Mass-Colell 2000), implies that under the typical characteristics of ad exchange markets, an advertiser can essentially follow a dynamic equilibrium bidding strategy while ensuring the best performance that can be guaranteed off-equilibrium.
Market design implications. While the major part of this paper focus on the perspective on the competing advertisers, our analysis has implications on the design and management of ad exchanges. Due to the primal feasibility constrains, the convergence of multipliers in the dual space (established in Theorem 4.3) does not necessarily guarantee convergence in performance, or any performance guarantees. However, the convergence of multipliers has an important practical implication for the ad exchange platform. Assuming the platform has access to the advertisers’ types, the convergence of multipliers provides a practical tool for predicting the bidding behavior of advertisers through the simple characterization of the vector $\mu^*$ that is given in (5). Given the vector $\mu^*$ and information on advertisers’ types, the vector function $\Psi(\cdot)$ defined in (6) may have practical importance to the platform as well, as the vector $\Psi(\mu^*)$ provides a simple approximation to the utility advertisers may expect in equilibrium. Moreover, the vector $\mu^*$ may be used to predict the auctioneer’s revenues. Since these results and predictions are established under the uniqueness of $\mu^*$, this may motivate the market designer to place restrictions on target expenditure rates of advertisers to guarantee that Assumption 4.1 holds. We further note that asymptotic regime can be approached by encouraging advertisers to manage larger budgets over larger time horizons (e.g., work with monthly budgets instead of daily budgets).

Several display ad auction platforms, including the ones operated by Facebook, Google, and Twitter, offer customized budget pacing services that manage budgets on behalf of advertisers. In this context, the equilibrium analysis and the conditions that allows adaptive pacing strategies to constitute an approximate Nash equilibrium in dynamic strategies have concrete implications, as these allow such platforms to offer “equilibrium” budget pacing services. Moreover, while platforms may use historical information on all advertisers for prediction purposes (as described above), our results suggest that platforms may design adaptive budget pacing algorithms that learn the correct multipliers without relying on such market information and with practically no loss of optimality.

A Proofs of main results

A.1 Proof of Theorem 3.1

We show that for any $\gamma < \bar{v}/p_k$ no admissible bidding strategy (including randomized ones) can guarantee asymptotic $\gamma$-competitiveness. In the proof we use Yao’s Principle, according to which in order analyze the worst-case performance of randomized algorithms, it suffices to analyze the expected performance of deterministic algorithms given distribution over inputs.
Preliminaries. In §2 we denoted by $B_k$ the class of admissible bidding strategies defined by the mappings $\{b^{\beta}_{k,t} : t = 1, \ldots, T\}$ together with the distribution $P_y$, and the target expenditure rate $\rho_k$. We now denote by $\tilde{B}_k \subset B_k$ the subclass of deterministic admissible strategies, defined by adjusting the histories to be $\tilde{H}_{k,t} := \sigma\left(\{v_{k,\tau}, b_{k,\tau}, z_{k,\tau}, u_{k,\tau}\}_{\tau=1}^{t-1}, v_{k,t}\right)$ for any $t \geq 2$, with $\tilde{H}_{k,1} := \sigma(v_{k,1})$. Then, $\tilde{B}_k$ is the subclass of bidding strategies $\beta \in B_k$ such that $b^{\beta}_{k,t}$ is measurable with respect to the filtration $\tilde{H}_{k,t}$ for each $t \in \{1, \ldots, T\}$.

To simplify notation we now drop the dependence on $k$. Given a target expenditure rate $\rho$, strategy $\beta$, vector of values $v$, vector of highest competing bids $d$, and a constant $\gamma \geq 1$, we define:

$$R^\gamma_\beta(v; d) = \frac{1}{T} \left( \pi^\mu(v; d) - \gamma \pi^\beta(v; d) \right).$$

We next adapt Yao’s principle (Yao, 1977) to our setting to establish a connection between the expected performance of any randomized strategy and the expected performance of the best deterministic strategy under some distribution over sequences of valuations.

Lemma A.1. (Yao’s principle) Let $E_v[\cdot]$ denote expectation with respect to some distribution over a set of valuation sequences $\{v^1, \ldots, v^m\} \in [0, \bar{v}]^{T \times m}$. Then, for any vector of competing bids $d' \in \mathbb{R}_+^T$ and for any bidding strategy $\beta \in B$,

$$\sup_{v \in [0, \bar{v}]^T} E^\beta \left[ R^\gamma_\beta(v; d) \right] \geq \inf_{\beta \in B} E_v \left[ R^\gamma_\beta(v; d') \right],$$

Lemma A.1 is an adaptation of Theorem 8.3 in Borodin and El-Yaniv (1998); For completeness we provide a proof for this Lemma in Appendix B. By Lemma A.1 to bound the worst-case loss of any admissible strategy (deterministic or not) relative to the best response in hindsight, it suffices to analyze the expected loss of deterministic strategies relative to the same benchmark, where valuations are drawn from a carefully selected distribution.

Worst-case instance. Fix $T \geq \bar{v}/\rho$ and a target expenditure rate $0 < \rho \leq \bar{v}$. Suppose that the advertiser is a priori informed that the vector of best competitors’ bids is

$$d^0 = \left(\underbrace{d, \ldots, d}_{m\left\lfloor \frac{T}{m}\right\rfloor \text{ auctions}}, \underbrace{d, \ldots, d}_{T_0 \text{ auctions}}\right),$$

where we decompose the sequence in $m := \lceil d/\rho \rceil$ batches of length $\lceil T/m \rceil$, and $T_0 := T - m \lfloor T/m \rfloor$ auctions are padded to the end of the sequence to handle cases when $T$ is not divisible by $m$. Suppose
that the advertiser also knows that the sequence of valuations $v$ is selected randomly according to a 
discrete distribution $\mathbf{p} = \{p_1, \ldots, p_m\}$ over the set of sequences $\mathcal{V} = \{v^1, v^2, \ldots, v^m\} \in [0, \bar{v}]^{T \times m}$, where:

$$v^1 = \left( v_1, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions}, v_2, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions}, \ldots, v_1, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions}, v_1, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions}, \right) \in T_0 \text{ auctions}$$

$$v^2 = \left( v_1, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions}, v_2, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions}, \ldots, v_1, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions}, v_1, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions}, \right) \in T_0 \text{ auctions}$$

$$v^{m-1} = \left( v_1, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions}, v_2, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions}, \ldots, v_1, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions}, v_1, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions}, \right) \in T_0 \text{ auctions}$$

$$v^m = \left( v_1, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions}, \ldots, \ldots, v_1, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions}, v_1, \ldots, v_{\left\lceil \frac{T}{m} \right\rceil} \text{ auctions} \right) \in T_0 \text{ auctions}$$

with $v_j = \varepsilon^{m-j}v + d$ for every $i \in \{1, \ldots, m\}$. We use indices $i \in \{1, \ldots, m\}$ to refer to valuation 
sequences and indices $j \in \{1, \ldots, m\}$ to refer to batches within a sequence (other than the last 
batch which always has zero utility). The parameters $\varepsilon, v,$ and $d$ are such that $\varepsilon \in (0, \bar{v}/2], \ \rho \leq d,$ 
$v > 0,$ and $v + d = \bar{v}$; precise values of these parameters will be determined later on.

Sequence $v^1$ represents a case where valuations gradually increase throughout the campaign
horizon (except the last $T_0$ auctions that allow cases where $T/m$ is not an integer); sequences $v^2, \ldots, v^m$ follow essentially the same structure of increasing net utilities, but introduce the risk of the net utilities going down to zero at different time points. Thus, the feasible set of valuations
present the advertiser with the following tradeoff: early auctions introduce return per unit of budget 
that is certain, but low. Later auctions introduce return per unit of budget that may be higher, 
but uncertain because the net utility may likely diminish to zero. In the rest of the proof, the
parameters of the instance above are tuned to maximize the worst-case loss that must incurred due 
to this tradeoff by any admissible bidding strategy.

**Useful subclass of deterministic strategies.** As we next show, under the structure of the worst-case instance described above, it suffices to analyze strategies that determine before the beginning of the campaign how many auctions to win at different stages of the campaign horizon. Define 
the set $\mathcal{X} := \left\{ x \in \{0, \ldots, \lceil T/m \rceil\}^m : \sum_{j=1}^{m} x_j \leq \lfloor B/d \rfloor \right\}$. Given $x \in \mathcal{X}$, we denote by $\beta^x \in \hat{B}$ a 
strategy that for each $j \in \{1, \ldots, m\}$ wins the first $x_j$ auctions in the $j$’th batch of $\lceil T/m \rceil$ auctions.
Lemma A.2. For any deterministic bidding strategy \( \beta \in \tilde{\cal B} \) there exists a vector \( x \in X \) such that for any \( v \in V \) one has \( \pi^\beta(d^0, v) = \pi(d^0, v) \).

Lemma A.2 implies that under the structure we defined, the minimal loss that is achievable by a deterministic strategy relative to the hindsight solution, is attained over the set \( X \). The proof of Lemma A.2 follows because every valuation sequence \( v_i \) is identical to \( v_1 \), thus indistinguishable, up to time \( \tau_i \), the first time that the utilities go down to zero in valuation sequence \( v_i \). Therefore, the bids of any deterministic strategy coincide up to time \( \tau_i \) under valuation sequences \( v_i \) and \( v_1 \). Because utilities after time \( \tau_i \) are zero, all bids under valuation sequence \( v_i \) after time \( \tau_i \) are irrelevant. Using the fact that all items in a batch have the same value, it suffices to look at the total number of auctions won by the strategy in each batch when the vector of valuations is \( v_1 \).

Analysis. Fix \( x \in X \) and consider the strategy \( \beta^x \). Define the net utilities matrix \( U \in \mathbb{R}^{m \times m} \):

\[
U = \begin{bmatrix}
  u_1 & u_2 & \ldots & u_{m-1} & u_m \\
  u_1 & u_2 & \ldots & u_{m-1} & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  u_1 & u_2 & 0 & \ldots & 0 \\
  u_1 & 0 & 0 & \ldots & 0
\end{bmatrix}
\]

with \( u_j = \varepsilon^{m-j}v \). The matrix \( U \) is invertible with rows representing different sequences in \( V \) and columns capturing the net utility of different batches (perhaps except the last \( T_0 \) auctions). Define the vector \( u = (u_m, u_{m-1}, \ldots, u_1)^t \in \mathbb{R}^m \) as the net utility of the most profitable batch of each sequence in \( V \). Define the following distribution over sequences of values:

\[
p' = \frac{e'U^{-1}}{e'U^{-1}e'},
\]

where \( e' = (1, \ldots, 1) \) and where \( U^{-1} \) denotes the inverse of \( U \). Some algebra shows that

\[
e'U^{-1} = \frac{1}{v} \left( 1, \frac{1}{\varepsilon} - 1, \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}, \ldots, \frac{1}{\varepsilon^{m-1}} - \frac{1}{\varepsilon^{m-2}} \right),
\]

and \( e'U^{-1}e' = 1/(v\varepsilon^{m-1}) \). This implies that \( \sum_{i=1}^m p_i = 1 \) and \( p_i \geq 0 \) for all \( i \in \{1, \ldots, m\} \), because \( \varepsilon \in (0, 1] \). Thus \( p \) is a valid probability distribution over inputs. Then, the expected performance-per-round of \( \beta^x \) is:
\[
\frac{1}{T} \mathbb{E}_v \left[ \pi^{\beta^*}(v; d^0) \right] = \frac{1}{T} \mathbf{p}' U \mathbf{x} = \frac{1}{T} v \epsilon^{m-1} \mathbf{e}' \mathbf{x} \overset{(a)}{\leq} v \epsilon^{m-1} \frac{\rho}{d},
\]

where (a) holds by Lemma [A.2] because \( \mathbf{e}' \mathbf{x} = \sum_{j=1}^m x_j \leq [B/d] \leq B/d = T \rho/d \). In addition, the expected performance-per-round of the hindsight solution satisfies:

\[
\frac{1}{T} \mathbb{E}_v \left[ \pi^u(v; d^0) \right] \overset{(a)}{\geq} \frac{1}{T} \left\lfloor \frac{T}{m} \right\rfloor \mathbf{p}' u = \frac{1}{T} \mathbf{m} v \epsilon^{m-1} (m - \epsilon(m - 1)) = \left( \frac{1 - \epsilon}{T} \right) v \epsilon^{m-1} (m - \epsilon(m - 1)),
\]

where: (a) holds since by the definition of the index \( m \) one has that \( \rho \geq d/m \) and therefore \( B = T \rho \geq T d/m \geq \lfloor T/m \rfloor d \), so the hindsight solution allows winning at least the \( \lfloor T/m \rfloor \) most profitable auctions; and (b) holds since \( \mathbf{p}' u = (\mathbf{e}' U^{-1} u)/(\mathbf{e}' U^{-1} \mathbf{e}) = v \epsilon^{m-1} (m - \epsilon(m - 1)) \).

Set \( \delta \in (0, 1) \) such that \( \gamma = \frac{1 - \delta}{\bar{v}/\rho} \). Setting \( d = \bar{v} - \epsilon \) and \( v = \epsilon = \frac{\delta \rho}{2(\bar{v} + 1) + 4} \), one has for any \( T \geq 4 \lfloor \bar{v}/\rho \rfloor / \delta \):

\[
\mathbb{E}_v \left[ R_\gamma^{\beta^*}(v; d^0) \right] \overset{(a)}{\geq} \epsilon^m \left( \frac{1}{m} - \frac{1}{T} \right) (m - \epsilon(m - 1)) - (1 - \delta) \frac{\bar{v}}{\bar{v} - \epsilon} \left( m - \frac{1}{m} + \frac{1}{\bar{v} - \epsilon} - \frac{1}{T} + \epsilon \frac{m - 1}{T} \right) \overset{(b)}{\geq} \epsilon^m \left( \delta - \frac{\bar{v} + 1}{\bar{v} - \epsilon} - \frac{m}{T} \right) \overset{(c)}{=} \left( \frac{\delta \bar{v}}{2(\bar{v} + 1) + \delta} \right)^m \left( \frac{\delta}{2} \right) \overset{(d)}{\geq} \frac{1}{4} \left( \frac{\bar{v}}{2(\bar{v} + 1) + 1} \right)^m \delta^{m+1} > 0,
\]

where: (a) holds from (A-1) and (A-2); (b) holds by using that \( \bar{v}/(\bar{v} - \epsilon) \geq 1 \) because \( 0 \leq \epsilon \leq \bar{v} \) for the first term in the parenthesis, using that \( (m - 1)/m \leq 1 \leq \bar{v}/(\bar{v} - \epsilon) \) in the second term, and dropping the last term because \( m \geq 2 \); (c) follows by setting \( \epsilon = \frac{\delta \bar{v}}{2(\bar{v} + 1) + \delta} \); and (d) holds because \( \delta/2 - m/T \geq \delta/4 \) for any \( T \geq 4 \lfloor \bar{v}/\rho \rfloor / \delta \) since \( m = \left\lfloor \frac{\bar{v} - \epsilon}{\rho} \right\rfloor \) by (A-2) and \( \epsilon \in (0, 1) \). This establishes that for any \( \gamma < \bar{v}/\rho \) there is a constant \( C > 0 \) such that for \( T \) large enough:

\[
\inf_{\beta \in \mathcal{B}} \sup_{v \in [0, \bar{v}]^T} \mathbb{E}_v \left[ R_\gamma^\beta(v; d) \right] \overset{(a)}{\geq} \inf_{\beta \in \mathcal{B}} \mathbb{E}_v \left[ R_\gamma^\beta(v; d^0) \right] \overset{(b)}{\geq} \inf_{x \in \mathcal{X}} \mathbb{E}_v \left[ R_\gamma^\beta(v; d^0) \right] \geq C,
\]

where (a) follows from Lemma [A.1] and (b) holds by Lemma [A.2]. Therefore, no admissible bidding
strategy can guarantee $\gamma$-competitiveness for any $\gamma < \tilde{v}/\rho$, concluding the proof.

A.2 Proof of Theorem 3.3

To simplify notation we drop the dependence on $k$. We consider an alternate process in which multipliers and bids continue being updated after the budget is depleted, but no reward is accrued after budget depletion. Denote by $\hat{\tau}^A := \inf\{t \geq 1 : \hat{B}_t^A < \tilde{v}\}$ the first auction in which the bid of the advertiser is constrained by the remaining budget, and by $\tau^A := \hat{\tau}^A - 1$ the last period in which bids are unconstrained. The dynamics of the adaptive pacing strategy are:

$$\mu_{t+1} = P_{[0,\mu]} (\mu_t - \epsilon (\rho - d_t x_t^A))$$

$$x_t^A = 1\{v_t - d_t \geq \mu_t d_t\}$$

$$\tilde{B}_{t+1}^A = \tilde{B}_t^A - d_t x_t^A.$$ 

Given the sequence of competing bids $d$ we denote by $z_t^A = d_t x_t^A$ the expenditure at time $t$ under the adaptive pacing strategy. Because the auction is won only if $v_t - d_t \geq \mu_t d_t$, the expenditure satisfies $z_t^A \leq v_t/(1 + \mu_t) \leq \tilde{v}$.

**Step 1: controlling the stopping time $\hat{\tau}^A$.** We first show that the adaptive pacing strategy does not run out of budget too early, that is, $\tau^A$ is close to $T$. From the update rule of the dual variables one has for every $t \leq \tau^A$,

$$\mu_{t+1} = P_{[0,\mu]} (\mu_t + \epsilon (z_t^A - \rho)) = P_{[0,\mu]} (\mu_t + \epsilon (z_t^A - \rho)) = \mu_t + \epsilon (z_t^A - \rho) - P_t,$$

where we define $P_t := \mu_t + \epsilon (z_t^A - \rho) - P_{[0,\mu]} (\mu_t + \epsilon (z_t^A - \rho))$ as the truncation error. Reordering terms and summing up to period $\tau^A$ one has:

$$\sum_{t=1}^{\tau^A} (z_t^A - \rho) = \sum_{t=1}^{\tau^A} \frac{1}{\epsilon} (\mu_{t+1} - \mu_t) + \sum_{t=1}^{\tau^A} P_t.$$

We next bound each term independently. For left-hand side of (A-3) we have

$$\sum_{t=1}^{\tau^A} (z_t^A - \rho) \overset{(a)}{=} B - \tilde{B}_{\tau^A+1}^A - \rho \tau^A \overset{(b)}{=} \rho (T - \tau^A) - \tilde{v},$$

where (a) holds since $\tilde{B}_{\tau^A+1}^A = B - \sum_{t=1}^{\tau^A} z_t^A$ and (b) uses $\tilde{B}_{\tau^A+1}^A \leq \tilde{v}$ and $\rho = B/T$. For the first
term in the right-hand side of (A-3) we have
\[
\sum_{t=1}^{\tau^A} \frac{1}{\epsilon} (\mu_{t+1} - \mu_t) = \frac{\mu_{\tau^A+1}^A}{\epsilon} - \frac{\mu_1^A}{\epsilon} \leq \frac{\bar{\mu}}{\epsilon},
\]
where the inequality follows because \(\mu_t \in [0, \bar{\mu}_k]\). We next bound the second term in the right-hand side of (A-3). The truncation error satisfies
\[
P_t = P_t^+ = (\mu_t + \epsilon (\bar{z}_t^A - \rho) - P_{[0,\bar{\mu}]} (\mu_t + \epsilon (\bar{z}_t^A - \rho)))^{+}
\]
\[
\overset{(a)}{=} (\mu_t + \epsilon (\bar{z}_t^A - \rho) - \bar{\mu}) \mathbb{1} \{\mu_t + \epsilon (\bar{z}_t^A - \rho) > \bar{\mu}\}
\]
\[
\overset{(b)}{\leq} \epsilon \bar{v} \mathbb{1} \{\mu_t + \epsilon (\bar{z}_t^A - \rho) > \bar{\mu}\}
\]
where \((a)\) holds since the truncation error is positive only if \(\mu_t + \epsilon (\bar{z}_t^A - \rho) > \bar{\mu}\), and \((b)\) holds since \(\mu_t \leq \bar{\mu}\) and since the deviation is bounded by \(\bar{z}_t^A - \rho \leq \bar{z}_t^A \leq \bar{v}\) as the expenditure is at most \(\bar{v}\).

We next show that there is no positive truncation error, or more formally, \(P_t^+ = 0\) whenever \(\epsilon \bar{v} \leq 1\). Consider the function \(f : \mathbb{R}^+ \rightarrow \mathbb{R}\) given by \(f(\mu) = \mu + (\epsilon \bar{v})/(1 + \mu)\). The function \(f(\cdot)\) is non-decreasing whenever \(\epsilon \bar{v} \leq 1\), and therefore
\[
\mu_t + \epsilon (\bar{z}_t^A - \rho) \overset{(a)}{\leq} \mu_t + \epsilon \left(\frac{\bar{v}}{1 + \mu_t} - \rho\right) = f(\mu_t) - \epsilon \rho
\]
\[
\overset{(b)}{\leq} f(\bar{\mu}) - \frac{\epsilon \bar{v}}{1 + \bar{\mu}} \overset{(c)}{=} \bar{\mu},
\]
where: \((a)\) holds since \(\bar{z}_t^A = d_t \mathbb{1} \{v_t - d_t \geq \mu_t d_t\} \leq v_t/(1 + \mu_t) \leq \bar{v}/(1 + \mu_t)\) as the payment is never greater than \(v_t/(1 + \mu_t)\); \((b)\) hold since \(f(\cdot)\) is non-decreasing and \(\mu_t \leq \bar{\mu}\), and since \(\bar{\mu} \geq \bar{v}/\rho - 1\).

Therefore, there is no positive truncation when \(\epsilon \bar{v} \leq 1\). Using these inequalities in (A-3) one has:
\[
T - \tau^A \leq \frac{\bar{\mu}}{\epsilon \rho} + \frac{\bar{v}}{\rho}, \tag{A-4}
\]
Therefore the adaptive pacing strategy does not run out of budget very early, and one has:
\[\pi^A(v; d) \geq \sum_{t=1}^{T \land \tau^A} (v_t - d_t)x^A_t\]

\[
\geq (a) \sum_{t=1}^{T} (v_t - d_t)x^A_t - \bar{v}(T - \tau^A)^+\\
\geq (b) \sum_{t=1}^{T} (v_t - d_t)x^A_t - \left(\frac{\bar{\mu} \bar{v}}{\epsilon \rho} + \frac{\bar{v}^2}{\rho} E_1\right),\tag{A-5}
\]

where (a) follows from \(0 \leq v_t - d_t \leq \bar{v}\), and (b) follows from \(\text{(A-4)}\).

**Step 2: bounding the regret.** We next bound the relative loss in terms of the potential value of auctions lost by the adaptive pacing strategy. Consider a relaxation of the hindsight problem \([\text{I}]\) in which the decision maker is allowed to take fractional items. In this relaxation, the optimal solution is a greedy strategy that sorts items in decreasing order of the ratio \((v_t - d_t)/d_t\) and wins items until the budget is depleted (where the “overflowing” item is fractionally taken). Let \(\mu^H\) be the sample-path dependent fixed threshold corresponding to the ratio of the “overflowing” item. Then all auctions won satisfy \((v_t - d_t) \geq \mu^H d_t\). We obtain an upper bound by assuming that the overflowing item is completely taken by the hindsight solution. The dynamics of this bound are:

\[x^H_t = 1\{v_t - d_t \geq \mu^H d_t\}\]

\[\tilde{B}^H_{t+1} = \tilde{B}^H_t - x^H_t \bar{v}^H.\]

Denote by \(\bar{\tau}^H := \inf\{t \geq 1 : \tilde{B}^H_t < 0\}\) the period in which the budget of the hindsight solution is depleted, and by \(\tau^H := \bar{\tau}^H - 1\). Then, one has:

\[\pi^H(v; d) - \pi^A(v; d) \leq \sum_{t=1}^{T \land \tau^H} x^H_t (v_t - d_t) - \sum_{t=1}^{T \land \tau^A} x^A_t (v_t - d_t)\]

\[
\leq (a) \sum_{t=1}^{T \land \tau^H} (v_t - d_t) (x^H_t - x^A_t) + E_1\tag{A-6}
\]

where (a) follow from \(\text{(A-5)}\), ignoring periods after \(\tau^H\). In addition, for any \(t \in \{1, \ldots, T \land \tau^H\}\):
\[(v_t - d_t) (x_t^H - x_t^A) \overset{(a)}{=} (v_t - d_t) (1 \{v_t - d_t \geq \mu^H d_t\} - 1 \{v_t - d_t \geq \mu_t d_t\}) \]
\[\overset{(b)}{=} (v_t - d_t) (1 \{\mu_t d_t > v_t - d_t \geq \mu^H d_t\} - 1 \{\mu^H d_t > v_t - d_t \geq \mu_t d_t\}) \]
\[\overset{(c)}{=} (v_t - d_t) 1 \{\mu_t d_t > v_t - d_t\} = (v_t - d_t) (1 - x_t^A), \quad (A-7)\]

where: (a) follows the threshold structure of the hindsight solution as well as the dynamics of the adaptive pacing strategy; (b) holds since \(x_t^H - x_t^A = 1\) if and only if the auction is won in the hindsight solution but not by the adaptive pacing strategy; and \(x_t^H - x_t^A = -1\) if and only if the auction is won by the adaptive pacing strategy but not in the hindsight solution; and (c) follows from discarding indicators. Putting (A-6) and (A-7) together one obtains a bound on the relative loss in terms of the potential value of the auctions lost by the adaptive pacing strategy:

\[\pi^H(v; d) - \pi^A(v; d) \leq \sum_{t=1}^{T \wedge \tau^H} (v_t - d_t) (1 - x_t^A) + E_1. \quad (A-8)\]

**Step 3: bounding the potential value of lost auctions.** We next bound the value of the auctions lost under the adaptive pacing strategy in terms of the value of alternative auctions won under the same strategy. We show that for all \(t = 1, \ldots, T:\)

\[(v_t - d_t) (1 - x_t^A) \leq \left(\frac{\bar{v}}{\rho} - 1\right) (v_t - d_t) x_t^A + \frac{\bar{v}}{\rho} \mu_t (\rho - z_t^A). \quad (A-9)\]

Note that when \(d_t > \bar{v}, (A-9)\) simplifies to \(v_t - d_t \leq \bar{v} \mu_t\), because \(x_t^A = 0\) and \(z_t^A = d_t x_t^A = 0\). The inequality then holds since \(v_t - d_t < 0\) and \(\mu_t \geq 0\). We next prove (A-9) holds also when \(d_t \leq \bar{v}\).

Using \(z_t^A = d_t 1\{v_t - d_t \geq \mu_t d_t\}\) one obtains:

\[\mu_t (\rho - z_t^A) \overset{(a)}{=} \mu_t \left(\frac{\rho}{\bar{v}} d_t - z_t^A\right) = \mu_t d_t \left(\frac{\rho}{\bar{v}} - 1 \{v_t - d_t \geq \mu_t d_t\}\right) \]
\[= \frac{\rho}{\bar{v}} \mu_t d_t 1 \{\mu_t d_t > v_t - d_t\} - \left(1 - \frac{\rho}{\bar{v}}\right) \mu_t d_t 1 \{v_t - d_t \geq \mu_t d_t\} \]
\[\overset{(b)}{\geq} \frac{\rho}{\bar{v}} (v_t - d_t) (1 - x_t^A) - \left(1 - \frac{\rho}{\bar{v}}\right) (v_t - d_t) x_t^A, \]

where (a) holds because \(d_t \leq \bar{v}\), and (b) holds because \(\rho \leq \bar{v}\) and using \(\mu_t d_t > v_t - d_t\) in the first term and \(\mu_t d_t \leq v_t - d_t\) in the second term. The claim follows from multiplying by \(\bar{v}/\rho\) and reordering terms.
Summing \((A-9)\) over \(t = 1 \ldots , T\) implies
\[
\sum_{t=1}^{T} (v_t - d_t) (1 - x_t^A) \leq \left( \frac{\bar{v}}{\rho} - 1 \right) \sum_{t=1}^{T} (v_t - d_t) x_t^A + \frac{\bar{v}}{\rho} \sum_{t=1}^{T} \mu_t (\rho - z_t^A). \tag{A-10}
\]

We next bound the second term in \((A-10)\). The update rule of the strategy implies that for any \(\mu \in [0, \bar{\mu}]\) one has:
\[
(\mu_{t+1} - \mu)^2 \leq (\mu_t - \mu - \epsilon (\rho - z_t^A))^2
= (\mu_t - \mu)^2 - 2\epsilon (\mu_t - \mu) (\rho - z_t^A) + \epsilon^2 (\rho - z_t^A)^2.
\]
where \((a)\) follows from a standard contraction property of the Euclidean projection operator. Reordering terms and summing over \(t = 1, \ldots , T\) we have for \(\mu = 0\):
\[
\sum_{t=1}^{T} \mu_t (\rho - z_t^A) \leq \sum_{t=1}^{T} \frac{1}{2\epsilon} \left( (\mu_t)^2 - (\mu_{t+1})^2 \right) + \frac{\epsilon}{2} (\rho - z_t^A)^2
= (a) \frac{(\mu_t^2)}{2\epsilon} - \frac{(\mu_{t+1})^2}{2\epsilon} + \sum_{t=1}^{T} \frac{\epsilon}{2} (\rho - z_t^A)^2
\leq (b) \frac{\mu^2}{2\epsilon} + \frac{\bar{v}^2}{2} T\epsilon, \tag{A-11}
\]
where \((a)\) follows from telescoping the sum, and \((b)\) follows from \(\mu_t \in [0, \bar{\mu}]\) together with \(\rho, z_t^A \in [0, \bar{v}]\). Combining \((A-10)\) and \((A-11)\) implies
\[
\sum_{t=1}^{T} (v_t - d_t) (1 - x_t^A) \leq \left( \frac{\bar{v}}{\rho} - 1 \right) \sum_{t=1}^{T} (v_t - d_t) x_t^A + \left( \frac{\bar{v} \mu^2}{2\rho} \epsilon + \frac{\bar{v}^3}{2\rho} T\epsilon \right). \tag{A-12}
\]

Step 4: putting everything together. Using the bound on regret derived in \((A-8)\) we have:
\[ \pi^H(v; d) - \pi^A(v; d) \leq \sum_{t=1}^{T} (v_t - d_t) (1 - \pi_t^A) + E_1 \]
\[ \leq \sum_{t=1}^{T} (v_t - d_t) (1 - \pi_t^A) + E_1 \]
\[ \leq \left( \bar{v} - 1 \right) \sum_{t=1}^{T} (v_t - d_t) \pi_t^A + \bar{v} E_1 + E_2, \]

where: (a) follows from adding all (non-negative) terms after \( \pi^H \) and since \( d_t \leq v_t \) for all \( t = 1, \ldots, T \); (b) follows from (A-12); and (c) follows from (A-5). Reordering terms we obtain that

\[ \pi^H(v; d) - \bar{v} \pi^A(v; d) \leq \bar{v} E_1 + E_2 = \frac{\bar{v}^2}{\rho^2} + \left( \frac{\bar{v}^2 \bar{\mu}}{\rho^2} + \frac{\bar{v} \bar{\mu}^2}{2 \rho} \right) \frac{1}{\epsilon} + \frac{\bar{v}^3}{2 \rho} T \epsilon, \]

and the result follows.

---

**A.3 Proof of Theorem 4.3**

We prove the result by first bounding the mean squared error when all bids are unconstrained, i.e., \( \tilde{B}_{k,t+1} \geq \bar{v} \) for every advertiser \( k \). We then argue that the adaptive pacing strategies do not run out of budget too early when implemented by all advertisers. We conclude by combining these results to upper bound the time average mean squared error.

**Step 1.** At a high level, this step adapts standard stochastic approximation results (see, e.g., Nemirovski et al. 2009) to the expenditure observations and to accommodate the possibility of different step sizes across agents. Fix some \( k \in \{1, \ldots, K\} \). The squared error satisfies the recursion

\[ |\mu_{k,t+1} - \mu_k^*|^2 = |P_{[0, \bar{\mu}_k]} (\mu_{k,t} - \epsilon_k (\rho_k - z_{k,t})) - \mu_k^*|^2 \]
\[ \leq |\mu_{k,t} - \mu_k^* - \epsilon_k (\rho_k - z_{k,t})|^2 \]
\[ = |\mu_{k,t} - \mu_k^*|^2 - 2 \epsilon_k (\mu_{k,t} - \mu_k^*) (\rho_k - z_{k,t}) + \epsilon_k^2 |\rho_k - z_{k,t}|^2, \]

where (a) follows from a standard contraction property of the Euclidean projection operator.

Given a vector of multipliers \( \bm{\mu} \), let \( G_k(\bm{\mu}) := \mathbb{E}_v \left[ 1 \{ (1 + \mu_{k,1}) d_{k,1} \leq v_{k,1} \} d_{k,1} \right] \) be the expected expenditure under the second-price auction allocation rule, with \( d_k = \max_{i \neq k} \{ v_i / (1 + \mu_i) \} \). Define \( \delta_{k,t} := \mathbb{E} \left[ (\mu_{k,t} - \mu_k^*)^2 1 \{ \tilde{B}_{k,t+1} \geq \bar{v}, \forall k \} \right] \) and \( \delta_t := \sum_{k=1}^{K} \delta_{k,t} \). Similarly, define \( \hat{\delta}_{k,t} := \delta_{k,t} / \epsilon_k \) and
\[ \hat{\delta}_t := \sum_{k=1}^{K} \hat{\delta}_{k,t}. \]

Taking expectations and dividing by \( \epsilon_k \) we obtain that

\[
\hat{\delta}_{k,t+1}^{(a)} \leq \hat{\delta}_{k,t} - 2E \left[ (\mu_{k,t} - \mu_k^*) (\rho_k - z_{k,t}) \right] + \epsilon_k \mathbb{E} \left[ |\rho_k - z_{k,t}|^2 \right] \\
= \hat{\delta}_{k,t} - 2E \left[ (\mu_{k,t} - \mu_k^*) (\rho_k - G_k(\mu_t)) \right] + \epsilon_k \mathbb{E} \left[ |\rho_k - z_{k,t}|^2 \right],
\]

(A-13)

where \((a)\) holds since remaining budgets monotonically decrease with \( t \), and by conditioning on the multipliers \( \mu_t \) and using the independence of \( V_{k,t} \) from the multipliers \( \mu_t \) to obtain that \( \mathbb{E}[Z_{k,t} | \mu_t] = G_k(\mu_t) \). For the second term in (A-13) one has:

\[
(\mu_{k,t} - \mu_k^*) (\rho_k - G_k(\mu_t)) = (\mu_{k,t} - \mu_k^*) (\rho_k - G_k(\mu^*) + G_k(\mu^*) - G_k(\mu_t)) \\
\geq (\mu_{k,t} - \mu_k^*) (G_k(\mu^*) - G_k(\mu_t)),
\]

where the inequality follows because \( \mu_{k,t} \geq 0 \) and \( \rho_k - G_k(\mu^*) \geq 0 \) and \( \mu_k^* (\rho_k - G_k(\mu^*)) = 0 \).

Summing over the different advertisers and Assumption 4.1 item (1) we obtain

\[
\sum_{k=1}^{K} (\mu_{k,t} - \mu_k^*) (\rho_k - G_k(\mu_t)) \geq \sum_{k=1}^{K} (\mu_{k,t} - \mu_k^*) (G_k(\mu^*) - G_k(\mu_t)) \geq \lambda \|\mu_t - \mu^*\|^2_2.
\]

In addition, for the third term in (A-13) we have \( \mathbb{E} \left[ |\rho_k - z_{k,t}|^2 \right] \leq \bar{v}^2 \), since \( \bar{v} \geq \rho_k \geq 0 \) and \( z_{k,t} \geq 0 \), and since the payment is at most the bid \( v_{k,t} / (1 + \mu_{k,t}) \leq v_{k,t} \leq \bar{v} \) because \( \mu_{k,t} \geq 0 \).

Denoting \( \bar{\epsilon} = \max_{k \in \{1, \ldots, K\}} \epsilon_k \), \( \xi = \min_{k \in \{1, \ldots, K\}} \epsilon_k \), we conclude by summing (A-13) over \( k \) that:

\[
\hat{\delta}_{t+1} \leq \hat{\delta}_t - 2\lambda \delta_t + \bar{\epsilon} K \bar{v}^2 \leq (1 - 2\lambda \xi) \hat{\delta}_t + K \bar{v}^2 \bar{\epsilon},
\]

where \((a)\) follows from \( \delta_t = \sum_{k=1}^{K} \delta_{k,t} = \sum_{k=1}^{K} \epsilon_k \hat{\delta}_{k,t} \geq \xi \hat{\delta}_t \) because \( \delta_{k,t} \geq 0 \). Lemma C.2 with \( a = 2\lambda \xi \leq 1 \) and \( b = \bar{\epsilon} K \bar{v}^2 \) implies that

\[
\hat{\delta}_t \leq \hat{\delta}_1 (1 - 2\lambda \xi)^{t-1} + \frac{K \bar{v}^2 \bar{\epsilon}}{2\lambda \xi}.
\]

Using that \( \delta_t \leq \bar{\epsilon} \hat{\delta}_t \) together with \( \hat{\delta}_1 \leq \delta_1 / \xi \leq K \bar{\mu}^2 / \xi \) because \( \mu_{k,t}, \mu_k^* \in [0, \bar{\mu}_k] \) and \( \bar{\mu} = \max_k \bar{\mu}_k \) we obtain that

\[
\delta_t \leq K \bar{\mu}^2 \frac{\bar{\epsilon} \bar{\xi}}{\xi} (1 - 2\lambda \xi)^{t-1} + \frac{K \bar{v}^2 \bar{\epsilon}^2}{2\lambda \xi}.
\]

(A-14)

**Step 2.** Let \( \tilde{\tau}_k \) be the first auction in which the remaining budget of advertiser \( k \) is less than \( \bar{v} \) (at the beginning of the auction), that is, \( \tilde{\tau}_k = \inf \{ t \geq 1 : \tilde{B}_{k,t} < \bar{v} \} \). Let \( \tau_k := \tilde{\tau}_k - 1 \) be the last
period in which the remaining budget of advertiser $k$ is greater than $\bar{v}$. Let $\tau = \min_{k=1,\ldots,K} \{\tau_k\}$. Since $v/(1+\mu) \leq \bar{v}$ for any $v \in [0,\bar{v}]$ and $\mu \geq 0$, for any period $t \leq \tau$ the bids of all advertisers are guaranteed to be $b_{k,t}^\Delta = v_{k,t}/(1+\mu_{k,t})$. Inequality (A-4) implies that for each bidder $k$, the stopping time satisfies:

$$T - \tau_k \leq \frac{\bar{\mu}_k}{\epsilon_k \rho_k} + \bar{v},$$

and therefore, denoting $\rho = \min_k \rho_k$ and $\bar{\mu} = \max_k \bar{\mu}_k$, one has:

$$T - \tau = \max_{k=1,\ldots,K} \{\tau_k\} = \max_{k=1,\ldots,K} \{T - \tau_k\} \leq \frac{\bar{\mu}}{\epsilon \rho} + \bar{v}.$$

(A-15)

**Step 3.** Putting everything together we obtain that

$$\sum_{t=1}^{T} \mathbb{E}_v [\|\mu_t - \mu^*\|_2^2] \leq \sum_{t=1}^{T} \mathbb{E}_v [\|\mu_t - \mu^*\|_2^2 \chi_{\{t \leq \tau\}} + \|\mu_t - \mu^*\|_2^2 \chi_{\{t > \tau\}}]

\leq \sum_{t=1}^{T} \delta_t + K \bar{\mu}^2 \mathbb{E}_v [(T - \tau)^+]

\leq K \bar{\mu}^2 \bar{\epsilon} \sum_{t=1}^{T} (1 - 2\lambda \xi)^{t-1} + \frac{TK \bar{v}^2 \bar{\epsilon}^2}{2\lambda \xi} + K \bar{\mu}^2 \left( \frac{\bar{\mu}}{\epsilon \rho} + \frac{\bar{v}}{\rho} \right)

\leq \frac{K \bar{\mu}^2 \bar{\epsilon}}{\lambda \xi^2} + \frac{TK \bar{v}^2 \bar{\epsilon}^2}{2\lambda \xi} + K \bar{\mu}^2 \left( \frac{\bar{\mu}}{\epsilon \rho} + \frac{\bar{v}}{\rho} \right),$$

where (a) follows from conditioning on the stopping time, (b) follows from the definition of $\delta_t$ and using that $\mu_t \in [0, \bar{\mu}_k]$, (c) follows from (A-14) and (A-15), and (b) follows from $\sum_{t=1}^{T} (1 - 2\lambda \xi)^{t-1} \leq \sum_{t=0}^{\infty} (1 - 2\lambda \xi)^t = \frac{1}{1-(1-2\lambda \xi)^t} \leq \frac{1}{\lambda \xi}$ because $1 - (1-x)^{1/2} \geq x/2$ for $x \in [0,1]$. The result then follows from Assumption 4.2.

**A.4 Proof of Theorem 4.4.**

In the proof we consider an alternate framework in which advertisers are also allowed to bid after depleting their budget. The main idea of the proof lies in analyzing the performance of a given advertiser, showing that its performance in the original framework (throughout its entire campaign) is close to the one it achieves in the alternate one before some advertiser runs out of budget.

**Preliminaries and auxiliary results.** Consider the sequence $\{(z_{k,t},u_{k,t})\}_{t \geq 1}$ of realized expenditures and utilities of advertiser $k$ in the alternate framework. Then, with the competing bid given by $d_{k,t} = \max_{i \neq k} v_{i,t}/(1+\mu_{i,t})$, one has under the second-price allocation rule $z_{k,t} = 1\{(1+\mu_{k,t})d_{k,t} \leq v_{k,t}\}d_{k,t}$, and $u_{k,t} = 1\{(1+\mu_{k,t})d_{k,t} \leq v_{k,t}\}(v_{k,t} - d_{k,t})$. Let $\tilde{B}_{k,t} = B_k - \sum_{s=1}^{t-1} z_{s,k}$ denote the
evolution of the $k^{th}$ advertiser’s budget at the beginning of period $t$ in the alternate framework.

Let $\tau = \inf\{t \geq 1 : \tilde{B}_{k,t+1} < \bar{v} \text{ for some } k = 1, \ldots, K\}$ be the last period in which the remaining budget of all advertisers is larger than $\bar{v}$. Since $v/(1 + \mu) \leq \bar{v}$ for any $v \in [0, \bar{v}]$ and $\mu \geq 0$, for any period $t \leq \tau$ the bids of all advertisers are guaranteed to be $b_{k,t}^\lambda = v_{k,t}/(1 + \mu_{k,t})$. Denoting $\rho = \min_k \rho_k$ and $\bar{\mu} = \max_k \bar{\mu}_k$, inequality (A-15) implies that

$$T - \tau \leq \frac{\bar{\mu}}{\epsilon \rho} + \frac{\bar{v}}{\rho}.$$

A key step in the proof involves showing that the utility-per-auction collected by the advertiser is “close” relative to $\Psi_k(\mu^*)$. The next result bounds the performance gap of the adaptive pacing strategy relative to $\Psi_k(\mu^*)$, in terms of the expected squared error of multiplier. Denote by $\mu_t \in \mathbb{R}^K$ the (random) vector such that $\mu_{i,t}$ is the multiplier used by advertiser $i$ at time $t$.

**Lemma A.3.** Let each advertiser $k$ follow the adaptive pacing strategy $A$ in the alternate framework where budgets are not enforced. Then, there exists a constant $C_3 > 0$ such that:

$$\mathbb{E}[u_{k,t}] \geq \frac{\Psi_k(\mu^*)}{T} - C_3 \left(\mathbb{E}[\|\mu_t - \mu^*\|^2]\right)^{1/2}.$$

We note that $C_3 = (2\bar{v} + \bar{\mu} \bar{v}^2 \bar{f})K^{1/2}$. The proof of Lemma A.3 is established by writing the utility in terms of the dual function and the complementary slackness condition, and then using that the dual and functions are Lipschitz continuous as argued in Lemma B.1. The proof of this result is deferred to Appendix B.

**Proving the result.** Let each advertiser $k$ follow the adaptive pacing strategy with step size $\epsilon_k$.

The performance of both the original and the alternate systems coincide until time $\tau$, and therefore:

$$\Pi_k^A \overset{(a)}{=} \mathbb{E} \left[\sum_{t=1}^{\tau \wedge T} u_{k,t}\right] \overset{(b)}{=} \mathbb{E} \left[\sum_{t=1}^{T} u_{k,t}\right] - \bar{v} \mathbb{E} [(T - \tau)^+] ,$$

where (a) follows from discarding all auctions after the time some advertiser runs out of budget; and (b) follows from $0 \leq u_{k,t} \leq \bar{v}$. Summing the lower bound on the expected utility-per-auction in Lemma A.3 over $t = 1, \ldots, T$ one has:
\[
\mathbb{E} \left[ \sum_{t=1}^{T} u_{k,t} \right] \geq \Psi_k(\mu^*) - C_3 \sum_{t=1}^{T} \left( \mathbb{E} \left[ ||\mu_t - \mu^*||^2 \right] \right)^{1/2} \\
\geq \Psi_k(\mu^*) - C_3 C_1^{1/2} \left( \frac{\epsilon}{\bar{\epsilon}} \right)^{1/2} \sum_{t=1}^{T} (1 - 2\lambda_\epsilon)^{(t-1)/2} - C_3 C_2^{1/2} \frac{\bar{\epsilon} \epsilon^{1/2}}{\epsilon^{1/2}} T \tag{A-16}
\]
\[
\geq \Psi_k(\mu^*) - C_3 C_1^{1/2} \frac{\epsilon^{1/2}}{\lambda} - C_3 C_2^{1/2} \frac{\bar{\epsilon} \epsilon}{\epsilon^{1/2}} T, \tag{A-17}
\]
where \((a)\) follows from Theorem 4.3 and \(\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}\) for \(x, y \geq 0\), and \((b)\) follows from \(\sum_{t=1}^{T} (1 - 2\lambda_\epsilon)^{(t-1)/2} \leq \sum_{t=0}^{\infty} (1 - 2\lambda_\epsilon)^{t/2} = \frac{1}{\lambda_\epsilon(1 - 2\lambda_\epsilon)^{1/2}} \leq \frac{1}{\lambda_\epsilon^2}\) because \(1 - (1 - x)^{1/2} \geq x/2\) for \(x \in [0, 1]\). We use the bound in \((A-15)\) to bound the truncated expectation as follows:
\[
\mathbb{E} \left[ (T - \tau)^+ \right] \leq \frac{\bar{v}\bar{\mu}}{\epsilon \rho} + \frac{\bar{v}^2}{\rho}. \tag{A-18}
\]
Combining \((A-16)\) and \((A-18)\) we obtain that
\[
\Pi_k^A \geq \mathbb{E} \left[ \sum_{t=1}^{T} u_{k,t} \right] - \bar{v}\mathbb{E} \left[ (T - \tau)^+ \right] \\
\geq \Psi_k(\mu^*) - C_3 C_1^{1/2} \frac{\epsilon^{1/2}}{\lambda} - C_3 C_2^{1/2} \frac{\bar{\epsilon} \epsilon}{\epsilon^{1/2}} T - \frac{\bar{v}^2}{\rho} - \frac{\bar{v}}{\rho}. \tag{A-19}
\]
The dependence of the payoff gap on the number of advertisers \(K\) is of the same as the dependencies of the products \(C_3 C_1^{1/2}\) and \(C_3 C_2^{1/2}\), which are of order \(K\). This concludes the proof. \(\square\)

### A.5 Proof of Theorem 5.2
As in the proof of Theorem 4.4, we consider an alternate framework in which advertisers are also allowed to bid when they have no remaining budget, without any utility gained. The performance of the deviating advertiser (advertiser \(k\)) in the original framework is equal to the one in the alternate framework until the first time some advertiser runs out of budget. Since the adaptive pacing strategy does not run out of budget too early, then main idea of the proof lies in analyzing the performance of the deviating advertiser in this alternate framework. We first argue that when the number of competitors is large, the expected squared error of the competitors’ multipliers relative to the vector \(\mu^*\) is small, because the impact of the deviating advertiser on its competitors is small. We then show that the benefit of deviating to any other strategy is small when competitors’ multipliers are “close to” \(\mu^*\).
Preliminaries and auxiliary results. Consider the sequence \( \{(z_{k,t}, u_{k,t})\}_{t \geq 1} \) of realized expenditures and utilities of advertiser \( k \) in the alternate framework, and let \( \{d_{k,t}^\beta\}_{t \geq 1} \) be the bids of advertiser \( k \). Then, with the competing bid given by \( d_{k,t} = \max_{\mu_k, m_k} = m_{k,t}, v_{i,t}/(1 + \mu_{i,t}) \), one has under the second-price allocation rule that \( z_{k,t} = 1 \{d_{k,t} \leq \beta_{k,t}\} d_{k,t} \), and \( u_{k,t} = 1 \{d_{k,t} \leq \beta_{k,t}\} (v_{k,t} - d_{k,t}) \). The competing bid faced by an advertiser \( i \neq k \) is given by \( d_{i,t} = \beta_{k,t} \max_{\mu_j, m_j = m_{j,t}} v_{j,t}/(1 + \mu_{j,t}) \). The history available at time \( t \) to advertiser \( k \) in the model described in Section 3 is defined by

\[
H_{k,t} := \sigma \left( \left\{ m_{k,\tau}, v_{k,\tau}, \beta_{k,\tau}, z_{k,\tau}, u_{k,\tau} \right\}_{\tau = 1}^{t-1}, m_{k,t}, v_{k,t}, y \right)
\]

for any \( t \geq 2 \), with \( H_{k,1} := \sigma (m_{k,1}, v_{k,1}, y) \). In what follows we provide few auxiliary results that we use in the proof. The complete proofs of these results are deferred to Appendix B.

Let \( \tau_i \) be the first auction in which the remaining budget of advertiser \( i \neq k \) is less than \( \bar{v} \) (at the beginning of the auction), that is, \( \tau_i = \inf\{t \geq 1 : \bar{B}_{i,t} < \bar{v}\} \), and let \( \tau = \inf_{i \neq k} \{\tau_i\} \). Since \( v/(1 + \mu) \leq \bar{v} \) for any \( v \in [0, \bar{v}] \) and \( \mu \geq 0 \), for any period \( t \leq \tau \) the bid is guaranteed to be \( b_{i,t} = v_{i,t}/(1 + \mu_{i,t}) \). Inequality (A-15) implies that the stopping time satisfies:

\[
T - \tau \leq \frac{\bar{\mu}}{\bar{\rho}} + \frac{\bar{v}}{\bar{\rho}}, \tag{A-20}
\]

with \( \bar{\rho} = \min_{i \neq k} \rho_i \) and \( \bar{\mu} = \max_{i \neq k} \mu_k \). First, we show that the mean squared errors of the estimated multipliers at period \( t \) can be bounded in terms of the minimum and maximum step sizes, the number of players and \( a_k = (a_{k,i})_{i \neq k} \in \mathbb{B}_k \). Denote by \( \mu_t \in \mathbb{R}^K \) the (random) vector such that \( \mu_{k,t} = \mu_k^* \) and \( \mu_{i,t} \) is the multiplier used by advertiser \( i \neq k \) at time \( t \).

Lemma A.4. Suppose that Assumption 4.1 holds and let \( \mu^* \) be the profile of multipliers defined by (3). Let each advertiser \( i \neq k \) follow the adaptive pacing strategy \( A \) with \( \bar{\epsilon} \leq 1/(2\lambda) \) and suppose that advertiser \( k \) uses some strategy \( \beta \in \mathcal{B}_k^{\geq 1} \). Then, there exist positive constants \( C_1, C_2, C_3 \) independent of \( T \) and \( K \), such that for any \( t \in \{1, \ldots, T\} \):

\[
\mathbb{E} \left[ \|\mu_t - \mu^*\|_2^2 \right] \leq C_1 K \frac{\bar{\epsilon}}{\epsilon} (1 - \lambda \bar{\epsilon})^{t-1} + C_2 K \frac{\bar{\epsilon}^2}{\epsilon} \|a_k\|_2^2 + C_3 \|a_k\|_2^2 \frac{\bar{\epsilon}^2}{\epsilon}.
\]

Lemma A.4 implies that the vector of multipliers \( \mu_t \) selected by advertisers under the adaptive pacing strategy converges in \( L_2 \) to the vector \( \mu^* \) under some assumptions on the step sizes and how often advertiser \( k \) interacts with its competitors. The next result bounds the Lagrangian utility-per-auction of the adaptive pacing strategy relative to \( \Psi_k(\mu^*) \), in terms of multipliers’ expected squared error.
Lemma A.5. Suppose that Assumption 4.1 holds and let $\mu^*$ be the profile of multipliers defined by (5). Let each advertiser $i \neq k$ follow the adaptive pacing strategy $A$. Then, there exists a constant $C_4 > 0$ such that for any $t \in \{1, \ldots, T\}$:

$$
\mathbb{E}[u_{k,t} - \mu_k^* z_{k,t} + \rho_k] \leq \frac{\Psi_k(\mu^*)}{T} + C_4 \|a_k\|_2 \cdot \left(\mathbb{E} \|\mu_t - \mu^*\|_2^2\right)^{1/2}.
$$

Proving the result. Let $\mu^*$ be the profile of multipliers defined by (5). Let each advertiser $i \neq k$ follow the adaptive pacing strategy with step size $\epsilon_k$, except advertiser $k$ who implements a strategy $\beta \in B^\text{cl}_k$. Both the original and the alternate systems coincide up to time $\tau$, and thus:

$$
\Pi^{\beta,A,k}_k \leq \mathbb{E} \left[ \sum_{t=1}^T u_{k,t} \right] + \bar{v} \mathbb{E} \left[ (T - \tau)^+ \right] \leq \sum_{t=1}^T \mathbb{E} \left[ u_{k,t} - \mu_k^* z_{k,t} + \rho_k \right] + \bar{v} \mathbb{E} \left[ (T - \tau)^+ \right],
$$

where (a) follows from adding all auctions after time $\tau$ in the alternate system, and using that the utility of each auction in the original system satisfy $0 \leq u_{k,t} \leq \bar{v}$; and (b) follows from adding the constraint $\sum_{t=1}^T z_{k,t} \leq B_k$ to the objective with a Lagrange multiplier $\mu_k^*$, because the strategy $\beta$ is budget-feasible in the alternate framework. Summing the lower bound on the expected utility-per-auction in Lemma A.5 over $t = 1, \ldots, T$ one has:

$$
\sum_{t=1}^T \mathbb{E} \left[ u_{k,t} - \mu_k^* z_{k,t} + \rho_k \right] \leq \Psi_k(\mu^*) + C_4 \|a_k\|_2 \sum_{t=1}^T \left(\mathbb{E} \|\mu_t - \mu^*\|_2^2\right)^{1/2}.
$$

The sum in the second term can be upper bounded by

$$
\sum_{t=1}^T \left(\mathbb{E} \|\mu_t - \mu^*\|_2^2\right)^{1/2} \leq C_1^{1/2} K^{1/2} \left(\frac{\bar{\epsilon}}{\bar{\xi}}\right)^{1/2} \sum_{t=1}^T (1 - \lambda \bar{\xi})^{(t-1)/2} + C_2^{1/2} K^{1/2} \frac{\bar{\epsilon}}{\bar{\xi}^{1/2}} T + C_3^{1/2} \|a_k\|_2 \frac{\bar{\epsilon} T}{\bar{\xi}}.
$$

where (a) follows from Lemma A.4 and $\sqrt{x+y+z} \leq \sqrt{x} + \sqrt{y} + \sqrt{z}$ for $x, y, z \geq 0$, and (b) follows from $\sum_{t=1}^T (1 - \lambda \bar{\xi})^{(t-1)/2} \leq \sum_{t=0}^\infty (1 - \lambda \bar{\xi})^{t/2} = \frac{1}{1 - (1 - \lambda \bar{\xi})^{1/2}} \leq \frac{2}{\lambda \bar{\xi}}$ because $1 - (1 - x)^{1/2} \geq x/2$ for $x \in [0, 1]$. Therefore, we obtain that

$$
\sum_{t=1}^T \mathbb{E} \left[ u_{k,t} - \mu_k^* z_{k,t} + \rho_k \right] \leq \Psi_k(\mu^*) + \frac{2C_4 C_1^{1/2}}{\lambda} \|a_k\|_2 K^{1/2} \frac{\bar{\epsilon}}{\bar{\xi}^{3/2}} T
$$

$$
+ C_4 C_2^{1/2} \|a_k\|_2 K^{1/2} \frac{\bar{\epsilon}}{\bar{\xi}^{1/2}} T + C_4 C_3^{1/2} \|a_k\|_2 \frac{\bar{\epsilon} T}{\bar{\xi}}.
$$

(A-21)
We use the bound in (A-20) to bound the truncated expectation as follows:

$$ \mathbb{E} [(T - \tau)^+] \leq \frac{\bar{\mu} + \bar{v}}{\rho}. \quad (A-22) $$

Combining (A-21) and (A-22) one obtains that

$$ \Pi_{k}^{\beta,A-k} \leq \mathbb{E} \left[ \sum_{t=1}^{T} u_{k,t} \right] + \bar{v} \mathbb{E} \left[ (T - \tau)^+ \right] \leq \Psi_{k}(\mu^*) + 2 C_{4} C_{1} / 2 \frac{\| a_{k} \|_{2} K^{1/2}}{\bar{\epsilon} T} + C_{4} C_{2} / 2 \frac{\| a_{k} \|_{2} K^{1/2}}{\bar{\epsilon} T} + \frac{\bar{\mu} \bar{v}}{\rho} + \bar{v}^{2} \rho.$$

The result follows from combining the last inequality with (A-19).

**B Proofs of key lemmas**

**Proof of Lemma A.1.** Let $\mathbb{P}\{v^i\}$ be the probability that valuation sequence $v^i$ is chosen, for $i = 1, \ldots, m$, and let $\beta$ be a (potentially randomized) strategy. We have:

$$ \sup_{\mathbb{P}\{v^i\}} \mathbb{E}^{\beta} \left[ R_{\gamma}^{\beta}(v; d) \right] \overset{(a)}{=} \sum_{i=1}^{m} \mathbb{P}\{v^i\} \sup_{d \in \mathbb{R}_{+}^{T}} \mathbb{E}^{\beta} \left[ R_{\gamma}^{\beta}(v; d) \right] \overset{(b)}{=} \sum_{i=1}^{m} \mathbb{P}\{v^i\} \mathbb{E}^{\beta} \left[ R_{\gamma}^{\beta}(v^i; d') \right] \overset{(c)}{=} \mathbb{E}^{\beta} \left[ \sum_{i=1}^{m} \mathbb{P}\{v^i\} R_{\gamma}^{\beta}(v^i; d') \right] \overset{(d)}{=} \inf_{\beta \in \mathcal{B}} \sum_{i=1}^{m} \mathbb{P}\{v^i\} R_{\gamma}^{\beta}(v^i; d'), $$

where: (a) follows because $\sum_{i} \mathbb{P}\{v^i\} = 1$; (b) follows because $v^i$ and $d'$ are feasible realizations; (c) follows from Fubini’s Theorem because the regret is bounded by $|R_{\gamma}^{\beta}(v; d)| + \gamma^{2} |\pi^{\beta}(v; d)| \leq \bar{v}(1 + \gamma)$ because no strategy (even in hindsight) can achieve more that $T\bar{v}$; and (d) follows because any randomized strategy can be thought of as a probability distribution over deterministic algorithms.

**Proof of Lemma A.2.** We use the worst-case instance structure detailed in the proof of Theorem 3.1. Fix any deterministic bidding strategy $\beta \in \mathcal{B}$. Since $\beta$ is deterministic one has $\pi^{\beta}(d, v) = \sum_{t=1}^{T} 1 \{ d_t \leq b^\beta_t \} (v_t - d_t)$ for any vectors $d$ and $v$ where $b^\beta_t$ is the bid dictated by $\beta$ at time $t$.

Let $v^i \in \mathcal{V}$ be the valuation sequence and $d^0 = (d, \ldots, d)$ the competing bid sequence chosen by the adversary. We denote by $b^i_t$ the bid at period $t$ under this input and $\beta$. We also denote the
corresponding expenditure by \( z^j_i := 1\{d \leq b^j_i\}d \), and the corresponding net utility by \( u^j_i := 1\{d \leq b^j_i\}(v^j_i - d) \). We further denote the history of decisions and observations under \( d^0 \), \( v^i \) and \( \beta \) by 
\[
\mathcal{H}^j_i := \sigma \left( \{v^j_{\tau}, b^j_{\tau}, z^j_{\tau}, u^j_{\tau}\}_{\tau=1}^{t-1} v^j_1 \right)
\]
for any \( t \geq 2 \), with \( \mathcal{H}^j_1 := \sigma (v^j_1) \).

We now define the sequence \( x \). For each \( j \in \{1, \ldots, m\} \), define:
\[
x_j := \sum_{t=(j-1)\left\lceil \frac{T}{m} \right\rceil + 1}^{\left\lfloor \frac{T}{m} \right\rfloor} 1\{d^0_t \leq b^j_t\},
\]
where we denote by \( b^j_t \) the bid at time \( t \) under history \( \mathcal{H}^j_1 \). Then, \( x_j \) is the number of auctions won by \( \beta \) throughout the \( j \)'th batch of \( \lfloor T/m \rfloor \) auctions when the vector of best competitors' bids is \( d^0 \) and the vector of valuations is \( v^1 \). Since \( d^0 \), \( v^1 \) and \( \beta \) are deterministic, so is \( x \), and clearly \( x \in \mathcal{X} \). Denote by \( \beta^x \) a strategy that for each \( j \in \{1, \ldots, m\} \) wins the first \( x_j \) auctions in the \( j \)'th batch (by bidding \( d^0 \) in these auctions).

We next show that \( \beta^x \) achieves the same performance as \( \beta \). First, assume that the vector of valuations is \( v^1 \). Then, by construction:
\[
\pi^\beta(d^0, v^1) = \sum_{t=1}^{T} 1\{d \leq b^j_t\}(v^j_t - d) = \sum_{j=1}^{m} x_j(v_j - d) = \pi^{\beta^x}(d^0, v^1),
\]
where \( v_j \) denotes the value of the \( j \)'th batch of the valuation sequence \( v^1 \). For any \( v^i \in V \setminus \{v^1\} \), define \( \tau^i := \inf \{t \geq 1 : v^j_t \neq v^1_t\} \) to be the first time that the valuation sequence is different than \( v^1 \). Let \( m^i \in \{1, \ldots, m\} \) be the number of batches that the valuation sequence \( v^i \) has in common with \( v^1 \). Since each batch has \( \lfloor T/m \rfloor \) items, we have that \( \tau^i = m^i \lfloor T/m \rfloor + 1 \). The sequences \( v^1 \) and \( v^i \) are identical, thus indistinguishable, up to time \( \tau^i \). Therefore, the bids of any deterministic strategy coincide up to time \( \tau^i \) under histories \( \mathcal{H}^j_1 \) and \( \mathcal{H}^j_i \). Then, one has:
\[
\pi^\beta(d^0, v^i) \overset{(a)}{=} \sum_{t=1}^{\tau^i-1} 1\{d \leq b^j_t\}(v^j_t - d) \overset{(b)}{=} \sum_{t=1}^{\tau^i-1} 1\{d \leq b^j_t\}(v^j_1 - d) \overset{(c)}{=} \sum_{j=1}^{m^i} x_j(v_j - d) = \pi^{\beta^x}(d^0, v^i),
\]
where \( (a) \) follows because all items in periods \( t \in \{\tau^i, \ldots, T\} \) have zero utility under \( v^i \), \( (b) \) follows because the valuation sequences and bids are equal during periods \( t \in \{1, \ldots, \tau^i - 1\} \), \( (c) \) follows from our definition of \( x \) and using that sequence \( v^1 \) has \( m^i \) batches with nonzero utility. We have thus established that \( \pi^{\beta^x}(d^0, v) = \pi^\beta(d^0, v) \) for any \( v \in V \). This concludes the proof. \( \square \)
The following result obtains some key characteristics of the model primitives under the matching model described in Section 5. We denote by $a_{k,i} = \mathbb{P}\{m_{k,t} = m_{i,t}\} = \sum_{m=1}^{M} \alpha_{k,m} \alpha_{i,m}$ the probability that advertisers $k$ and $i$ compete in the same auction at a given time period. Given a vector of multipliers $\mu$, we denote $\Psi_k(\mu) := T (\mathbb{E}_\nu [\{v_{k,1} - (1 + \mu_k)d_{k,1}\} + \mu_k \rho_k])$ the dual performance under $\mu$ with $d_{k,1} = \max \left\{ \max_{i \neq k} \left\{ m_{i,1} \right\} v_{i,1}/(1 + \mu_i) \right\}$. In addition, we denote by $G_k(\mu) := \mathbb{E}_\nu [\{1 + \mu_k,1\}d_{k,1} \leq v_{k,1}1\}d_{k,1}]$ the expected expenditure under the second-price auction allocation rule. Results for the original model how by putting $M = 1$ and $\alpha_{k,m} = 1$.

**Lemma B.1.** Suppose that for each advertiser $k$ the valuation density satisfies $f_k(x) \leq \bar{f} < \infty$ for all $x \in [0, \bar{v}]$. Then:

(i) Fix $\mu_{k,t} = (\mu_{i,t})_{i \neq k} \in \mathbb{R}^{K-1}$, the competitor’s multipliers at time $t$. The maximum competing bid $d_{k,t} = \max_{i \neq k} \{v_{i,t}/(1 + \mu_{i,t})\}$ is integrable over $[0, \bar{d}_{k,t}]$ where $\bar{d}_{k,t} = \bar{v}/(1 + \min_{i \neq k} \mu_{i,t})$, with cumulative distribution function $H_k(x; \mu_{k,t}) = \prod_{i \neq k} \left(1 - a_{k,i} \bar{F}_i((1 + \mu_{i,t})x)\right)$.

(ii) $\Psi_k(\cdot)$ is Lipschitz continuous. In particular, for any $\mu \in \mathcal{U}$ and $\mu' \in \mathcal{U}$, one has
\[
\frac{1}{T} |\Psi_k(\mu) - \Psi_k(\mu')| \leq \bar{v} |\mu_k - \mu'_k| + \bar{v} \sum_{i \neq k} a_{k,i} |\mu_i - \mu'_i|.
\]

(iii) $G_k(\cdot)$ is Lipschitz continuous. In particular, for any $\mu \in \mathcal{U}$ and $\mu' \in \mathcal{U}$, one has
\[
|G_k(\mu) - G_k(\mu')| \leq \bar{v}^2 \bar{f} |\mu_k - \mu'_k| + 2\bar{v}^2 \bar{f} \sum_{i \neq k} a_{k,i} |\mu_i - \mu'_i|.
\]

**Proof of Lemma B.1.** We prove the three parts of the Lemma.

(i). Using that the values $v_{i,t}$ are independent across advertisers and identical across time, we can write the cumulative distribution function as
\[
H_k(x; \mu_{k,t}) = \mathbb{P}\left\{ \max_{i \neq k, m_{k,t} = m_{i,t}} \frac{v_{i,t}}{1 + \mu_{i,t}} \leq x \right\}
\]
\[
= (a) \prod_{i \neq k} \left( \mathbb{P}\{m_{k,t} \neq m_{i,t}\} + \mathbb{P}\{m_{k,t} = m_{i,t}\} \mathbb{P}\left\{ \frac{v_{i}}{1 + \mu_{i,t}} \leq x \right\} \right)
\]
\[
= (b) \prod_{i \neq k} \left(1 - a_{k,i} \bar{F}_i((1 + \mu_{i,t})x)\right),
\]
\[
\text{where (a) follows by conditioning on whether advertiser } i \neq k \text{ participates in the same auction that advertiser } k, \text{ and (b) follows from } a_{k,i} = \mathbb{P}\{m_{k,t} = m_{i,t}\}. \text{ Because } F_i(\cdot) \text{ has support } [0, \bar{v}] \text{ we} \]

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conclude that the support is \([0, \bar{d}_{k,t}]\) with \(\bar{d}_{k,t} = \max_{i \neq k} \bar{v}/(1 + \mu_{i,t}) = \bar{v}/(1 + \min_{i \neq k} \mu_{i,t})\).

(ii). Denote by \(\bar{\Psi}_k(\mu) = \Psi_k(\mu)/T\). For every realized vectors \(v = \{v_i\}_i\) and \(m = \{m_i\}_i\), the function \((v_k - (1 + \mu_k)d_k)^+\) is differentiable in \(\mu_k\) with derivative \(-d_k 1\{v_k \geq (1 + \mu_k)d_k\}\), except in the set \(\{(v, m) : v_k = (1 + \mu_k)d_k\}\) that has measure zero because values are absolutely continuous with support \([0, \bar{v}]\) and independent. As the derivative is bounded by \(d_k\), which is integrable since \(d_k \leq \bar{v}\) from Item (i), we conclude by Leibniz’s integral rule that:

\[
\frac{\partial \bar{\Psi}_k(\mu)}{\partial \mu_k} = \rho_k - \mathbb{E}[d_k 1\{v_k \geq (1 + \mu_k)d_k\}] = \rho_k - G_k(\mu),
\]

which implies that \(|\frac{\partial \bar{\Psi}_k(\mu)}{\partial \mu_k}| \leq \bar{v}\) because \(\rho_k, G_k(\mu) \in [0, \bar{v}]\).

Fix an advertiser \(i \neq k\). Recall that the maximum competing bid faced by advertiser \(k\) is \(d_k = \max_{i \neq k, m_k = m_i} \{v_i/(1 + \mu_i)\}\). Let \(d_{k\setminus i} = \max_{j \neq k, i, m_k = m_j} \{v_j/(1 + \mu_j)\}\) be the maximum competing bid faced by advertiser \(k\) with advertiser \(i\) excluded. By conditioning on whether advertiser \(i\) and \(k\) compete in the same auction, we can write the random function \((v_k - (1 + \mu_k)d_k)^+\) as

\[
(v_k - (1 + \mu_k)d_k)^+ = \begin{cases} 
(v_k - (1 + \mu_k)d_{k\setminus i} \lor \frac{v_i}{1 + \mu_i})^+, & \text{if } m_{k,t} = m_{i,t}, \\
(v_k - (1 + \mu_k)d_{k\setminus i})^+, & \text{if } m_{k,t} \neq m_{i,t}.
\end{cases}
\]

We obtain that the function \((v_k - (1 + \mu_k)d_k)^+\) is differentiable in \(\mu_i\), with derivative

\[
\frac{v_i}{1 + \mu_i}(1 + \mu_k)^2 \left\{ \frac{v_k}{1 + \mu_k} \geq \frac{v_i}{1 + \mu_i} \geq d_{k\setminus i}, m_k = m_i \right\}
\]

except in the sets \(\{(v, m) : \frac{v_k}{1 + \mu_k} = \frac{v_i}{1 + \mu_i} \leq d_{k\setminus i}, m_k = m_i\}\) and \(\{(v, m) : \frac{v_k}{1 + \mu_k} \geq \frac{v_i}{1 + \mu_i} = d_{k\setminus i}, m_k = m_i\}\). Again, these sets have measure zero because values are absolutely continuous with support \([0, \bar{v}]\) and independent. Because the derivative is bounded by \(v_k/(1 + \mu_i)\), which is integrable since \(v_k \leq \bar{v}\), we conclude by Leibniz’s integral rule that:

\[
\frac{\partial \bar{\Psi}_k(\mu)}{\partial \mu_i} = \mathbb{E}\left[ \frac{v_k}{1 + \mu_i} \left\{ \frac{v_k}{1 + \mu_k} \geq \frac{v_i}{1 + \mu_i} \geq d_{k\setminus i}, m_k = m_i \right\} \right] \\
\leq \mathbb{E}\left[ \frac{v_k}{1 + \mu_i} 1\{m_k = m_i\} \right] \leq \bar{v}a_{k,i},
\]

where the first inequality follows because \(\frac{v_i}{1 + \mu_i} \leq \frac{v_k}{1 + \mu_k}\) and dropping part of the indicator, and the last inequality follows because \(v_k \in [0, \bar{v}]\) and \(\mu_i \geq 0\). This concludes the proof.

(iii). We show that \(G_k(\mu)\) is Lipschitz continuous by bounding its derivatives. Since values are
independent across advertisers we can write the expected expenditure as

\[
G_k(\mu) = \int_0^\beta x f_k((1 + \mu_k)x) dH_k(x; \mu_k) \\
= \int_0^\beta ((1 + \mu_k)x f_k((1 + \mu_k)x) - \bar{F}_k((1 + \mu_k)x)) H_k(x; \mu_k) dx,
\]

where the second equation follows from integration by parts. Using the first expression for the expected expenditure we obtain that

\[
\frac{\partial G_k}{\partial \mu_k}(\mu) = -\int_0^\beta x^2 f_k((1 + \mu_k)x) \mathbf{1}\{(1 + \mu_k)x \leq \bar{v}\} H_k(x; \mu_k) dx,
\]

where we used Leibniz rule because \(x \bar{F}_k((1 + \mu_k)x)\) is differentiable w.r.t. \(\mu_k\) almost everywhere with derivative that is bounded by \(\bar{v}^2 \bar{f}\). Therefore, one obtains

\[
\left|\frac{\partial G_k}{\partial \mu_k}(\mu)\right| \leq \bar{v}^2 \bar{f}.
\]

Using the second expression for the expected expenditure we obtain for \(i \neq k\) that

\[
\frac{\partial G_k}{\partial \mu_i}(\mu) = \int_0^\beta ((1 + \mu_k)x f_k((1 + \mu_k)x) - \bar{F}_k((1 + \mu_k)x)) \frac{\partial H_k}{\partial \mu_i}(x; \mu_k) dx,
\]

where we used Leibniz rule as \(H_k(x; \mu_k)\) is differentiable w.r.t. \(\mu_i\) almost everywhere with derivative

\[
\frac{\partial H_k}{\partial \mu_i}(x; \mu_k) = a_{k,i} x f_i((1 + \mu_i)x) \mathbf{1}\{(1 + \mu_i)x \leq \bar{v}\} \prod_{j \neq k, i} (1 - a_{k,j} \bar{F}_j((1 + \mu_j)x)),
\]

which is bounded from above by \(a_{k,i} \bar{v} \bar{f} \mathbf{1}\{(1 + \mu_i)x \leq \bar{v}\}\). Therefore, one obtains

\[
\left|\frac{\partial G_k}{\partial \mu_i}(\mu)\right| \leq 2a_{k,i} \bar{v}^2 \bar{f},
\]

and the result follows.

**Proof of Lemma A.3.** Fix a profile of advertiser types \(\theta\). Let each advertiser \(k\) follow the adaptive pacing strategy, and define \(\delta_t = \sum_{k=1}^K \mathbb{E}_{v} [||\mu_{k,t} - \mu^*_k||^2]\). Denote \(\Psi_k(\mu) = \Psi_k(\mu)/T\), and let \(\hat{\Psi}_{k,t}(\mu) = (v_{k,t} - (1 + \mu_{k,t})d_{k,t})^+ + \mu_k \rho_k\) be normalized empirical dual objective function for advertiser \(k\), with \(d_{k,t} = \max_{i \neq k} v_{i,t}/(1 + \mu_{i,t})\). Using the structure of the adaptive pacing strategy, the utility of advertiser \(k\) from the \(t^{th}\) auction can be written as:
where \( (\sum_{k=1}^{K} a_k G_k(t) - \rho_k) \) follows from the linearity of expectation and conditioning on the multipliers \( \mu_t \). Thus, one has:

\[
\Psi_k(\mu_t) \geq \Psi_k(\mu^*) - \bar{v} \| \mu_t - \mu^* \|_1 ,
\]

for all \( t = 1, \ldots, T \) because \( a_{k,i} \in [0, 1] \). Since the expenditure is Lipschitz continuous (Lemma B.1 item (iii)) one has:

\[
\sum_{k=1}^{K} \sum_{i=1}^{n} \left| y_{i} \right| \leq \left( n \sum_{i=1}^{n} y_{i}^2 \right)^{1/2} ,
\]

and \( \sum_{i=1}^{n} y_{i}^2 \) is \( \mu_{i,t} G_i(t) \) the (random) vector such that \( \mu_{k,t} = \mu_{k,i}^* \) and \( \mu_{i,t} \) is the multiplier used by advertiser \( i \neq k \) at time \( t \). Fix an advertiser \( i \neq k \). Define \( \delta_{k,t} := \mathbb{E} \left[ (\mu_{k,t} - \mu_{k,i}^*)^2 \right] \) and \( \delta_t := \sum_{i \neq k} \delta_{i,t} \). Similarly, define \( \hat{\delta}_{i,t} := \delta_{i,t}/\epsilon_i \) and \( \hat{\delta}_t := \sum_{i \neq k} \hat{\delta}_{i,t} \). We obtain from (A-13) in the proof of Theorem 4.3 that:

\[
\mathbb{E} \left[ u_{k,t} \right] \geq \Psi_k(\mu^*) - (2\bar{v} + \bar{v}^2 \bar{f}) K^{1/2} \hat{\delta}_t^{1/2} .
\]

This concludes the proof. \( \square \)

**Proof of Lemma A.4.** Denote by \( \mu_t \in \mathbb{R}^K \) the (random) vector such that \( \mu_{k,t} = \mu_{k,i}^* \) and \( \mu_{i,t} \) is the multiplier used by advertiser \( i \neq k \) at time \( t \). Fix an advertiser \( i \neq k \). Define \( \delta_{k,t} := \mathbb{E} \left[ (\mu_{k,t} - \mu_{k,i}^*)^2 \right] \) and \( \delta_t := \sum_{i \neq k} \delta_{i,t} \). Similarly, define \( \hat{\delta}_{i,t} := \delta_{i,t}/\epsilon_i \) and \( \hat{\delta}_t := \sum_{i \neq k} \hat{\delta}_{i,t} \). We obtain from (A-13) in the proof of Theorem 4.3 that:
where the equality follows by conditioning on \( \mathbf{\mu}_t \). The third term satisfies \( \mathbb{E}[|\rho_i - z_{i,t}|^2] \leq v^2 \), since \( \rho_i, z_{i,t} \in [0, \bar{v}] \). Proceeding to bound the second term, recall that the payment of advertiser \( i \) is \( z_{i,t} = 1\{d_{i,t} \leq v_{i,t}/(1 + \mu_{i,t})\}d_{i,t} \) where the competing bid faced by the advertiser is given by

\[
d_{i,t} = \begin{cases} b_{k,t} \lor d_{i\backslash k,t}, & \text{if } m_{k,t} = m_{i,t}, \\ d_{i\backslash k,t}, & \text{if } m_{k,t} \neq m_{i,t}, \end{cases}
\]

where \( d_{i\backslash k,t} = \max_{j \neq k,i,m_{j,t}=m_i} \{v_{j,t}/(1 + \mu_{j,t})\} \) denotes the maximum competing bid faced by advertiser \( i \) when advertiser \( k \) is excluded. Recall that for a fixed vector \( \mathbf{\mu} \in \mathbb{R}_+^K \), the expected expenditure function is given by \( G_i(\mathbf{\mu}) = \mathbb{E}\{\hat{d}_i \{d_i \leq v_i/(1 + \mu_i)\}\} \) where \( \hat{d}_i = \max_{j \neq i,m_j = m_i} \{v_{j,t}/(1 + \mu_j)\} \). For the second term in (B-3) one has:

\[
(\mu_{i,t} - \mu_i^*) (\rho_i - \mathbb{E}[z_{i,t} | \mathbf{\mu}_t]) = (\mu_{i,t} - \mu_i^*) (\rho_i - G_i(\mathbf{\mu}^*) + G_i(\mathbf{\mu}^*) - G_i(\mathbf{\mu}_t) + G_i(\mathbf{\mu}_t) - \mathbb{E}[z_{i,t} | \mathbf{\mu}_t]) \\
\geq (\mu_{i,t} - \mu_i^*) (G_i(\mathbf{\mu}^*) - G_i(\mathbf{\mu}_t)) - |\mu_{i,t} - \mu_i^*| \cdot |G_i(\mathbf{\mu}_t) - \mathbb{E}[z_{i,t} | \mathbf{\mu}_t]|,
\]

where the inequality follows because \( \mu_{i,t} \geq 0 \) and \( \rho_i - G_i(\mathbf{\mu}^*) \geq 0 \) and \( \mu_i^* (\rho_i - G_i(\mathbf{\mu}^*)) = 0 \), together with \( xy \geq -|x| \cdot |y| \) for \( x, y \in \mathbb{R} \). Because values are independent, and advertisers \( i \) and \( k \) compete only when \( m_{k,t} = m_{i,t} \), we obtain that

\[
|G_i(\mathbf{\mu}_t) - \mathbb{E}[z_{i,t} | \mathbf{\mu}_t]| = | \mathbb{E}\{[\hat{d}_i \{d_i \leq b_{i,t}\} - d_{i,t} \{d_i \leq b_{i,t}\}] \} 1\{m_{k,t} = m_{i,t}\} | \leq \bar{v} a_{k,i},
\]

where the equality follows because the bid of advertiser \( i \neq k \) is \( b_{i,t} = v_{i,t}/(1 + \mu_{i,t}) \), and \( d_{i,t} = b_{k,t} \lor d_{i\backslash k,t} \) and \( \tilde{d}_{i,t} = (v_{k,t}/(1 + \mu^*_k)) \lor d_{i\backslash k,t} \) when \( m_{k,t} = m_{i,t} \); and the inequality follows because the expenditure is at most the bid and \( b_{i,t} \leq \bar{v} \) together with \( \mathbb{P}\{m_{k,t} = m_{i,t}\} = a_{k,i} \). Summing over the different advertisers and using Assumption 4.1 item (1) we obtain by

\[
\sum_{i \neq k} (\mu_{i,t} - \mu_i^*) (\rho_i - \mathbb{E}[z_{i,t} | \mathbf{\mu}_t]) \geq \sum_{i=1}^K (\mu_{i,t} - \mu_i^*) (G_i(\mathbf{\mu}^*) - G_i(\mathbf{\mu}_t)) - \bar{v} \sum_{i \neq k} a_{k,i} |\mu_{i,t} - \mu_i^*| \\
\geq \lambda \|\mathbf{\mu}_t - \mathbf{\mu}^*\|_2^2 - \bar{v}\|\mathbf{a}_k\|_2 \cdot \|\mathbf{\mu}_t - \mathbf{\mu}^*\|_2,
\]

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where the last inequality follows from Cauchy-Schwarz inequality. Denoting \( \bar{\epsilon} = \max_{k \in \{1, \ldots, K\}} \epsilon_k \), \( \xi = \min_{k \in \{1, \ldots, K\}} \epsilon_k \), we conclude by summing \((A-13)\) over \( k \) that:

\[
\hat{\delta}_{t+1}^{(a)} \leq \hat{\delta}_t - 2\lambda \hat{\delta}_t + \bar{\epsilon} K \bar{v}^2 + 2\bar{v} \| \mathbf{a}_k \|_2 \hat{\delta}_t^{1/2} + (1 - 2\lambda \bar{\epsilon}) \hat{\delta}_t + K \bar{v}^2 \bar{\epsilon} + 2\bar{v} \| \mathbf{a}_k \|_2 \hat{\delta}_t^{1/2} \hat{\delta}_t^{1/2} ,
\]

where \((a)\) follows because \( \mathbb{E} \| \mathbf{\mu}_t - \mathbf{\mu}^* \|_2 \leq \left( \mathbb{E} \| \mathbf{\mu}_t - \mathbf{\mu}^* \|_2^2 \right)^{1/2} = \delta_t^{1/2} \) by Jensen’s Inequality, and \((b)\) follows from \( \hat{\delta}_t = \sum_{i \neq k} \delta_{i,t} = \sum_{i \neq k} \epsilon_i \hat{\delta}_{i,t} \geq \epsilon \hat{\delta}_t \) because \( \delta_{i,t} \geq 0 \) and \( \delta_t \leq \bar{\epsilon} \hat{\delta}_t \). Lemma \( C.3 \) with \( a = 2\lambda \xi \leq 1 \), \( b = \bar{\epsilon} K \bar{v}^2 \) and \( c = 2\bar{v} \| \mathbf{a}_k \|_2 \bar{\epsilon}^{1/2} \) implies that

\[
\hat{\delta}_t \leq \hat{\delta}_1 \frac{(1 - \lambda \xi)^{t-1}}{\xi} + \frac{K \bar{v}^2 \bar{\epsilon}}{\lambda} \frac{\bar{v}^2 \| \mathbf{a}_k \|_2 \bar{\epsilon}^{1/2}}{\bar{v}^2 \| \mathbf{a}_k \|_2 \bar{\epsilon}^{1/2} \bar{\epsilon}^{1/2}} .
\]

Using that \( \delta_t \leq \bar{\epsilon} \hat{\delta}_t \) together with \( \hat{\delta}_1 \leq \delta_1 / \xi \leq K \bar{\mu}^2 / \xi \) because \( \mu_{i,t}, \mu_i^* \in [0, \bar{\mu}_k] \) and \( \bar{\mu} = \max_k \bar{\mu}_k \) we obtain that

\[
\delta_t \leq K \bar{\mu}^2 \bar{\epsilon} \left( 1 - \lambda \xi \right)^{t-1} + \frac{K \bar{v}^2 \bar{\epsilon}}{\lambda} \frac{\bar{v}^2 \| \mathbf{a}_k \|_2 \bar{\epsilon}^2}{\bar{v}^2 \| \mathbf{a}_k \|_2 \bar{\epsilon}^2} .
\]

This concludes the proof.

\[\Box\]

**Proof of Lemma \( A.5 \).** Let each advertiser \( i \neq k \) follow the adaptive pacing strategy. Denote \( \bar{\Psi}_k(\mathbf{\mu}) := \Psi_k(\mathbf{\mu}) / T \), and by \( \mathbf{\mu}_t \in \mathbb{R}^K \) the (random) vector such that \( \mu_{k,t} = \mu_k^* \) and \( \mu_{i,t} \) is the multiplier of the adaptive pacing strategy of advertiser \( i \neq k \) at time \( t \). Based on the second-price allocation rule, the Lagrangian utility of advertiser \( k \) from the \( t \)th auction can be written as:

\[
u_{k,t} - \mu_k^* z_{k,t} + \rho_k = 1\{d_{k,t} \leq b_{k,t}^\beta\} \left( v_{k,t} - (1 + \mu_k^*) d_{k,t} \right) + \rho_k \leq (v_{k,t} - (1 + \mu_k^*) d_{k,t})^+ + \rho_k ,
\]

where the first equality follows because \( z_{k,t} = 1\{d_{k,t} \leq b_{k,t}^\beta\} d_{k,t} \) and \( u_{k,t} = 1\{d_{k,t} \leq b_{k,t}^\beta\} (v_{k,t} - d_{k,t}) \), and the inequality because \( x \leq x^+ \) for all \( x \in \mathbb{R} \) and dropping the indicator that advertiser \( k \) wins the auction. Taking expectations we obtain:

\[
\mathbb{E} \left[ u_{k,t} - \mu_k^* z_{k,t} + \rho_k \right] \leq \mathbb{E} \left[ \mathbb{E} \left[ (v_{k,t} - (1 + \mu_k^*) d_{k,t})^+ \left| \mathbf{\mu}_t \right. \right] + \rho_k \right] \overset{(b)}{=} \mathbb{E} \left[ \bar{\Psi}_k(\mathbf{\mu}_t) \right] ,
\]

where \((a)\) follows from conditioning on \( \mathbf{\mu}_t \), and \((b)\) holds since \( \{v_{k,t}\}_{k=1}^K \) are independent of the multipliers \( \mathbf{\mu}_t \). Since the dual objective is Lipschitz continuous (Lemma \( B.1 \), item \((ii)\) one has:
\[ \bar{\Psi}_k(\mu_t) \leq \bar{\Psi}_k(\mu^*) + \bar{v} \sum_{i \neq k} a_{k,i} |\mu_{i,t} - \mu^*_i| \leq \bar{\Psi}_k(\mu^*) + \bar{v} \left\| \mu_t - \mu^* \right\|_2, \]

where the last inequality follows from Cauchy-Schwarz inequality, for all \( t = 1, \ldots, T \). The result follows by taking expectations and using Jensen’s Inequality.

\[ \square \]

C Additional auxiliary analysis

Proposition C.1. (Uniqueness of steady state) Suppose Assumption 4.1 holds. Then there exists a unique vector of multipliers \( \mu^* \in \mathcal{U} \) defined by (5).

Proof. We first establish that selecting a multiplier outside of \([0, \bar{\mu}_k]\) is a dominated strategy for advertiser \( k \). Notice that for every \( \mu - \mu_k \) and \( x > \bar{\mu}_k \) we have that

\[ \Psi_k(x, \mu_{-k}) = \mathbb{E}_v \left[ v_{k,1} - (1 + x)d_{k,1} \right]^+ + x\rho_k \geq x\rho_k > \bar{v} \geq \Psi_k(0, \mu_{-k}), \]

where (a) follows from dropping the first term, (b) holds since by Assumption 4.1 one has that \( \rho_k \geq \bar{v}/\bar{\mu}_k \) from and \( x > \bar{\mu}_k \), and (c) follows from \( 0 \leq v_{k,t} \leq \bar{v} \). Thus every \( x > \bar{\mu}_k \) in the dual problem is dominated by \( x = 0 \), and the equilibrium multipliers lie in the set \( \mathcal{U} \). Define \( G_k(\mu) := \mathbb{E}_v \left[ \mathbf{1}\{ (1 + \mu_k)d_{k,1} \leq v_{k,1} \} d_{k,1} \right] \) to be the expected expenditure under the second-price auction allocation rule. Assumption 4.1 implies that:

\[ (\mu - \mu^*)^T(G(\mu^*) - G(\mu)) > 0, \quad (B-4) \]

for all \( \mu \in \mathcal{U} \) such that \( \mu \neq \mu^* \). To prove uniqueness, suppose that there exists another equilibrium multiplier \( \mu \in \mathcal{U} \) such that \( \mu \neq \mu^* \). From (B-4) one has:

\[ 0 < \sum_{k=1}^{K} (\mu_k - \mu^*_k)(G_k(\mu^*) - \rho_k + \rho_k - G_k(\mu)) \]

\[ \overset{(a)}{=} \sum_{k=1}^{K} \mu_k (G_k(\mu^*) - \rho_k) - \mu_k^*(\rho_k - G_k(\mu)), \]

where (a) follows from \( \mu_k(\rho_k - G_k(\mu)) = 0 \) and \( \mu_k^*(G_k(\mu^*) - \rho_k) = 0 \) by (5). As \( \mu_k, \mu_k^* \geq 0 \) and \( G_k(\mu^*), G_k(\mu) \leq \rho_k \), we obtain that the right hand-side is non-positive, contradicting (B-4).

\[ \square \]

Lemma C.2. Let \( \{\delta_t\}_{t \geq 1} \) be a sequence of numbers such that \( \delta_t \geq 0 \) and \( \delta_{t+1} \leq (1 - a)\delta_t + b \) with \( b \geq 0 \) and \( 0 \leq a \leq 1 \). Then,
\[ \delta_t \leq (1 - a)^{t-1}\delta_1 + \frac{b}{a} \leq e^{-a(t-1)}\delta_1 + \frac{b}{a}. \]

**Proof.** We prove the result by induction. The result trivially holds for \( t = 1 \) because \( a, b \geq 0 \). For \( t > 1 \), the recursion gives

\[ \delta_{t+1} \leq (1 - a) \delta_t + b \leq (1 - a)^t \delta_1 + \frac{b}{a} \leq e^{-at} \delta_1 + \frac{b}{a}, \]

where the second inequality follows from the induction hypothesis and the fact that \( 1 - a \geq 0 \) for \( t \geq t_0 \), and the last inequality because \( 1 - a \leq e^{-a} \) for \( a \in \mathbb{R} \). \( \square \)

**Lemma C.3.** Let \( \{\delta_t\}_{t \geq 1} \) be a sequence of numbers such that \( \delta_t \geq 0 \) and \( \delta_{t+1} \leq (1 - a)\delta_t + b + c\delta_t^{1/2} \) with \( c \geq 0, b \geq 0 \) and \( 0 \leq a \leq 1 \). Then,

\[ \delta_t \leq (1 - a/2)^{t-1}\delta_1 + \frac{2b}{a} + \frac{c^2}{a}. \]

**Proof.** The square root term can be bounded as follows

\[ c\delta_t^{1/2} = \frac{c}{a^{1/2}} a^{1/2} \delta_t^{1/2} \leq \frac{c^2}{2a} + a\delta_t, \]

because \( xy \leq (x^2 + y^2)/2 \) for \( x, y \in \mathbb{R} \) by the AM-GM inequality. Using this bound, we can rewrite the inequality in the statement as

\[ \delta_{t+1} \leq (1 - a)\delta_t + b + c\delta_t^{1/2} \leq (1 - a/2)\delta_t + b + c^2/2a. \]

The result then follows from Lemma C.2. \( \square \)

### C.1 Stability analysis

We first show that the first part of Assumption 4.1 can be implied by the diagonal strict concavity condition defined in Rosen (1965). Indeed, since the set \( \mathcal{U} \) is compact and since the vector function \( G(\cdot) \) is bounded in \( \mathcal{U} \), to verify that the first part of Assumption 4.1 holds, it suffices to show that \((\mu - \mu')^T(G(\mu') - G(\mu)) > 0\) for all \( \mu, \mu' \in \mathcal{U} \). The latter is equivalent to the diagonal strict concavity assumption of Rosen (1965). Furthermore, denote by \( J_G : \mathbb{R}_+^K \to \mathbb{R}^{K \times K} \) the Jacobian matrix of the vector function \( G \), that is, \( J_G(\mu) = \left( \frac{\partial G_k}{\partial \mu_i}(\mu) \right)_{k,i} \). Then, by Theorem 6 of Rosen (1965), it is sufficient to show that the symmetric matrix \( J_G(\mu) + J_G^T(\mu) \) is negative definite.
We next provide an analytical expressions for $G(\mu)$ for two advertisers with valuations that are independently uniformly distributed and exponentially distributed to demonstrate numerically that the latter condition holds in this case.

**Example C.4. (Two bidders with uniform valuations)** Assume $K = 2, \mathcal{U} = [0,1]^2$, and $v_{k,t} \sim U[0,1]$, i.i.d. for all $k \in \{1,2\}$ and $t \in \{1,\ldots,T\}$. One obtains:

$$G_1(\mu) = \int_0^1 \int_0^{\min\{1+\mu_2,\mu_1 \}} \frac{x_2}{1+\mu_2} dx_2 dx_1 = \frac{1}{6(1+\mu_1)^2} \left( \frac{1}{2(1+\mu_2)} - \frac{1+\mu_1}{3(1+\mu_2)^2} \right).$$

Therefore:

$$G(\mu) = 1 \{ \mu_2 \leq \mu_1 \} \left[ \frac{1+\mu_2}{6(1+\mu_1)^2} \right] + 1 \{ \mu_1 < \mu_2 \} \left[ \frac{1}{2(1+\mu_2)} - \frac{1+\mu_1}{3(1+\mu_2)^2} \right].$$

Following this expression, one may validate the first part of Assumption 4.1 by creating a grid of $\mu_{i,j} \in \mathcal{U}$, and for a given grid calculate the maximal monotonicity constant $\lambda$ for which the condition holds. For example, for a $10 \times 10$ grid (with $\|\mu_{i,j} - \mu_{i,j+1}\| = \|\mu_{i,j} - \mu_{i+1,j}\|$ is 0.1 for all $i = 0, 1, \ldots, 9$ and $j = 0, 1, \ldots, 9$) the latter condition holds with $\lambda = 0.047$. Similarly, for a $20 \times 20$ grid (with $\|\mu_{i,j} - \mu_{i,j+1}\| = \|\mu_{i,j} - \mu_{i+1,j}\|$ is 0.05 for all $i = 0, 1, \ldots, 19$ and $j = 0, 1, \ldots, 19$) the latter condition holds with $\lambda = 0.045$.

**Example C.5. (Two bidders with exponential valuations)** Assume $K = 2, \mathcal{U} = [0,1]^2$, and $v_{k,t} \sim \exp(1)$, i.i.d. for all $k \in \{1,2\}$ and $t \in \{1,\ldots,T\}$. One obtains:

$$G_1(\mu) = \int_0^\infty \int_0^{\min\{1+\mu_2,\mu_1 \}} \frac{x_2 e^{-x_2} e^{-x_1}}{1+\mu_2} dx_2 dx_1 = \frac{1+\mu_2}{(2+\mu_2+\mu_1)^2}.$$ 

Following same lines as in Example C.5 one may validate the first part of Assumption 4.1. For example, for a $10 \times 10$ grid (with $\|\mu_{i,j} - \mu_{i,j+1}\| = \|\mu_{i,j} - \mu_{i+1,j}\|$ is 0.1 for all $i = 0, 1, \ldots, 9$ and $j = 0, 1, \ldots, 9$) the latter condition holds with $\lambda = 0.066$. Similarly, for a $20 \times 20$ grid (with $\|\mu_{i,j} - \mu_{i,j+1}\| = \|\mu_{i,j} - \mu_{i+1,j}\|$ is 0.05 for all $i = 0, 1, \ldots, 19$ and $j = 0, 1, \ldots, 19$) the latter condition holds with $\lambda = 0.065$.

The following result expands and complements the above examples by showing that when the number of players is large, the stability assumption holds in symmetric settings in which every
advertiser participates in each auction with the same probability and all advertisers have the same
distribution of values. In particular, the monotonicity constant $\lambda$ of the expenditure function $G$ is
shown to be independent of the number of players.

**Proposition C.6. (Stability in symmetric settings)** Consider a symmetric setting in which
advertisers participate in each auction with probability $\alpha_{i,m} = 1/M$, advertisers values are drawn
from a continuous density $f(\cdot)$, and the ratio of number of auctions to number of players $\kappa := K/M$
is fixed. Then, there exist $K \in \mathbb{N}$ and $\lambda > 0$ such that for all $K \geq K$ there exists a set $U \subset \mathbb{R}_+^K$
with $0 \in U$ such that $G$ is $\lambda$-strongly monotone over $U$.

**Proof.** We prove the result in three steps. First, we argue that a sufficient condition for $\lambda$-strong
monotonicity of $G$ over $U$ is that $\lambda$ is a lower bound on the minimum eigenvalue of the symmetric
part of the Jacobian of $-G$ over $U$. Second, we characterize the Jacobian of the vector function $G$
in the general case. Third, we show that in the symmetric case $G$ is strongly monotone around
$\mu = 0$ by bounding the minimum eigenvalue of the symmetric part of the Jacobian of $-G$ at zero.

**Step 1.** We denote by $J_G : \mathbb{R}_+^K \to \mathbb{R}^{K \times K}$ the Jacobian matrix of the vector function $G$, that is,$J_G(\mu) = \left( \frac{\partial G_k}{\partial \mu_i}(\mu) \right)_{k,i}$. Let $\lambda$ be a lower bound on the minimum eigenvalue of the symmetric part
of $-J_G(\mu)$ over $U$. That is, $\lambda$ satisfies
$$\lambda \leq \min_{\|x\|_2 = 1} -\frac{1}{2} x^T \left( J_G(\mu) + J_G(\mu)^T \right) x$$
for all $\mu \in U$. Lemma B.1, item (iii) shows that $G(\mu)$ is differentiable in $\mu$. Thus, by the mean
value theorem there exists some $\xi \in \mathbb{R}^K$ in the segment between $\mu$ and $\mu'$ such that for all $\mu, \mu' \in U$
one has $G(\mu') = G(\mu) + J_G(\xi)(\mu' - \mu)$. Therefore,
$$(\mu - \mu')^T (G(\mu') - G(\mu)) = -\frac{1}{2} (\mu - \mu')^T \left( J_G(\xi) + J_G(\xi)^T \right) (\mu - \mu') \geq \lambda \| \mu - \mu' \|_2^2 ,$$
since $\xi \in U$. Hence, a sufficient condition for $\lambda$-strong monotonicity of $G$ over $U$ is that $\lambda$ is a lower
bound on the minimum eigenvalue of the symmetric part of the Jacobian of $-G$ over $U$.

**Step 2.** Some definitions are in order. Let $H_{k,i}(x; \mu_{-k,i}) = \prod_{j \neq i,k} \left( 1 - a_{k,j} F_j((1 + \mu_j)x) \right)$ and
$\ell_k(x) = x f_k(x) \mathbf{1 \{ x \leq \bar{v} \}}$. Additionally, we denote
$$\gamma_{k,i} = \int_0^\bar{v} \ell_k \left( (1 + \mu_k)x \right) \ell_i \left( (1 + \mu_i)x \right) H_{k,i}(x; \mu_{-k,i}) \, dx ,$$
$$\omega_{k,i} = \int_0^\bar{v} F_k \left( (1 + \mu_k)x \right) \ell_i \left( (1 + \mu_i)x \right) H_{k,i}(x; \mu_{-k,i}) \, dx .$$

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We first determine the partial derivatives of the cumulative distribution function $H_k$. One has

$$
\frac{\partial H_k}{\partial x}(x; \mu_k) = \sum_{i \neq k} a_{k,i} (1 + \mu_i) f_i ((1 + \mu_i)x) 1 \{(1 + \mu_i)x \leq \bar{v}\} H_{k,i}(x; \mu_{k,i}) ,
$$

$$
\frac{\partial H_k}{\partial \mu_i}(x; \mu_k) = a_{k,i} x f_i ((1 + \mu_i)x) 1 \{(1 + \mu_i)x \leq \bar{v}\} H_{k,i}(x; \mu_{k,i}) .
$$

Using equation (B-1) one obtains that

$$
\frac{\partial G_k}{\partial \mu_k}(\mu) = - \int_0^\bar{v} x^2 f_k ((1 + \mu_k)x) 1 \{(1 + \mu_k)x \leq \bar{v}\} dH_k(x; \mu_k)
$$

$$
= - \sum_{i \neq k} \frac{a_{k,i}}{1 + \mu_k} \int_0^\bar{v} \ell_k ((1 + \mu_k)x) \ell_i ((1 + \mu_i)x) H_{k,i}(x; \mu_{k,i}) dx
$$

$$
= - \sum_{i \neq k} \frac{a_{k,i}}{1 + \mu_k} \gamma_{k,i} .
$$

Using equation (B-2) we have

$$
\frac{\partial G_k}{\partial \mu_i}(\mu) = \int_0^\bar{v} ((1 + \mu_k)x f_k ((1 + \mu_k)x) - \bar{F}_k ((1 + \mu_k)x)) \frac{\partial H_k}{\partial \mu_i}(x; \mu_k) dx
$$

$$
= \frac{a_{k,i}}{1 + \mu_i} \int_0^\bar{v} \ell_k ((1 + \mu_k)x) \ell_i ((1 + \mu_i)x) H_{k,i}(x; \mu_{k,i}) dx
$$

$$
- \frac{a_{k,i}}{1 + \mu_i} \int_0^\bar{v} \bar{F}_k ((1 + \mu_k)x) \ell_i ((1 + \mu_i)x) H_{k,i}(x; \mu_{k,i}) dx
$$

$$
= \frac{a_{k,i}}{1 + \mu_i} (\gamma_{k,i} - \omega_{k,i}) .
$$

**Step 3.** Consider a symmetric setting in which each advertiser participates in each auction with the same probability and all advertisers have the same distribution of values. By symmetry we obtain that $a_{k,i} = 1/M$, because $\alpha_{i,m} = 1/M$ for all advertiser $i$ and auction $m$. Evaluating at $\mu = 0$ we obtain that $\gamma_{k,i} = \gamma = \frac{\gamma}{M} \int_0^\bar{v} \ell(x)^2 (1 - \bar{F}(x)/M)^{K-2} dx$ for all $k \neq i$ and $\omega_{k,i} = \omega = \frac{\gamma}{M} \int_0^\bar{v} \bar{F}(x) \ell(x)(1 - \bar{F}(x)/M)^{K-2} dx$ for all $k \neq i$. Therefore, $\frac{\partial G_k}{\partial \mu_k}(0) = - \frac{K-1}{M} \gamma$ and $\frac{\partial G_k}{\partial \mu_i}(0) = \frac{1}{M} (\gamma - \omega)$. The eigenvalues of $- (J_G(0) + J_G(0)^T) / 2$ are $\nu_1 = \frac{K-1}{M} \omega$ with multiplicity 1 and $\nu_2 = \frac{K\gamma - \omega}{M}$ with multiplicity $K - 1$.

Assume further that the expected number of players per auction $\kappa := K/M$ is fixed, which implies that the number of auctions is proportional to the number of players. Because $(1 - \bar{F}(x)/M)^{K-2}$ converges to $e^{-K\bar{F}(x)}$ as $K \to \infty$ and the integrands are bounded, Dominated Convergence Theorem implies that
\[
\lim_{K \to \infty} \nu_1 = \kappa \int_0^\infty \bar{F}(x)\ell(x)e^{-\kappa \bar{F}(x)}\,dx > 0,
\]
and
\[
\lim_{K \to \infty} \nu_2 = \kappa \int_0^\infty \ell(x)^2e^{-\kappa \bar{F}(x)}\,dx > 0.
\]
Hence, there exist \( K \in \mathbb{N} \) and \( \lambda' > 0 \) such that for all \( K \geq K \) the minimum eigenvalue value of 
\(- (J_G(0) + J_G(0)^T) / 2\) is at least \( \lambda' > 0 \). Because densities are continuous, one obtains that \( J_G(\mu) \) is continuous in \( \mu \). Since the eigenvalues of a matrix are continuous functions of its entries, we conclude that there exists \( \lambda \in (0, \lambda'] \) such that for each \( K \geq K \) there exists a set \( \mathcal{U} \subset \mathbb{R}^K \) with \( 0 \in \mathcal{U} \) such that \( G \) is \( \lambda \)-strongly monotone over \( \mathcal{U} \).

References


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