Abstract

We formulate and analyze a stylized dynamic model of a price-taking firm that manages production and capacity, uses only internal financing, and faces stochastic market environments. The firm has two operationally independent production facilities, each of which makes two products, and a cash reserve which finances all operations and dividend issuance. Each period the firm chooses the amount of dividend to issue and at each facility it chooses production quantities and amounts of capacity to augment or divest. Relevant market data are exogenous and evolve stochastically. We completely characterize the optimal policy and the endogenous values of the capacities and cash reserve, and show that they invite a real-option interpretation. We find that internal financing creates a spillover between the endogenous values of the two operationally independent facilities and we specify how this leads to interdependence of their optimal policies. We show that an “invest/stay put/divest” (ISD) policy remains optimal for partially irreversible investments, but internal financing changes the ISD thresholds. If the exogenous data are intertemporally independent, an internally financed firm is less likely to issue dividends or to expand capacity than if it were in a perfect capital market. As the market becomes more volatile, the endogenous values of capacities and cash increase, and the firm becomes more reluctant to issue dividends.
1 Introduction

This research is motivated by two observations. First, many businesses have multiple production facilities, make a range of products, and face stochastic market environments. Thus, management teams dynamically adjust productive capacity and product mix in response to market changes. Second, firms face financial frictions in external capital markets and, thus, they may prefer to finance their capacity investments using internal funds, according to pecking-order theory (Myers and Majluf 1984) and as empirically documented (Lamont 1997, Shin and Park 1999).

We build a stylized model to shed light on the joint effect of these empirical regularities. We ask, how should a firm that uses internal financing manage its production and capacity-portfolio in a stochastic market? How would the use of internal financing affect the optimal policy? In other words, compared with one in a perfect capital market, how should a firm that uses internal financing manage its production and capacity portfolio differently? Since businesses commonly make completely different products for different markets at different facilities, does the consequent operational independence across different facilities permit the firm to make optimal production plans in a decentralized manner?

To answer these research questions, we formulate a Markov decision process (MDP) model of a price-taking firm with two production facilities, each of which makes two products. Capacities at the facilities depreciate due both to the passage of time and to the usage of capacities for production. To highlight the effects of internal financing, we assume that the firm has a cash reserve which finances all of its operations and dividend issuance (see e.g. Lamont 1997, Shin and Park 1999 for empirical evidence of internal capital markets). Each period, the firm chooses the amount of each product to produce, the amount of each facility’s capacity to divest, the amount of cash to invest in capacity expansion at each facility, and the amount of dividend to issue to maximize the expected value of the present value of its dividends. The product profit margins, yields on capacity investments, and market prices for divested capacities are stochastic and evolve over time.

We solve the MDP analytically and show that its value function is linear in the cash balance and the facilities’ capacities. This result implies that the market value of the firm is homogeneous of degree one with respect to its cash and capacity levels. Furthermore, despite their complexity, the value function and optimal policy admit an intuitive real option interpretation. At each production
facility, each unit of capacity can be viewed as a real option that enables the production of two products and divestment. Each unit of cash can be viewed as a real option that enables investment in two facilities, dividend issuance, and liquidity buildup. Given the dynamic setting, exercising a real option does not simply generate capacity or cash, but generates real options in capacity and cash which can be exercised further. Thus, the firm dynamically manages three types of mutually embedded real options in two facilities and the cash reserve. We quantify the endogenous values of the real options, namely the endogenous values of capacity and cash, and show that the optimal policy exercises these options optimally.

The analysis reveals the following effects of internal financing. First, the facilities have interdependent endogenous values, although they may produce completely different products and serve unrelated markets and are thus operationally independent. The intuition is that real options in one facility can generate real options in cash, which can generate real options in the other facility when investment occurs. Thus, under internal financing, the value of each facility spills over and affects the value of the other through the cash reserve. In perfect capital markets, this value spillover vanishes and endogenous values of operationally independent facilities are unrelated.

Due to the spillover, the policy at a facility affects not only its own endogenous value, but also the other facility’s endogenous value and subsequently its operations. Therefore, value spillover leads to interdependence of optimal policies at different facilities. Lamont (1997) provides empirical evidence of interdependence among investment decisions. We analytically characterize this interdependence among all decisions and show how it reaches deeper levels of operations such as the product mix decisions at different facilities.

Interdependence of facility policies necessitates coordination within the firm. While coordination is generally difficult for multi-division companies, our result implies that headquarters can achieve it in a decentralized way. If headquarters provides each facility’s manager the endogenous values of the firm’s cash and capacity for payoff evaluation, a locally optimal decision is optimal for the firm as a whole. Such use of endogenous values is consistent with the use of internal prices (e.g., transfer prices in Eccles 1983) as decentralized integrative devices in multi-division firms.

Besides inducing interdependence among facilities, internal financing has the following policy implications. First, it affects the capacity management policy for partially irreversible investments (i.e., the capacity investment is costly to reverse). In a perfect capital market, the optimal capacity
management policy has an “invest/stay put/divest” (ISD) structure (this literature is discussed in §2). We show that, while the same structure is optimal in the presence of financial frictions, the ISD thresholds are determined by the endogenous values of capacity and internal cash, which are higher than their counterparts in perfect capital markets.

Second, when the exogenous data in the model are intertemporally independent and identically distributed (i.i.d.), the firm is less likely to issue a dividend or invest in capacity expansion than if it could access a perfect capital market. This offers a possible explanation for the commonly observed “cash hoarding” phenomenon (Bates et al. 2009) and is consistent with the empirical findings in Almeida et al. (2004) and Han and Qiu (2007).

Finally, as the exogenous environment becomes riskier, an internally financed firm is more reluctant to issue dividends. The reason is that endogenous values of capacity increase with greater exogenous risk, and that makes investments more valuable. Thus, the firm is more likely to retain internal cash for financing than to issue it as dividends.

The paper has other results. (i) When the exogenous data are i.i.d., the optimal policy has a multi-dimensional threshold structure, and (ii) production-induced depreciation leads to a “coupling” between production and divestment: the condition under which production is optimal depends on the divestment decision.

In the remainder of the paper, §2 reviews the relevant literature and §3 presents and discusses the model. Section 4 solves the model, presents a real option interpretation of the results, and discusses the effects of internal financing. Section 5 builds upon the results in §4 and further examines the structure of the optimal policy. Section 6 studies policy implications of riskier or better market environments, and §7 discusses extensions of the model. The summary of the paper in section 8 includes testable hypotheses that are suggested by the paper’s results.

2 Related Literature

A large literature formulates stochastic models for capacity management; see Van Mieghem (2003) for an extensive review. However, few works address the dynamic management of multiple facilities and, of those, only Eberly and Van Mieghem (1997) uses a stylized model to derive qualitative insights, which makes it particularly relevant here. That paper and ours differ in several
ways. First, operational details are absent in Eberly and Van Mieghem (1997) which considers only capacity adjustment. In contrast, this paper has operational features such as product mix decisions and capacity depreciation that permit us to explore in depth the policy implications of internal financing, and to delineate the linkage between optimal production and divestment decisions. Second, Eberly and Van Mieghem (1997) considers a firm free of financial frictions, whereas we incorporate the effects of financial frictions in the form of internal financing.

A strand of literature in operations management and economics considers capacity investment under partial irreversibility. Most of it studies a single-dimensional capacity and proves the optimality of an “invest/stay put/divest” (ISD) policy under the implicit assumption of a perfect capital market (see e.g. Abel and Eberly 1996, Sobel 1970). Eberly and Van Mieghem (1997) extends the result to a model with multi-dimensional capacity. This paper shows that an ISD policy for multi-dimensional capacity remains optimal for a firm in an imperfect capital market.

In the operations management literature, most studies of stochastic capacity investment with financial decisions are in one-period settings. In recent examples, including Boyabatlı and Toktay (2011), Chod and Zhou (2014) and Boyabatlı et al. (2015), the primary concern is the effect of financial frictions on the choice between flexible and dedicated capacity. The goal of this paper is quite different. We do not endogenize the choice of the financing vehicle or model the financial frictions behind internal financing. Rather, the use of internal financing is given, and we study how it affects the optimal policy.

Internal financing and internal capital markets have been studied both empirically (Lamont 1997, Carpenter and Petersen 2002) and theoretically in finance and economics. The theoretical strand of this literature typically considers the effects of internal capital markets on project investments using agency models. For example, Stein (1997) studies the economic value created by internal capital markets through centralization when project managers and headquarters managers derive private benefits. Inderst and Müller (2003) examines the effects of internal fund centralization on the agency problem between equity and debt holders. This paper is quite different because it is concerned with the operational implications of internal financing.

There is a large finance literature on dynamic coordination of a firm’s financial and investment decisions. These papers are usually richly detailed on the finance side but much less so on the operations side. Many include depreciation due to the passage of time, but capacity is for a single
resource and production decisions are generally absent. In Mauer and Triantis (1994), an exception, production is a decision variable that can be either “on” or “off” with fixed costs to switch from one status to the other, but the capacity investment decision is one-shot. In Cortazar et al. (1998), another exception, the decision maker chooses instantaneous production and investment rates for a single resource subject to an exogenous upper bound on cumulative investment. Our paper has an operational focus and does not model the dynamic adjustment of the firm’s capital structure.

Some valuable insights in this paper stem from a real option perspective on the valuation and management of firm assets. This viewpoint connects our work with the large literature that applies real option theory to firm decisions (Dixit and Pindyck 1994). Studies that are particularly relevant have non-lumpy investment in capacity expansion. Pindyck (1988) is an early example devoid of financial considerations. Other works (Boyle and Guthrie 2003, Sundaresan et al. 2015, Asvanunt et al. 2010) integrate financing and investment decisions in a real option framework, but consider only lumpy investments, with the number and size of investments exogenously given. This paper, in contrast, endogenizes the number and size of capacity investments and, thus, deepens the understanding of the real option nature of assets and the associated optimal decisions.

3 Model

Consider a discrete-time multi-period model of a firm that is risk-neutral and relies solely on internal funds for financing over the planning horizon. It faces stochastic external data: product profits, investment yields, and market prices of divested capacity. In each period it decides how much to produce, how much existing capacity to divest, how much to invest in capacity expansion, and how much of a dividend to issue. The model is formulated in §3.1, its optimal policy and value function are defined in §3.2, and important modeling assumptions are discussed in §3.3.

3.1 Model notation and assumptions

The firm has three assets: the cash reserve and two production facilities. Both facilities have flexible production capacity and can make two products. Henceforth, \( j \in \{1, 2\} \) and \( k \in \{1, 2\} \) denote the facility and product, respectively; “product \((j, k)\)” signifies “product \(k\) produced at facility \(j\)”, and “capacity \(j\)” is shorthand for “capacity at facility \(j\)”. The paper makes no assumption
about whether the products made by facilities 1 and 2 are identical. The two facilities may make completely different products and serve distinct markets.

At the beginning of period $t = 1, 2, \ldots$, the firm observes internal data which are capacity $j$, denoted $K_{jt}$ ($j = 1, 2$), and the balance of the cash reserve, denoted $W_t$. Also, it observes external data which are the profit margin $p_{jkt}$ per unit of product $(j, k)$, the market price $r_{jt}$ per unit of divested capacity $j$, and $y_{jt}$ which is the amount of capacity $j$ installed per unit of cash invested to expand capacity $j$. Henceforth, $y_{jt}$ and $r_{jt}$ are called “investment yield” and “divestment price”, respectively, and they satisfy $r_{jt}, y_{jt} > 0$.

After observing the asset levels and exogenous data, the firm chooses the amount of product $(j, k)$ to produce, $q_{jkt}$, which is denominated in units of capacity $j$ $(j, k = 1, 2)$; the fraction of capacity $j$ to divest, $d_{jt}$ (mnemonic for “divestment”); the amount of cash to invest in expanding capacity $j$, $i_{jt}$ (mnemonic for “investment”); and the amount of cash to issue as dividend, $x_t$. The chronology is as follows: production, investment, and dividend issuance occur at the beginning of the period; divestment occurs at the end of the period; all earnings from production and divestment are collected at the end of the period; capacity installed from investment is ready to use at the beginning of the next period.

The first constraint on decisions $(q_{jkt}, d_{jt}, i_{jt}, x_t : j, k = 1, 2)$ is that the total production quantity at facility $j$ $(j = 1, 2)$ cannot exceed the capacity available:

$$\sum_{k=1}^{2} q_{jkt} \leq K_{jt}. \quad (1)$$

Second, capacity depreciates during period $t$ after production occurs but before divestment; $(q_{jkt} : j, k = 1, 2)$ is constrained to guarantee non-negative post-depreciation capacities. The model accounts for two forms of depreciation: natural depreciation, in which case the capacity diminishes simply due to the passage of time; and production-induced depreciation, in which case the capacity diminishes due to its use for production. Given production quantities $q_{jkt} (j, k = 1, 2)$, capacity $K_{jt}$ depreciates to $\theta_j K_{jt} - \sum_{k=1}^{2} \lambda_{jk} q_{jkt}$ at the end of period $t$. Here, $\theta_j \in (0, 1]$ and $\lambda_{jk} \in [0, 1]$ reflect natural depreciation and production-induced depreciation, respectively. When $\theta_j = 1$ and $\lambda_{jk} = 0$,
the capacity does not depreciate at all. To guarantee nonnegative post-depreciation capacity,

$$\sum_{k=1}^{2} \lambda_{jk} q_{jkt} \leq \theta_j K_{jt} \quad (j = 1, 2).$$

(2)

Natural depreciation is typically slight, so $\lambda_{jk} \leq \theta_j$ $(j, k = 1, 2)$ is assumed for simplicity until §7 where the effects of $\lambda_{j1} > \theta_j > \lambda_{j2}$ for $j = 1, 2$ are considered.

Third, the level of the cash reserve must be adequate to finance production, investment, and dividend issuance. Assume that the production cost is paid at the end of the period after the revenue is collected. If the profit margins are nonnegative, the cash reserve finances only the investment and dividend decisions, and must satisfy $i_{1t} + i_{2t} + x_t \leq W_t$. If the product margins are negative, the cash reserve must also cover the net loss from production. Thus, the financing constraint is

$$i_{1t} + i_{2t} + x_t + \sum_{j=1}^{2} \sum_{k=1}^{2} (p_{jkt})^+ q_{jkt} \leq W_t,$$

(3)

where $(a)^+ = \max\{a, 0\}$ for any scalar $a$. If inequality (3) is not binding, namely the cash is not used up in period $t$, then the left-over amount is carried into the next period.

Finally, divestment decision $d_{jt}$, which is the fraction of capacity $j$ to divest at the end of period $t$, satisfies

$$0 \leq d_{jt} \leq 1.$$

(4)

Divestment occurs at the end of the period after capacity depreciates, so $d_j$ is the fraction of post-depreciation capacity that is divested. Since the amount of capacity installed by investing $i_{jt}$ in facility $j$ is $y_{jt} i_{jt}$ (given investment yield $y_{j}$), the amount of capacity $j$ available at the beginning of period $t + 1$ is

$$K_{j,t+1} = y_{jt} i_{jt} + (1 - d_{jt})(\theta_j K_{jt} - \lambda_{j1} q_{j1t} - \lambda_{j2} q_{j2t}).$$

(5)

During period $t$, the firm spends $i_{jt}$ $(j = 1, 2)$ in capacity investment, and $x_t$ in dividend issuance. It also collects $p_{jkt} q_{jkt}$ $(j, k = 1, 2)$ from producing $q_{jkt}$ units of product $(j, k)$ with profit margin $p_{jkt}$, and collects $r_j d_j (\theta_j K_{jt} - \lambda_{j1} q_{j1t} - \lambda_{j2} q_{j2t})$ from divesting capacity $j$ with divestment
price $r_{jt}$. Thus, the cash level at the beginning of period $t + 1$ is

$$W_{t+1} = W_t - x_t - \sum_{j=1}^{2} i_{jt} + \sum_{j=1}^{2} r_{jt}d_{jt} \left( \theta_j K_{jt} - \lambda_j q_{jt1} - \lambda_j q_{j2t} \right) + \sum_{j=1}^{2} \sum_{k=1}^{2} p_{jkt} q_{jkt}. \quad (6)$$

For simplicity, (6) ignores interest on the cash reserve, but all analysis and insights in the paper remain valid when interest is included.

For ease of exposition, henceforth let $e_t$ denote the eight-dimensional vector $(p_{jkt}, r_{jt}, y_{jt} : j, k = 1, 2)$ of exogenous data, where $e$ is a mnemonic for “exogenous”. Assume that $e_1, e_2, \ldots$ is an exogenous Markov process with state space $\Omega \subseteq \mathbb{R}^8$. Let $\zeta_e$ denote the random vector (r.v.) that has the same distribution as $e_{t+1}$ given that $e_t = e$. The remainder of the paper uses $p_{jkt}^e, r_{jt}^e$, and $y_{jt}^e$ to emphasize the connection between $e$ and its components.

### 3.2 Optimal policy and the value function

The vectors $a_t = (q_{jkt}, i_{jt}, d_{jt}, x_t : j, k = 1, 2)$ and $s_t = (K_{1t}, K_{2t}, W_t)$ are the respective action and endogenous state in period $t$. The history up to the beginning of period $t$ is $H_t := (s_1, e_1, a_1, s_2, e_2, a_2, ..., s_{t-1}, e_{t-1}, a_{t-1}, s_t, e_t)$. A policy is a non-anticipative decision rule based on all partial histories. That is, for each $H_t$ and $t$, a policy specifies $a_t$ satisfying constraints (1)–(4).

Let $N$ denote the length of the planning horizon, where $N$ is presumed to be large. The goal of the firm is to maximize its financial value, namely, the expected present value (EPV) of the dividends over $N$ periods (see Lucas 1978; Brodie et al. 1995, p. 326; and Cochrane 2005). Let $\beta$ denote the single-period discount factor. The present value of the dividends, beginning in period $\tau$, is

$$\Pi_\tau := \sum_{t=\tau}^{N} \beta^{t-\tau} x_t. \quad (7)$$

Policy $\pi^*$ is defined to be optimal if $\mathbb{E}_{\pi^*|H_\tau}(\Pi_\tau) \geq \mathbb{E}_{\pi|H_\tau}(\Pi_\tau)$ for all partial histories $H_\tau$, all $\tau \in \{1, 2, \ldots, N\}$, and all policies $\pi$ ($\mathbb{E}$ denotes the expectation operator).

The state space of the endogenous state $s_t = (K_{1t}, K_{2t}, W_t)$ is $\mathbb{R}^3_+$ (where $\mathbb{R}^+_+ = [0, \infty)$). Define the value function $V_N(K_1, K_2, W, e)$ of the MDP as the expected value of $\Pi_1$ under an optimal
policy \( \pi^* \) as a function of the initial history \( H_1 = (K_1, K_2, W, e) \):

\[
V_N(K_1, K_2, W, e) = \mathbb{E}_{\pi^*|H_1=(K_1,K_2,W,e)}[\Pi_1] \quad (K_1, K_2, W, e) \in \mathbb{R}^3_+ \times \Omega.
\] (8)

Henceforth, we refer to \( V_N(K_1, K_2, W, e) \) as the value of the firm. The remainder of the paper solves the optimization problem

\[
V_n(K_1, K_2, W, e) = \max_{a_1, a_2, \ldots} \left\{ \mathbb{E} \left( \sum_{t=1}^{n} \beta^{t-1} x_t \right) : (1) - (6) \text{ for all } t = 1, 2, \ldots, n \right\},
\] (9)

for \( n = 1, 2, \ldots, N \), and discusses the optimal policy and the value of the firm. Proofs of all formal results are in the Electronic Companion. This model is a finite-horizon Markov decision process; for brevity, henceforth it is called the MDP.

### 3.3 Model discussion

**Self-financing assumption and exclusion of bankruptcy risk.** This paper seeks to understand how the very fact of using internal finance affects the optimal production and capacity management policy. To highlight the effect of internal financing (and to subdue other channels, such as taxes, bankruptcy, information asymmetry, and moral hazard, through which financial frictions exert influences), the model assumes that the cash reserve finances all operations in the firm, and that the firm is all-equity financed. This is the self-financing assumption.

Similarly, the exclusion of bankruptcy risk is intentional. Several features in the model ensure that the firm is not faced with bankruptcy risk. First, it lacks debt and fixed costs, so it cannot be driven into bankruptcy by an inability to make payments. Second, although the profit margin \( p_{jkt}^e \) may be negative, the following lemma implies that it is sub-optimal to produce a positive amount of output when \( p_{jkt}^e < 0 \). Therefore, under an optimal policy, the profits from production are always nonnegative.

**Lemma 1.** An optimal policy specifies \( q_{jkt} = 0 \) if \( p_{jkt}^e < 0 \) \((j, k = 1, 2)\).

Third, investment yields are non-negative and depreciation cannot lead to negative capacity, so cash and capacity levels are always nonnegative. Therefore, the modeled firm is immune from bankruptcy risks. While firms in reality may voluntarily elect to file for bankruptcy, this possibility
is beyond the scope of the paper.

In summary, the self-financing assumption and the exclusion of bankruptcy risk generate clean insights on the effects of internal financing and they preserve the analytical tractability of the model. Incorporating other financial apparatuses such as debt financing would introduce additional channels through which the operational and financial decisions interact, thus obscuring the main research focus.

**Demand and inventory.** Demand is absent from the model for analytical tractability, and it makes the model more descriptive of firms in commodity industries such as oil and natural gas where short-run “demand” is ill-defined; a firm can sell as much product as it has available without affecting the price. Similarly, the model ignores inventories to maintain analytical tractability and to highlight the interaction of dynamic capacity adjustment and production.

**Salvage value.** It follows from (9) that \( V_0(\cdot, \cdot, \cdot, \cdot) \equiv 0 \), although an arguably better model of the firm’s salvage value is \( r_1^e K_1 + r_2^e K_2 + W \). The paper employs the former (zero) salvage value instead of the latter for two reasons. First, we wish to understand a firm’s normal operations over long planning horizons, i.e., \( N \) is large, rather than to study the optimal dissolution of a firm. Therefore, the expected discounted salvage value has a negligible effect on optimal decisions early in the going. Second, if \( V_0(K_1, K_2, W, e) = r_1^e K_1 + r_2^e K_2 + W \), all of the qualitative results would remain valid but the exposition and proofs would be more cumbersome. In fact, any salvage value function that is affine in \( K_1, K_2 \), and \( W \) would preserve the main results of the paper. However, nonlinear salvage value functions would destroy important analytical properties of the MDP.

**Markov-modulated exogenous uncertainty.** The model assumes that the profit margins \( p_{jkt}^e \), investment yields \( y_{jt}^e \), and divestment prices \( r_{jt}^e \) \((j, k = 1, 2)\) form an eight-dimensional Markov process. In other words, \( p_{jkt}^e, y_{jt}^e, \) and \( r_{jt}^e \) are components of \( e_t \) with \( e_1, e_2, ..., e_N \) being an observable discrete-time Markov process. Thus, the exogenous parameters are Markov-modulated. A more general Markov-modulated assumption would permit \( p_{jkt}^e, y_{jt}^e, \) and \( r_{jt}^e \) to be general monotone functions of an exogenous Markov process. All results in the paper hold if a suitable general Markov-modulated model is used for \( (p_{jkt}^e, y_{jt}^e, r_{jt}^e : j, k = 1, 2) \).
4 Value Function, Optimal Policy, and Linkages Among Decisions

Recall that \( a = (q_{jk}, d_j, i_j, x : j, k = 1, 2) \) is the action vector of the MDP, \((K_1, K_2, W) \in \mathbb{R}^3_+ \) is the endogenous state, and \( e \in \Omega \) is the exogenous state. A dynamic program that corresponds to optimization problem (9) is, using \( V_0(\cdot, \cdot, \cdot) \equiv 0 \), for \( n = 1, 2, \ldots, N \) and \((K_1, K_2, W, e) \in \mathbb{R}^3_+ \times \Omega \),

\[
V_n(K_1, K_2, W, e) = \max_{a \in \mathbb{R}^9_+} \left\{ x + \beta \mathbb{E}[V_{n-1}(K'_1, K'_2, W', \zeta_e)] : d_1, d_2 \leq 1, \quad \sum_{k=1}^2 q_{1k} \leq K_1, \quad \sum_{k=1}^2 q_{2k} \leq K_2, \quad \sum_{k=1}^2 \lambda_{jk} q_{jk} \leq \theta_j K_j \quad (j = 1, 2), \quad i_1 + i_2 + x \leq W \right\},
\]

where \( K'_j = y^e_{ij} + (1 - d_j) \left( \theta_j K_j - \sum_{k=1}^2 \lambda_{jk} q_{jk} \right) \quad (j = 1, 2) \),

\[
W' = W - x - \sum_{j=1}^2 i_j + \sum_{j=1}^2 r_{ij}^e d_j \left( \theta_j K_j - \sum_{k=1}^2 \lambda_{jk} q_{jk} \right) + \sum_{j,k=1}^2 p_{jk}^e q_{jk},
\]

in which \( K'_1, K'_2, \) and \( W' \) are the subsequent asset levels, \( \zeta_e \) is the subsequent exogenous state given current state \( e \), and the constraints in (10a) correspond to (1)–(4). Since Lemma 1 asserts that the optimal production quantity \( q_{jk} = 0 \) if \( p_{jk}^e < 0 \), (3) simplifies to \( i_1 + i_2 + x \leq W \).

In the following subsections, §4.1 characterizes the value function and elicits economic insights regarding the endogenous values of capacity and cash. Section 4.2 specifies an optimal policy, and §4.3 presents a real-option interpretation of the results, and discusses the effects of internal financing.

4.1 Value function, value of the firm, and endogenous values of capacity and cash

For expository simplicity, the remainder of the paper no longer specifies sets \( \Omega \) and \( \{1, 2, \ldots, N\} \) when referring to \( e \) and \( n \), respectively. It should be understood that all statements involving \( e \) and \( n \) are valid for all \( e \in \Omega \) and \( n \in \{1, 2, \ldots, N\} \) unless otherwise noted.

Theorem 1. The value function is

\[
V_n(K_1, K_2, W, e) = f_{1n}(e) K_1 + f_{2n}(e) K_2 + g_n(e) W, \quad (K_1, K_2, W, e) \in \mathbb{R}^3_+ \times \Omega, \quad (11)
\]
where \( f_{1n}, f_{2n}, \) and \( g_n \) are real-valued functions on \( \Omega \) that satisfy the following recursion with \( f_{10}(\cdot) \equiv f_{20}(\cdot) \equiv g_{0}(\cdot) \equiv 0 \) and \( j \in \{1, 2\} \):

\[
  f_{jn}(e) = \max \left\{ B_{jn1}(e), B_{jn2}(e), B_{jn3}(e), B_{jn4}(e), B_{jn5}(e), B_{jn6}(e) \right\}, \\
  g_n(e) = \max \left\{ \beta \mathbb{E}[g_{n-1}(\zeta_e)], 1, \beta y_1^e \mathbb{E}[f_{1,n-1}(\zeta_e)], \beta y_2^e \mathbb{E}[f_{2,n-1}(\zeta_e)] \right\};
\]

in which \( B_{jn}(l = 1, 2, \ldots, 6) \) are defined as follows:

\[
  B_{jn1}(e) = \beta \theta_j \mathbb{E}[f_{j,n-1}(\zeta_e)], \\
  B_{jn2}(e) = \beta \left[ (\theta_j - \lambda_{j1}) \mathbb{E}[f_{j,n-1}(\zeta_e)] + p_{j1}^e \mathbb{E}[g_{n-1}(\zeta_e)] \right], \\
  B_{jn3}(e) = \beta \left[ (\theta_j - \lambda_{j2}) \mathbb{E}[f_{j,n-1}(\zeta_e)] + p_{j2}^e \mathbb{E}[g_{n-1}(\zeta_e)] \right], \\
  B_{jn4}(e) = \beta \theta_j r_j^e \mathbb{E}[g_{n-1}(\zeta_e)], \\
  B_{jn5}(e) = \beta \left[ p_{j1}^e + (\theta_j - \lambda_{j1}) r_j^e \right] \mathbb{E}[g_{n-1}(\zeta_e)], \\
  B_{jn6}(e) = \beta \left[ p_{j2}^e + (\theta_j - \lambda_{j2}) r_j^e \right] \mathbb{E}[g_{n-1}(\zeta_e)].
\]

From (11), \( V_n(\cdot, \cdot, \cdot, e) \) depends linearly on state variables \( K_1, K_2, \) and \( W \). Thus, Theorem 1 implies that the value of the firm is the sum of the values of its separately-evaluated assets: the capacities of production facilities 1 and 2, and the cash reserve. Furthermore, the marginal values of the capacities and cash reserve (\( f_{1n}(e), f_{2n}(e), \) and \( g_n(e) \)) are the same as the average values. Hereafter, \( f_{jn}(e) \) and \( g_n(e) \) are called unit values of capacity \( j \) (\( j = 1, 2 \)) and cash.

From (12) and (13), the unit values depend functionally solely on exogenous state \( e \) (and on the length of the remaining horizon). Recall that \( e \) is an eight-dimensional vector with its elements being product profit margins, investment yields, and divestment prices. Therefore, the unit values change when any of these elements change and, consequently, the value of the firm as a whole changes. This reflects the intuition that the firm’s value depends on the exogenous state of the wider economy.

Equations (12) and (13) yield important economic insights on the dependence of the unit values on the exogenous state; that is the focus of the remainder of this subsection. From (12), the unit value of capacity is the maximum of six terms. The first maximand, \( B_{jn1}(e) = \beta \theta_j \mathbb{E}[f_{j,n-1}(\zeta_e)] \), has the following interpretation. If one unit of capacity \( j \) is not used for production and is not divested
in the current period, then it depreciates to $\theta_j$ units next period, when each unit of capacity $j$ is worth $f_{j,n-1}(\zeta_e)$. Thus, using discount factor $\beta$, $B_{jn1}$ is the expected discounted payoff from one unit of capacity $j$ if it is used neither for production nor divestment in the current period, with $n$ periods remaining in the planning horizon. In other words, $B_{jn1}$ is the expected payoff if the production-divestment decision for a unit of capacity $j$ is $(q_{j1}, q_{j2}, d_j) = (0, 0, 0)$.

Similarly, each of $B_{jn2}(e), ..., B_{jn6}(e)$ can be interpreted as the expected discounted payoff of a production-divestment decision $(q_{j1}, q_{j2}, d_j)$ for a unit of capacity $j$ ($j = 1, 2$), with $n$ periods remaining in the planning horizon. Table 1 lists the six maximands and their corresponding production-divestment decisions for a unit of capacity $j$, i.e., $K_j = 1$.

**Table 1:** $f_{jn}(e)$ maximands and the corresponding production-divestment decisions $(q_{j1}, q_{j2}, d_j)$ for a unit of capacity $j$: $K_j = 1$ ($j = 1, 2$)

<table>
<thead>
<tr>
<th>Maximand</th>
<th>$(q_{j1}, q_{j2}, d_j)$ whose exp. disc. * payoff is the maximand</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{jn1}(e) = \beta \theta_j E[f_{j,n-1}(\zeta_e)]$</td>
<td>(0, 0, 0)</td>
</tr>
<tr>
<td>$B_{jn2}(e) = \beta [(\theta_j - \lambda_{j1}) E[f_{j,n-1}(\zeta_e)] + p_{j1} E[g_{n-1}(\zeta_e)]]$</td>
<td>($K_j = 1, 0, 0$)</td>
</tr>
<tr>
<td>$B_{jn3}(e) = \beta \lambda_{j2}^2 E[g_{n-1}(\zeta_e)] + \beta (\theta_j - \lambda_{j2}) E[f_{j,n-1}(\zeta_e)]$</td>
<td>(0, $K_j = 1, 0$)</td>
</tr>
<tr>
<td>$B_{jn4}(e) = \beta \theta_j r_{j}^e E[g_{n-1}(\zeta_e)]$</td>
<td>(0, 0, 1)</td>
</tr>
<tr>
<td>$B_{jn5}(e) = \beta [p_{j1} + (\theta_j - \lambda_{j1}) r_{j1}^e E[g_{n-1}(\zeta_e)]]$</td>
<td>($K_j = 1, 0, 1$)</td>
</tr>
<tr>
<td>$B_{jn6}(e) = \beta [p_{j2} + (\theta_j - \lambda_{j2}) r_{j2}^e E[g_{n-1}(\zeta_e)]]$</td>
<td>(0, $K_j = 1, 1$)</td>
</tr>
</tbody>
</table>

*: “exp. disc.” stands for “expected discounted”.

Now consider the six triplets of production-divestment decisions in Table 1. Recall that the production-divestment constraints at facility $j$ are $q_{j1}, q_{j2} \geq 0$, $q_{j1} + q_{j2} \leq K_j$, $\lambda_{j1} q_{j1} + \lambda_{j2} q_{j2} \leq \theta_j K_j$ ($\lambda_{j1}, \lambda_{j2} \leq \theta_j$), and $0 \leq d_j \leq 1$ (see (1), (2), and (4)). When $K_j = 1$, these constraints yield a polyhedron with six extreme points which are precisely the six $(q_{j1}, q_{j2}, d_j)$ triplets in Table 1. Thus, henceforth the triplets in Table 1 are termed extremal decisions. Equation (12) implies that the (endogenous) unit value of capacity $j$ is the highest expected discounted payoff that can be generated by an extremal production-divestment decision for a unit of capacity $j$.

Similarly, each of the four maximands in (13) is the expected discounted payoff of an investment-dividend triplet $(i_1, i_2, x)$ for a unit of cash, with $n$ periods remaining in the planning horizon. Table 2 summarizes this correspondence. Since $(i_1, i_2, x)$ must satisfy $i_1 + i_2 + x \leq W$ and $i_1, i_2, x \geq 0$, the four $(i_1, i_2, x)$ triplets in Table 2 comprise the set of extremal investment-dividend decisions for the cash reserve. Therefore, (13) implies that the (endogenous) unit value of cash is the highest
expected discounted payoff that can be generated by an extremal dividend-investment decision for a unit of cash.

Table 2: $g_n(e)$ maximands and the corresponding investment-dividend decisions $(i_1, i_2, x)$ for a unit of cash: $W = 1$

<table>
<thead>
<tr>
<th>Maximand</th>
<th>$(i_1, i_2, x)$ whose exp. disc. * payoff is the maximand</th>
<th>Maximand</th>
<th>$(i_1, i_2, x)$ whose exp. disc. * payoff is the maximand</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta \mathbb{E}[g_{n-1}(\zeta_e)]$</td>
<td>$(0, 0, 0)$</td>
<td>$\beta y_1^e \mathbb{E}[f_{2, n-1}(\zeta_e)]$</td>
<td>$(0, W = 1, 0)$</td>
</tr>
<tr>
<td>$\beta y_1^e \mathbb{E}[f_{1, n-1}(\zeta_e)]$</td>
<td>$(W = 1, 0, 0)$</td>
<td></td>
<td>$(0, 0, W = 1)$</td>
</tr>
</tbody>
</table>

\*: “exp. disc.” stands for “expected discounted”.

4.2 Optimal policy

This section characterizes a Markov optimal policy and uses the following notation. In state $(K_1, K_2, W, e) \in \mathcal{R}_+^2 \times \Omega$ with $n$ periods remaining in the planning horizon, $Q_{jkn}(K_1, K_2, W, e) \in \mathcal{R}_+$ $(j, k = 1, 2)$ denotes an optimal amount of product $(j, k)$ to produce, $D_{jn}(K_1, K_2, W, e) \in [0, 1]$ denotes an optimal fraction of capacity $j$ to divest at the end of the period, $I_{jn}(K_1, K_2, W, e) \in \mathcal{R}_+$ denotes an optimal amount of cash to invest in capacity $j$ expansion $(j = 1, 2)$, and $X_n(K_1, K_2, W, e) \in \mathcal{R}_+$ denotes an optimal amount of cash to issue as dividend. Define functions $\tilde{Q}_{jkn}, \tilde{D}_{jn}, \tilde{I}_{jn}$, and $\tilde{X}_n$ on $\Omega$ with values 0 or 1 according to Tables 3 and 4.

Table 3: Table of $(\tilde{Q}_{j1n}(e), \tilde{Q}_{j2n}(e), \tilde{D}_{jn}(e)), e \in \Omega$ $(j = 1, 2)$

<table>
<thead>
<tr>
<th>$(\tilde{Q}<em>{j1n}(e), \tilde{Q}</em>{j2n}(e), \tilde{D}_{jn}(e))$</th>
<th>Condition</th>
<th>$(\tilde{Q}<em>{j1n}(e), \tilde{Q}</em>{j2n}(e), \tilde{D}_{jn}(e))$</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0, 0)$</td>
<td>$f_{jn}(e) = B_{jn1}(e)$</td>
<td>$(0, 0, 1)$</td>
<td>$f_{jn}(e) = B_{jn4}(e)$</td>
</tr>
<tr>
<td>$(1, 0, 0)$</td>
<td>$f_{jn}(e) = B_{jn2}(e)$</td>
<td>$(1, 0, 1)$</td>
<td>$f_{jn}(e) = B_{jn5}(e)$</td>
</tr>
<tr>
<td>$(0, 1, 0)$</td>
<td>$f_{jn}(e) = B_{jn3}(e)$</td>
<td>$(0, 1, 1)$</td>
<td>$f_{jn}(e) = B_{jn6}(e)$</td>
</tr>
</tbody>
</table>

Resolution of ties: if $f_{jn}(e) = B_{jnl}(e) = B_{jn'l'}(e)$ for $l \neq l'$ $(l, l' \in \{1, 2, \ldots, 6\})$, $(\tilde{Q}_{j1n}(e), \tilde{Q}_{j2n}(e), \tilde{D}_{jn}(e))$ takes the value associated with the smaller $l'$. For example, if $B_{jn2}(e) = B_{jn4}(e) > B_{jn5}(e)$ for $l' \in \{1, 3, 5, 6\}$, then $(\tilde{Q}_{j1n}(e), \tilde{Q}_{j2n}(e), \tilde{D}_{jn}(e)) = (1, 0, 0)$.

Table 4: Table of $(\tilde{I}_{1n}(e), \tilde{I}_{2n}(e), \tilde{X}_n(e)), e \in \Omega$

<table>
<thead>
<tr>
<th>$(\tilde{I}<em>{1n}(e), \tilde{I}</em>{2n}(e), \tilde{X}_n(e))$</th>
<th>Condition</th>
<th>$(\tilde{I}<em>{1n}(e), \tilde{I}</em>{2n}(e), \tilde{X}_n(e))$</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0, 0)$</td>
<td>$g_n(e) = \beta \mathbb{E}[g_{n-1}(\zeta_e)]$</td>
<td>$(0, 1, 0)$</td>
<td>$g_n(e) = \beta y_1^e \mathbb{E}[f_{2, n-1}(\zeta_e)]$</td>
</tr>
<tr>
<td>$(1, 0, 0)$</td>
<td>$g_n(e) = \beta y_1^e \mathbb{E}[f_{1, n-1}(\zeta_e)]$</td>
<td>$(0, 0, 1)$</td>
<td>$g_n(e) = 1$</td>
</tr>
</tbody>
</table>

Resolution of ties: if two or more conditions in the table hold simultaneously, $(\tilde{I}_{1n}(e), \tilde{I}_{2n}(e), \tilde{X}_n(e))$ takes the value that is earlier in the sequence $(0, 0, 0), (1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$. For example, if $g_n(e) = \beta y_1^e \mathbb{E}[f_{1, n-1}(\zeta_e)] = 1$, then $(\tilde{I}_{1n}(e), \tilde{I}_{2n}(e), \tilde{X}_n(e)) = (1, 0, 0)$ instead of $(0, 0, 1)$. 

15
Theorem 2. The following policy is optimal:

\[
Q_{jk}(K_1, K_2, W, e) = K_j \times \tilde{Q}_{jk}(e), \quad D_{jn}(K_1, K_2, W, e) = \tilde{D}_{jn}(e), \quad (j, k = 1, 2) \tag{15a}
\]

\[
I_{jn}(K_1, K_2, W, e) = W \times \tilde{I}_{jn}(e), \quad X_n(K_1, K_2, W, e) = W \times \tilde{X}_n(e), \quad (j = 1, 2). \tag{15b}
\]

Linearity of the optimal policy resonates with linearity of the value function (Theorem 1). Just as the coefficients in the value function are the unit values of each unit of capacity and cash, the coefficients in (15) can be viewed as an optimal disposition of each unit of capacity and cash. Specifically, \(\tilde{Q}_{jk}(e)\) is the optimal amount of product \((j, k)\) to produce with one unit of capacity \(j\), \(\tilde{D}_{jn}(e)\) is the optimal fraction of a unit of capacity \(j\) to divest at the end of the period, and \(\tilde{I}_{jn}(e)\) and \(\tilde{X}_n(e)\) are the optimal amounts to invest and distribute as a dividend with one unit of cash. Theorem 2 implies that knowledge of \((\tilde{Q}_{jk}, \tilde{D}_{jn}, \tilde{I}_{jn}, \tilde{X}_n : j, k = 1, 2)\), which is termed the unit policy, is equivalent to knowledge of an optimal policy. The unit policy, which is more parsimonious than an optimal policy, is examined in the remainder of the paper to gain insights regarding the optimal policy.

From Table 3, \((\tilde{Q}_{j1n}, \tilde{Q}_{j2n}, \tilde{D}_{jn})\) is extremal, and each of its possible values corresponds to a maximand \(B_{jn}(e)\) in (12) \((l = 1, \ldots, 6)\) (see Table 1 and associated discussion). Thus, a comparison of Tables 3 and 1 implies that the unit production-divestment policy specifies the decision that yields the highest expected payoff among all extremal decisions for a unit of capacity \(j\).

Similarly, Tables 4 and 2 imply that the unit investment-dividend policy \((\tilde{I}_{1n}, \tilde{I}_{2n}, \tilde{X}_n)\) specifies the decision that yields the highest expected payoff among all extremal decisions for a unit of cash.

Although the unit production policy \((\tilde{Q}_{j1n}, \tilde{Q}_{j2n})\) asserts that, in a period, it is not optimal to make both products at the same facility it does not imply that the firm should install dedicated instead of flexible capacity. The optimality of making product \((j, 1)\) or \((j, 2)\) depends on the market environment \(e\), which changes from period to period. Therefore, in general, a firm with flexible capacity would be better off than one with dedicated capacity because it can adapt optimally to changing market conditions. Given the absence of demand in the model, this result implies that investing in flexible capacity is advantageous even without the commonly known demand pooling benefit. This is consistent with the finding in the single-period perfect capital market model in Van Mieghem (1998).
The “bang-bang” feature of \((\tilde{Q}_{j1n}, \tilde{Q}_{j2n})\) is due to the assumption \(\lambda_{j1}, \lambda_{j2} \leq \theta_j\), which causes the extreme points of \(\{(q_{j1}, q_{j2}) : q_{j1}, q_{j2} \geq 0, q_{j1} + q_{j2} \leq K_j, \lambda_{j1}q_{j1} + \lambda_{j2}q_{j2} \leq \theta_jK_j\}\) to be \((0, 0)\), \((0, K_j)\), and \((K_j, 0)\). As shown in §7, if \(\lambda_{j1} > \theta_j > \lambda_{j2}\), then also \(\left(\frac{\theta_j - \lambda_{j2}}{\lambda_{j1} - \lambda_{j2}}, \frac{\lambda_{j1} - \theta_j}{\lambda_{j1} - \lambda_{j2}}K_j\right)\) is an extreme point. The extremal property of the optimal policy persists and it can be optimal to produce both products in the same period.

This subsection concludes with a brief discussion of the mathematical properties of the model that give rise to Theorems 1 and 2. As a preliminary, note that the nonlinearities in dynamical equations (5) and (6) can be eliminated by re-defining the divestment decision as amounts of capacity to divest instead of fractions. This yields an equivalent model with linear dynamical equations (see the beginning of the Electronic Companion), linear rewards and linear constraints on the actions. Furthermore, the constraints on the actions have the property that each action variable is affected by at most one endogenous state variable. These features lead to the linearity of the value function and the optimal policy. Ning and Sobel (2017) examine in depth the class of decomposable affine MDPs that have this property.

4.3 A real-option perspective and effects of internal financing

A real-option perspective. The preceding interpretations of the value function and the unit policy in §§4.1 and 4.2 point to an integrated view of Theorems 1 and 2 from a real-option perspective. Take capacity 1 as an example. In any period, each unit of capacity at facility 1 is an option to make product (1,1), to make product (1,2), or to divest. The optimal exercising policy should be the action that generates the highest payoff. Theorems 1 and 2 confirm this intuition and state that, in any period, an optimal exercising policy \((\tilde{Q}_{11n}, \tilde{Q}_{12n}, \tilde{D}_{1n})\) for the real option embedded in a unit of capacity 1 generates the highest payoff. The (endogenous) value of this option, \(f_{1n}(\cdot)\), is the payoff generated by \((\tilde{Q}_{11n}, \tilde{Q}_{12n}, \tilde{D}_{1n})\).

Similarly, in any period, each unit of cash represents an option on investing in facility 1, investing in facility 2, distributing a dividend, and holding liquidity. Theorems 1 and 2 state that, in any period, an optimal exercising policy \((\tilde{I}_{1n}, \tilde{I}_{2n}, \tilde{X}_n)\) for the real option embedded in a unit of cash generates the highest payoff. The (endogenous) value of this option, \(g_n(\cdot)\), is the payoff generated by \((\tilde{I}_{1n}, \tilde{I}_{2n}, \tilde{X}_n)\).

A notable feature of the real options in capacity and cash is that their exercising payoffs, \(f_{jn}(\cdot)\)
and $g_n(\cdot)$, are real options themselves. This reflects the dynamic nature of the model. In a multi-period setting, exercising a real option does not simply generate capacity or cash, but generates more real options in capacity and cash which can be exercised further. Thus, the real options are embedded in each other, and their endogenous values are interdependent.

The following corollary is consistent with the intuition that the value of a real option increases with its duration.

**Corollary 1.** For $j = 1, 2$, $f_{j,n-1}(e) \leq f_{j,n}(e)$, and $g_{n-1}(e) \leq g_n(e)$ with $g_1(\cdot) \equiv 1$.

**Effects of internal financing on the endogenous values of capacity and cash.** From a real-option perspective, (12) and (13) can be viewed as pricing equations for real options in a unit of capacity 1, capacity 2 and cash. Specifically, (13) implies that the worth of a dollar in the firm’s cash reserve may exceed one, namely, $g_n(e) \geq 1$. This result highlights the effect of capital market frictions, in the form of internal financing, on the value of internal cash. In a perfect capital market, the firm can access external capital freely as if it were its own cash reserve. A dollar in the external capital market is always worth its fair value, which is a dollar. Thus, the unit value of internal cash should always be one as well.

In the presence of financial frictions, however, the firm’s access to external capital is restricted and it uses internal financing. Thus, the endogenous value of internal cash should exceed one to account for its value as a financing tool. Equation (13) formalizes this intuition and implies that the financing value of internal cash is the payoff of immediate investment or liquidity buildup.

Furthermore, (12) implies that the unit values of capacities 1 and 2 depend on each other through $g_n(\cdot)$. This is an interesting result because it implies that, although the two facilities operate independently in possibly separate markets, the value of one facility spills over and affects the value of the other through the shared cash reserve. Note that this value spillover between $f_{1n}(\cdot)$ and $f_{2n}(\cdot)$ is nil if the financial market is perfect. In that case, the unit value of internal cash $g_n(\cdot) \equiv 1$, and, consequently, $f_{1n}(\cdot)$ and $f_{2n}(\cdot)$ do not affect each other (see (12)).

**Policy implications of internal financing.** From Tables 3 and 4, unit policies $(\tilde{Q}_{j1n}, \tilde{Q}_{j2n}, \tilde{D}_{jn})$ $(j = 1, 2)$ and $(\tilde{I}_{1n}, \tilde{I}_{2n}, \tilde{X}_n)$ depend on endogenous values $f_{jn}(\cdot)$ and $g_n(\cdot)$. Thus, closely intertwined endogenous values yield interdependent unit policies, indicating that all decisions must be coordinated. Specifically, due to value spillover, the production policy of facility 1 depends not only
on its own market \((p_{11}^1, p_{12}^1, r_1^1, y_1^1)\) and depreciation features \((\theta_1, \lambda_{11}, \lambda_{12})\), but also on the market served by facility 2 \((p_{21}^2, p_{22}^2, r_2^2, y_2^2)\) and facility 2’s features \((\theta_2, \lambda_{21}, \lambda_{22})\). This is in stark contrast with the case in a perfect capital market, where the facilities are not linked and the two production facilities could be managed independently. Coordination of high-level investment decisions under internal financing has been observed empirically by Lamont (1997). Our result predicts that such coordination should occur also in low-level operations such as production, and specifies how it arises from the interdependent endogenous values.

Tables 3 and 4 imply that the coordination of all decisions can be achieved in a decentralized manner, provided that the correct endogenous values are used to compute the payoffs at each facility. This result implies that, in practice, headquarters can set the endogenous values of assets as internal prices or transfer prices (Eccles 1983) among individual facilities to achieve coordination.

## 5 Unit policy

This section examines the unit policy \((\tilde{Q}_{jkn}, \tilde{D}_{jn}, \tilde{I}_{jn}, \tilde{X}_n : j, k = 1, 2)\). Section 5.1 illuminates the effects of internal financing on capacity management when investments are partially irreversible. It also reveals the tension between production and divestment due to production-induced depreciation. Section 5.2 considers the case in which the exogenous process \(e_1, e_2, \ldots\) is a sequence of i.i.d. random vectors (r.v.s). It shows that the unit policy has a multi-dimensional threshold structure, and that an internally financed firm is less likely to issue dividends or invest than its counterpart in a perfect capital market.
5.1 General exogenous process: Effects of internal financing and depreciation

The following definitions of subsets of $\Omega$ ($j = 1, 2$), are useful to characterize the unit policy.

\[
D_{jn} = \{ e \in \Omega : r_j^e \mathbb{E}[g_{n-1}(\zeta_e)] - \mathbb{E}[f_{j,n-1}(\zeta_e)] > 0 \}, \tag{16a}
\]

\[
C_j^1 = \{ e \in \Omega : p_{j1}^e - \lambda_j r_j^e > 0, \quad p_{j1}^e - \lambda_j r_j^e \geq p_{j2}^e - \lambda_j r_j^e \}, \tag{16b}
\]

\[
C_j^2 = \{ e \in \Omega : p_{j2}^e - \lambda_j r_j^e > 0, \quad p_{j1}^e - \lambda_j r_j^e < p_{j2}^e - \lambda_j r_j^e \}, \tag{16c}
\]

\[
P_{jn}^1 = \{ e \in \Omega : p_{j1}^e \mathbb{E}[g_{n-1}(\zeta_e)] - \lambda_j \mathbb{E}[f_{j,n-1}(\zeta_e)] > 0, \quad p_{j1}^e \mathbb{E}[g_{n-1}(\zeta_e)] - \lambda_j \mathbb{E}[f_{j,n-1}(\zeta_e)] \geq p_{j2}^e \mathbb{E}[g_{n-1}(\zeta_e)] - \lambda_j \mathbb{E}[f_{j,n-1}(\zeta_e)] \}, \tag{16d}
\]

\[
P_{jn}^2 = \{ e \in \Omega : p_{j2}^e \mathbb{E}[g_{n-1}(\zeta_e)] - \lambda_j \mathbb{E}[f_{j,n-1}(\zeta_e)] > 0, \quad p_{j1}^e \mathbb{E}[g_{n-1}(\zeta_e)] - \lambda_j \mathbb{E}[f_{j,n-1}(\zeta_e)] < p_{j2}^e \mathbb{E}[g_{n-1}(\zeta_e)] - \lambda_j \mathbb{E}[f_{j,n-1}(\zeta_e)] \}, \tag{16e}
\]

\[
W_n^c = \{ e \in \Omega : \beta y_n^e \mathbb{E}[f_{j,n-1}(\zeta_e)] < 1, \quad \beta \mathbb{E}[g_{n-1}(\zeta_e)] < 1 : j = 1, 2 \}, \tag{16f}
\]

\[
W_n^1 = \{ e \in \Omega : \mathbb{E}[g_{n-1}(\zeta_e)] - y_1^e \mathbb{E}[f_{1,n-1}(\zeta_e)] < 0, \quad y_1^e \mathbb{E}[f_{1,n-1}(\zeta_e)] \geq y_2^e \mathbb{E}[f_{2,n-1}(\zeta_e)], \quad 1 \leq \beta y_1^e \mathbb{E}[f_{1,n-1}(\zeta_e)] \}; \tag{16g}
\]

\[
W_n^2 = \{ e \in \Omega : \mathbb{E}[g_{n-1}(\zeta_e)] - y_2^e \mathbb{E}[f_{2,n-1}(\zeta_e)] < 0, \quad y_1^e \mathbb{E}[f_{1,n-1}(\zeta_e)] < y_2^e \mathbb{E}[f_{2,n-1}(\zeta_e)], \quad 1 \leq \beta y_2^e \mathbb{E}[f_{2,n-1}(\zeta_e)] \}; \tag{16h}
\]

These sets and their complements, denoted by superscript $c$, form seven partitions of $\Omega$ that play important roles in characterizing the unit policy: \{\(D_{jn}, D_{jn}^c\), \{\(C_j^1\), \(C_j^2\), \((C_j^1 \cup C_j^2)^c\)\}, \{\(P_{jn}^1\), \(P_{jn}^2\), \((P_{jn}^1 \cup P_{jn}^2)^c\)\}, and \{\(W_n^c\), \(W_n^1\), \(W_n^2\), \((W_n^c \cup W_n^1 \cup W_n^2)^c\)\} ($j = 1, 2$). Let \(1(\cdot)\) denote the indicator function (which equals 1 if the argument is true and 0 otherwise).

**Proposition 1.** The unit policy \((\hat{Q}_{jkn}, \hat{D}_{jn}, \hat{I}_{jn}, \hat{X}_n : j, k = 1, 2)\) satisfies

\[
\hat{Q}_{jkn}(e) = 1\left( e \in (D_{jn} \cap C_j^k) \cup (D_{jn}^c \cap D_{jn}) \right), \tag{17}
\]

\[
\hat{D}_{jn}(e) = 1(e \in D_{jn}), \tag{18}
\]

\[
\hat{I}_{jn}(e) = 1(e \in W_n^1), \tag{19}
\]

\[
\hat{X}_n(e) = 1(e \in W_n^c). \tag{20}
\]

The unit production policy in (17) has the following interpretation. If divestment is optimal,
i.e., if \( e \in \mathcal{D}_{jn} \), then make product \((j, k)\) if \( e \in \mathcal{C}_j^k \). Otherwise, if it is optimal not to divest, i.e., if \( e \in \mathcal{D}_{jn}^c \), then make product \((j, k)\) if \( e \in \mathcal{P}_{jn}^k \). Since \( \mathcal{C}_j^k \) and \( \mathcal{P}_{jn}^k \) are different (see (16)), (17) implies that the condition for making product \((j, k)\) changes with the divestment decision.

The dependence of the production condition on divestment is driven by production-induced depreciation, \( \lambda_{jk} \). When \( \lambda_{jk} > 0 \), production reduces the amount of capacity available for divestment at the end of the period. Thus, when the divestment price is high, it may be optimal to forego production profit so as to reserve more capacity for divestment. This creates a tension between production and divestment decisions, and leads to different conditions for production with and without divestment, as in (17). From (16), \( \mathcal{C}_j^k \) and \( \mathcal{P}_{jn}^k \) are identical if production-induced depreciation is nil.

Another consequence of production-induced depreciation is that it may not be optimal to make the product with the higher profit margin. For example, if \( p_{e1}^j > p_{e2}^j \) and \( \lambda_{j1} \gg \lambda_{j2} \), then, from (16b–16e), it would be optimal either to make \((j, 2)\) or to make nothing at all.

Proposition 1 leads to the following result for capacity adjustment decisions.

**Corollary 2.** For \( j = 1, 2 \), if \( \tilde{D}_{jn}(e) = 1 \), then \( \mathbb{E}[f_{j,n-1}(\zeta_e)]/\mathbb{E}[g_{n-1}(\zeta_e)] < \frac{r_j^e}{y_j^e} \). If \( \tilde{I}_{jn}(e) = 1 \), then \( \mathbb{E}[f_{j,n-1}(\zeta_e)]/\mathbb{E}[g_{n-1}(\zeta_e)] > \frac{1}{y_j^e} \).

Corollary 2 implies that the capacity adjustment policy has an “invest/stay put/divest” (ISD) structure when investment is partially irreversible. To see this, note that strict partial irreversibility means that divestment revenue is lower than investment cost, namely, \( r_j^e y_j^e < 1 \). According to Corollary 2, divestment is optimal only when \( \mathbb{E}[f_{j,n-1}(\zeta_e)]/\mathbb{E}[g_{n-1}(\zeta_e)] < \frac{r_j^e}{y_j^e} \). Investment is optimal only when \( \mathbb{E}[f_{j,n-1}(\zeta_e)]/\mathbb{E}[g_{n-1}(\zeta_e)] > \frac{1}{y_j^e} \). Since \( r_j^e < 1/y_j^e \) for partially irreversible investments, this creates a gap in the ratio over which the firm is inactive.

Corollary 2 extends a result in Abel and Eberly (1996) to a capacity-portfolio setting with financial constraints. In perfect capital markets, the expected marginal value of cash \( \mathbb{E}[g_{n-1}(\zeta_e)] = 1 \), which causes the conditions in Corollary 2 to coincide with that in Abel and Eberly (1996). Although in the presence of financial frictions the marginal value of cash may exceed one, Corollary 2 shows that the ISD structure still applies to the ratio of the expected marginal values of capacity and cash.
5.2 I.i.d. exogenous process: Threshold policy and effects of internal financing

This subsection considers the unit policy when the exogenous process \(e_1, e_2, \ldots\) consists of i.i.d. random vectors. In this case, the expected unit values of capacity and cash, \(\mathbb{E}[f_{jn}(\zeta_e)] (j = 1, 2)\) and \(\mathbb{E}[g_n(\zeta_e)]\), are constant with respect to \(e\). Let \(\zeta\) denote the random vector that is common to the i.i.d. exogenous process, and let \(F_{jn} = \mathbb{E}[f_{j,n-1}(\zeta)] (j = 1, 2)\) and \(G_n = \mathbb{E}[g_{n-1}(\zeta)]\).

**Threshold policy.** The next corollary states that the unit divestment policy has a threshold structure with respect to divestment price, \(r_j^e\).

**Corollary 3.** Under an i.i.d. exogenous process, the unit divestment policy for capacity \(j\) \((j = 1, 2)\) is \(\bar{D}_{jn} = 1(r_j^e > F_{jn}/G_n)\).

From Corollary 3, it is optimal to divest if and only if the divestment price is sufficiently high. Since \(G_n \geq 1\) from (13), the threshold condition is equivalent to \(r_j^e G_n > F_{jn}\). Note that \(r_j^e G_n\) is the expected payoff from divesting a unit of capacity \(j\), and \(F_{jn}\) is the expected payoff of retaining a unit of capacity \(j\) into next period. Thus, \(F_{jn}/G_n\) is the divestment price at which divesting nothing or maximally yields the same expected payoff. Therefore, Corollary 3 states that divestment is optimal only when the divestment price exceeds this indifference threshold.

The following corollary implies that the production policy for facility \(j\) \((j = 1, 2)\) has a threshold structure with respect to the divestment price \(r_j^e\) and profit margins \(p_{j1}^e\) and \(p_{j2}^e\).

**Corollary 4.** Under an i.i.d. exogenous process, the unit production policy for capacity \(j\) is

\[
\begin{align*}
\hat{Q}_{j1n}(e) &= 1\left(r_j^e > \frac{F_{jn}}{G_n}, p_{j1}^e > \lambda_1 r_j^e, p_{j1}^e + p_{j2}^e \geq (\lambda_1 - \lambda_2) r_j^e \right) \\
&\quad + 1\left(r_j^e \leq \frac{F_{jn}}{G_n}, p_{j1}^e > \lambda_1 \frac{F_{jn}}{G_n}, p_{j1}^e - p_{j2}^e \geq (\lambda_1 - \lambda_2) \frac{F_{jn}}{G_n} \right), \quad (21a) \\
\hat{Q}_{j2n}(e) &= 1\left(r_j^e > \frac{F_{jn}}{G_n}, p_{j1}^e > \lambda_1 r_j^e, (p_{j1}^e - p_{j2}^e) < (\lambda_1 - \lambda_2) r_j^e \right) \\
&\quad + 1\left(r_j^e \leq \frac{F_{jn}}{G_n}, p_{j1}^e > \lambda_1 \frac{F_{jn}}{G_n}, p_{j1}^e - p_{j2}^e < (\lambda_1 - \lambda_2) \frac{F_{jn}}{G_n} \right). \quad (21b)
\end{align*}
\]

Consider \(\hat{Q}_{j1n}(e)\) to see that (21) corresponds to a threshold production policy for product \((j, 1)\).

From the first line of (21a), \(\hat{Q}_{j1n}(e) = 1\) if \(r_j^e > F_{jn}/G_n, p_{j1}^e > \lambda_1 r_j^e,\) and \(p_{j1}^e - p_{j2}^e \geq (\lambda_1 - \lambda_2) r_j^e\).

These inequalities determine a set that is the truncation of a convex cone and a hyperplane in the three-dimensional set of \((r_j^e, p_{j1}^e, p_{j2}^e)\). Similarly, the conditions in the second line of (21a) form a
set that is the truncation of a convex cone and two hyperplanes. Thus, the production policy has a multi-dimensional threshold structure.

Corollary 5 implies that the unit investment and dividend policy has a threshold structure with respect to investment yields $y_1^e$ and $y_2^e$.

**Corollary 5.** Under an i.i.d. exogenous process, the unit investment and dividend policy is

\[
\begin{align*}
\tilde{I}_{1n}(e) & = 1(y_1^e > G_n/F_1n, \ y_2^e/y_1^e < F_1n/F_2n, \ y_1^e \geq 1/(\beta F_1n)), & (22a) \\
\tilde{I}_{2n}(e) & = 1(y_2^e > G_n/F_2n, \ y_2^e/y_1^e \geq F_1n/F_2n, \ y_2^e \geq 1/(\beta F_2n)), & (22b) \\
\tilde{X}_n(e) & = 1(y_1^e < 1/(\beta F_1n), \ y_2^e < 1/(\beta F_2n), \ \beta G_n < 1). & (22c)
\end{align*}
\]

The denominator $F_{jn}$ ($j = 1, 2$) in (22) is positive because, from (12), $f_{jn}(e) \geq B_{jn4}(e) = \beta \theta_j r_j^e G_{n-1}$ in which $\beta, \theta_j, r_j > 0$ and $G_{n-1} \geq 1$. From (22), dividend issuance is optimal only when both the investment yields $y_j^e$ ($j = 1, 2$) and the expected unit value of cash $G_n$ are low. Otherwise, if investment yields are sufficiently high, it is optimal to invest instead of issuing a dividend. In this case, capacities $1$ and $2$ compete for investment. From (22a) and (22b), all cash is used to expand capacity $1$ if $y_2^e/y_1^e < F_1n/F_2n$, and otherwise it is used to expand capacity $2$. If the yields are low and the expected value of cash $G_n$ is high, namely if $e \in \mathcal{W}_n^0$ where

\[
\mathcal{W}_n^0 = \{ e : y_1^e \leq G_n/F_1n, \ y_2^e \leq G_n/F_2n, \ \beta G_n-1 \geq 1 \}, \quad (23)
\]

then $\tilde{I}_{1n}(e) = \tilde{I}_{2n}(e) = \tilde{X}_n(e) = 0$, and all cash is retained as liquidity for use in the next period.

**Policy implications of internal financing.** Equations (22) and (23) allow a comparison of the dividend and investment policies of an internally financed firm and its counterpart which lives in a perfect capital market. As discussed earlier, absent financial frictions, the unit value of cash is one. Thus, $\beta G_n < 1$ for all $n$ in (23), indicating that $\mathcal{W}_n^0 = \emptyset$. This implies that the firm in a perfect capital market should not hold internal cash and should either invest its cash earnings or distribute them as dividends. This result is intuitive because in this case the firm can access external capital freely, rendering an internal cash reserve unnecessary.

The dividend-investment policy changes dramatically when the firm faces substantial financial frictions and finances all decisions internally. In this case, Corollary 1 implies that the expected
unit value of cash $G_n$ satisfies $G_{n-1} \leq G_n$. Since $G_2 = 1$, early on in the planning horizon, namely when $n$ is large, the condition $\beta G_n < 1$ in (22c) will be violated, indicating that $\tilde{X}_n(e) = 0$. This suggests that financial frictions, in the form of internal financing, make a firm less likely to issue dividends as it strives to exploit the financing value of its internal cash reserve.

A high value of internal cash also makes the firm more cautious with investments. To see this, note that $G_n = 1$ in a perfect capital market, so the investment policy in (22) reduces to
$$\tilde{I}_1^n(e) = 1(y_1^e > 1/(\beta F_1^n), \ y_2^e/y_1^e < F_1^n/F_2^n) \text{ and } \tilde{I}_2^n(e) = 1(y_2^e > 1/(\beta F_2^n), \ y_2^e/y_1^e \geq F_1^n/F_2^n).$$
Thus, investment occurs when the investment yield is greater than the threshold value $1/(\beta F_j^n)$ ($j = 1, 2$).

When $G_n$ is sufficiently high such that $\beta G_n \geq 1$ under financial frictions, the investment policy becomes $\tilde{I}_1^n(e) = 1(y_1^e > G_n/F_1^n, \ y_2^e/y_1^e < F_1^n/F_2^n)$ and $\tilde{I}_2^n(e) = 1(y_2^e > G_n/F_1^n, \ y_2^e/y_1^e \geq F_1^n/F_2^n).$ Thus, investment occurs only when the investment yield is greater than the threshold value $G_n/F_{jn}$ ($j = 1, 2$). But given $G_n \geq 1/\beta$, this threshold value is higher than the threshold $1/(\beta F_j^n)$ in a perfect capital market.

Thus, internal financing makes the firm less likely to issue dividends and may also make it more cautious in making investments. As a result, the firm maintains a large cash reserve, offering a possible explanation for the commonly observed “cash hoarding” phenomenon (Bates et al. 2009).

6 Effects of exogenous uncertainty on firm value and optimal policy

How do the unit values and optimal policy depend on exogenous risk? Exogenous risk in the model is specified by the collection of one-step r.v.s $\{\zeta_e : e \in \Omega\}$ that determine the process $e_1, e_2, \ldots$ (together with the initial state $e_1$). Thus, this section examines how the unit values and optimal policy change when the random vectors become stochastically larger or riskier. Henceforth, we refer to $\zeta_e$ as one-step transition r.v.s.

Let $\{\zeta'_e : e \in \Omega\}$ denote a set of one-step transition r.v.s, with at least some elements different from the original set $\{\zeta_e : e \in \Omega\}$. Let $V'_n, f'_{jn} \ (j = 1, 2)$, and $g'_n$ denote the value function, unit value of capacity $j$, and unit value of cash under $\{\zeta'_e : e \in \Omega\}$. Under the assumption that $\Omega = \mathbb{R}^8$, $\zeta'_e$ is said to be stochastically larger than $\zeta_e$ (Shaked and Shanthikumar 1994, p. 114)
if \( P\{\zeta_e \leq a\} \geq P\{\zeta'_e \leq a\} \) for all \( a \in \mathbb{R}^8 \). Note that \( \Omega = \mathbb{R}^8 \) includes as a special case \( P\{\zeta_e \geq 0\} = 1 \). Intuitively, a stochastically larger r.v. means that, starting from the same exogenous state, next period’s profit margin, divestment price, and investment yield are stochastically higher. For example, it implies \( P\{r_j^{\zeta'_e} \leq b\} \leq P\{r_j^{\zeta_e} \leq b\} \) for all \( b \in \mathbb{R} \) in which \( r_j^{\zeta'_e} \) and \( r_j^{\zeta_e} \) denote next period’s divestment price for facility \( j \) if the current exogenous state is \( e \). Thus, the primed exogenous market is more favorable for the firm. The insight provided by the following proposition is that the firm value and the unit values increase when the market is more favorable.

**Proposition 2.** If \( \Omega = \mathbb{R}^8 \) and \( \zeta'_e \) is stochastically larger than \( \zeta_e \) for all \( e \in \Omega \), then \( f_{jn}(e) \leq f'_{jn}(e) \), \( g_n(e) \leq g'_n(e) \) and \( V_n(K_1, K_2, W, e) \leq V'_n(K_1, K_2, W, e) \) \( ((K_1, K_2, W, e) \in \mathbb{R}_+^3 \times \Omega, j = 1, 2) \).

Next, consider the effects of increased riskiness of the one-step transition r.v.s, i.e., when the exogenous environment is more volatile. A definition of comparative riskiness (Rothschild and Stiglitz 1970) is that \( \zeta'_e \) is riskier than \( \zeta_e \) if there is a random variable \( Z_e \) with sample space \( \Omega \) such that \( \zeta'_e = \zeta_e + Z_e \), \( \mathbb{E}(Z_e) = 0 \) (0 denotes the zero vector in \( \mathbb{R}^8 \)), and \( \zeta_e \) and \( Z_e \) are independent. The intuition is that \( Z_e \) is “noise” that makes \( \zeta'_e \) “fuzzier” than \( \zeta_e \). (See Machina and Pratt 1997 and chapter 12 of Shaked and Shanthikumar 1994 for other definitions of comparative riskiness.) The insight provided by the following proposition is that a riskier market yields higher unit values.

**Proposition 3.** If \( \Omega \) is a convex set and \( \zeta'_e \) is riskier than \( \zeta_e \) for all \( e \in \Omega \), then \( f_{jn}(e) \leq f'_{jn}(e) \), \( g_n(e) \leq g'_n(e) \), and \( V_n(K_1, K_2, W, e) \leq V'_n(K_1, K_2, W, e) \) \( ((K_1, K_2, W, e) \in \mathbb{R}_+^3 \times \Omega, j = 1, 2) \).

Proposition 3 is consistent with the real-option nature of the firm’s assets. Heightened exogenous riskiness implies greater variances of profit margins, divestment price, and investment yields. Therefore, the payoffs of the real options embedded in capacity and cash have greater variance and, thus, the real options become more valuable according to standard option-pricing results (Björk 2009 Proposition 9.5).

Propositions 2 and 3 have important policy implications when the exogenous process is i.i.d. In this case, let \( \zeta \) denote the one-step transition r.v. which is invariant with respect to \( e \in \Omega \). Recall that \( W^e_n \) denotes the set of exogenous states in which it is optimal to issue a dividend. Let \( W'_n \) denote its counterpart when \( \zeta \) is replaced by \( \zeta' \) which is stochastically larger or riskier. Corollary 6 states that, under \( \zeta' \), the firm is less likely to distribute dividends.
Corollary 6. Under an i.i.d. exogenous process, if $\zeta$ is replaced by $\zeta'$:

1. If $\Omega = \mathbb{R}^8$ and $\zeta'$ is stochastically larger than $\zeta$, then $W_n^{\zeta'} \subseteq W_n^\zeta$.

2. If $\Omega$ is a convex set and $\zeta'$ is riskier than $\zeta$, then $W_n^{\zeta'} \subseteq W_n^\zeta$.

Corollary 6 can be understood from a real-option perspective. Issuing a dividend, investing, and holding liquidity are competing ways to exercise the real option embedded in cash. Issuing one dollar as dividend yields one dollar of immediate reward. Thus, the payoff of dividend issuance is always the face value of the dividend, regardless of the exogenous process. Investment and holding liquidity, on the other hand, yield real options in capacity and cash. Thus, the payoffs of investment and holding liquidity increase as the exogenous process becomes stochastically larger or riskier. Consequently, the firm becomes less likely to issue a dividend.

On a more fundamental level, dividend issuance permanently removes capital from the firm, whereas investment and holding liquidity retain it. Corollary 6 confirms the intuition that, as the exogenous environment becomes stochastically larger or riskier, the firm is more likely to circulate capital internally to exploit future favorable opportunities.

7 Extensions

Three or more facilities and products. For expository simplicity, this paper models a firm with only two facilities, each of which can produce just two products. The modeling framework easily extends to any number of facilities at which arbitrarily many facility-dependent numbers of products are produced. The same qualitative insights would remain valid.

Simultaneous production of both products and a different depreciation assumption. The optimal policy in Theorem 2 never stipulates that a facility make both products in the same period. The remainder of this section shows that this feature is due to the depreciation assumption, $\lambda_{jk} \leq \theta_j$ ($j = 1, 2$). Under the different assumption $\lambda_{j1} > \theta_j > \lambda_{j2}$, it can be optimal to make a mix of both products at a facility in the same period, and all qualitative insights in §4 remain valid. The value function is linear in $K_1$, $K_2$, and $W$, and there is an extremal optimal policy.

Let $\tilde{V}_n(\cdot, \cdot, \cdot)$ denote the value function under the assumption $\lambda_{j1} > \theta_j > \lambda_{j2}$ ($j = 1, 2$).
Theorem 3. The value function is

\[ \tilde{V}_n(K_1, K_2, W, e) = \tilde{f}_{1n}(e)K_1 + \tilde{f}_{2n}(e)K_2 + \tilde{g}_n(e)W, \quad (K_1, K_2, W, e) \in \mathbb{R}_+^4 \times \Omega, \]  

(24)

where \( \tilde{f}_{1n}, \tilde{f}_{2n}, \) and \( \tilde{g}_n \) are real-valued functions on \( \Omega \) that satisfy the following recursion with \( \tilde{f}_{10}(\cdot) \equiv \tilde{f}_{20}(\cdot) \equiv \tilde{g}_0(\cdot) \equiv 0 \) and \( j \in \{1, 2\} \):

\[ \tilde{f}_{jn}(e) = \max \{ \tilde{B}_{jn1}(e), \tilde{B}_{jn2}(e), \tilde{B}_{jn3}(e), \tilde{B}_{jn4}(e), \tilde{B}_{jn5}(e), \tilde{B}_{jn6}(e), \tilde{B}_{jn7}(e) \}, \]  

(25)

\[ \tilde{g}_n(e) = \max \{ \beta E[\tilde{g}_{n-1}(\zeta_e)], 1, \beta g^1_n E[\tilde{f}_{1n-1}(\zeta_e)], \beta g^2_n E[\tilde{f}_{2n-1}(\zeta_e)] \}, \]  

(26)

in which \( \tilde{B}_{jn1} \) (\( l = 1, 2, \ldots, 7 \)) are defined as follows:

\[ \tilde{B}_{jn1}(e) = \beta \theta_j \beta E[\tilde{f}_{j,n-1}(\zeta_e)], \]  

\[ \tilde{B}_{jn2}(e) = \beta [(\theta_j - \lambda_{j2}) \beta E[\tilde{f}_{j,n-1}(\zeta_e)] + p^e_{j1} \beta E[\tilde{g}_{n-1}(\zeta_e)]], \]  

\[ \tilde{B}_{jn3}(e) = \beta [(\theta_j - \lambda_{j2}) \beta E[\tilde{f}_{j,n-1}(\zeta_e)] + p^e_{j2} \beta E[\tilde{g}_{n-1}(\zeta_e)]], \]  

\[ \tilde{B}_{jn4}(e) = \beta \theta_j e \beta E[\tilde{g}_{n-1}(\zeta_e)], \]  

\[ \tilde{B}_{jn5}(e) = \beta [p^e_{j1} + (\theta_j - \lambda_{j1}) r^e_j] \beta E[\tilde{g}_{n-1}(\zeta_e)], \]  

\[ \tilde{B}_{jn6}(e) = \beta [p^e_{j2} + (\theta_j - \lambda_{j2}) r^e_j] \beta E[\tilde{g}_{n-1}(\zeta_e)], \]  

\[ \tilde{B}_{jn7}(e) = \beta \frac{p^e_{j1} - p^e_{j2} + (\lambda_{j1} p^e_{j1} - \lambda_{j2} p^e_{j2})}{\lambda_{j1} - \lambda_{j2}} \beta E[\tilde{g}_{n-1}(\zeta_e)] + \beta (2\theta_j - \lambda_{j1} - \lambda_{j2}) \beta E[\tilde{f}_{j,n-1}(\zeta_e)]. \]  

(27a)  

(27b)  

(27c)  

(27d)

In comparison with Theorem 1, Theorem 3 has an additional maximand \( \tilde{B}_{jn7}(e) \) for \( \tilde{f}_{jn} \) (compare (12) and (25)). The next theorem shows that \( \tilde{B}_{jn7}(e) \) corresponds to an optimal policy that makes both products at facility \( j \). Let \( (\tilde{Q}^u_{jkn}(e), \tilde{D}^u_{jkn}(e), \tilde{I}^u_{jn}(e), \tilde{X}^u_{jn}(e) : j, k = 1, 2) \) denote an optimal policy and define \( (\tilde{Q}^u_{j1n}(e), \tilde{D}^u_{j1n}(e), \tilde{I}^u_{j1n}(e), \tilde{X}^u_{j1n}(e) : j = 1, 2) \) as in Tables 5 and 6. Notice in Table 5 that \( \tilde{Q}^u_{j1n}(e) > 0 \) and \( \tilde{Q}^u_{j2n}(e) > 0 \) when \( \tilde{B}_{jn7}(e) \) is the maximal maximand in (25).

<table>
<thead>
<tr>
<th>( (\tilde{Q}^u_{j1n}(e), \tilde{Q}^u_{j2n}(e), \tilde{D}^u_{j1n}(e)) )</th>
<th>Condition</th>
<th>( (\tilde{Q}^u_{j1n}(e), \tilde{Q}^u_{j2n}(e), \tilde{D}^u_{j1n}(e)) )</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (0, 0, 0) )</td>
<td>( \tilde{f}<em>{jn}(e) = \tilde{B}</em>{jn1}(e) )</td>
<td>( (0, 0, 1) )</td>
<td>( \tilde{f}<em>{jn}(e) = \tilde{B}</em>{jn4}(e) )</td>
</tr>
<tr>
<td>( (1, 0, 0) )</td>
<td>( \tilde{f}<em>{jn}(e) = \tilde{B}</em>{jn2}(e) )</td>
<td>( (1, 0, 1) )</td>
<td>( \tilde{f}<em>{jn}(e) = \tilde{B}</em>{jn5}(e) )</td>
</tr>
<tr>
<td>( (0, 1, 0) )</td>
<td>( \tilde{f}<em>{jn}(e) = \tilde{B}</em>{jn3}(e) )</td>
<td>( (0, 1, 1) )</td>
<td>( \tilde{f}<em>{jn}(e) = \tilde{B}</em>{jn6}(e) )</td>
</tr>
<tr>
<td>( \left( \frac{\theta_j - \lambda_{j2}}{\lambda_{j1} - \lambda_{j2}}, \frac{\lambda_{j1} - \theta_j}{\lambda_{j1} - \lambda_{j2}}, 0 \right) )</td>
<td>( \tilde{f}<em>{jn}(e) = \tilde{B}</em>{jn7}(e) )</td>
<td>( \tilde{f}<em>{jn}(e) = \tilde{B}</em>{jn7}(e) )</td>
<td>( \tilde{f}<em>{jn}(e) = \tilde{B}</em>{jn7}(e) )</td>
</tr>
</tbody>
</table>

Resolution of ties: if \( \tilde{f}_{jn}(e) = \tilde{B}_{jn7}(e) = \tilde{B}_{jn7'}(e) \) for \( l \neq l' \) (\( l, l' \in \{1, 2, \ldots, 7\} \)), \( (\tilde{Q}^u_{j1n}(e), \tilde{Q}^u_{j2n}(e), \tilde{D}^u_{j1n}(e)) \) takes the value associated with the smaller \( l' \).
Table 6: Table of \( (\bar{I}_{1n}(e), \bar{I}_{2n}(e), \bar{X}_n(e)), e \in \Omega \)

<table>
<thead>
<tr>
<th>((\bar{I}<em>{1n}(e), \bar{I}</em>{2n}(e), \bar{X}_n(e)))</th>
<th>Condition</th>
<th>((\bar{I}<em>{1n}(e), \bar{I}</em>{2n}(e), \bar{X}_n(e)))</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 0))</td>
<td>(\bar{g}<em>n(e) = \beta \mathbb{E}[\bar{f}</em>{n-1}(\zeta_e)])</td>
<td>((0, 1, 0))</td>
<td>(\bar{g}<em>n(e) = \beta y_e \mathbb{E}[\bar{f}</em>{2,n-1}(\zeta_e)])</td>
</tr>
<tr>
<td>((1, 0, 0))</td>
<td>(\bar{g}<em>n(e) = \beta y_1 \mathbb{E}[\bar{f}</em>{1,n-1}(\zeta_e)])</td>
<td>((0, 0, 1))</td>
<td>(\bar{g}_n(e) = 1)</td>
</tr>
</tbody>
</table>

Resolution of ties: if two or more conditions in the table hold simultaneously, \( (\bar{I}_{1n}(e), \bar{I}_{2n}(e), \bar{X}_n(e)) \) takes the value that is earlier in the sequence \( (0, 0, 0), (1, 0, 0), (0, 1, 0), \) and \( (0, 0, 1) \).

**Theorem 4.** The following policy is optimal:

\[
\begin{align*}
\bar{Q}_{jkn}(K_1, K_2, W, e) &= K_j \times \bar{Q}_{jkn}(e), \\
\bar{D}_{jn}(K_1, K_2, W, e) &= \bar{D}_{jn}(e), \\
\bar{I}_{jn}(K_1, K_2, W, e) &= W \times \bar{I}_{jn}(e), \\
\bar{X}_n(K_1, K_2, W, e) &= W \times \bar{X}_n(e),
\end{align*}
\]

\((j, k = 1, 2)\) \hspace{1cm} (28a)

\[
\begin{align*}
\bar{g}_n(e) &= \beta \mathbb{E}[\bar{f}_{n-1}(\zeta_e)], \\
\bar{g}_n(e) &= \beta y_e \mathbb{E}[\bar{f}_{2,n-1}(\zeta_e)], \\
\bar{g}_n(e) &= \beta y_1 \mathbb{E}[\bar{f}_{1,n-1}(\zeta_e)], \\
\bar{g}_n(e) &= 1.
\end{align*}
\]

\((j = 1, 2)\) \hspace{1cm} (28b)

**8 Summary**

This paper studies the dynamic production and capacity management of a price-taking firm that uses only internal financing and lives in a stochastic market environment. The firm has two production facilities, each of which can make two products; and an internal cash reserve, which is the only source of financing during the planning horizon.

We analytically characterize the value function and optimal policy, and show that they admit a real option interpretation in spite of their complexity. We show that the use of internal financing induces interdependence among facilities, even if they make completely different products and serve distinct markets. We analytically characterize how the endogenous values of the facilities are related, which results in value spillover, and how the optimal policies at different facilities are linked. When capacity investments are partially irreversible, we characterize the effects of internal financing on the ISD structure of the capacity management policy. When the exogenous data are i.i.d., we show that internal financing makes the firm (i) less likely to issue a dividend or invest than if it could access a perfect capital market, and (ii) less likely to issue a dividend as the exogenous environment becomes more favorable or riskier.

The results in this paper generate several testable hypotheses. First, with the use of internal financing, the optimality of the product mix in one facility can directly affect the optimality of the product mix in the other. This can be tested using the operations of companies that serve different markets to see if, after controlling for macro factors, a change in the product mix in one market,
e.g., East Asia, is accompanied by a change in the other market, e.g., North America.

Another testable hypothesis is that firms are less likely to issue dividends when the market becomes more volatile. While in practice, dividend issuance is more complicated than is modeled in the paper, one could potentially test this result using aggregate measures such as the fraction of companies that issue dividends in an industry, and investigate the effects of market volatility on this fraction.

The third testable hypothesis is that firms that face serious financial frictions have larger cash reserves than firms that access capital markets more readily. This is consistent with the precautionary purpose of holding cash in the finance literature, which has been empirically verified by Almeida et al. (2004) and Han and Qiu (2007).

Several directions are worth exploring in the future. This paper did not spell out the precise dependence of the optimal policy on the exogenous state. It would be interesting to investigate how the different partitions of the exogenous state space \( \Omega \) in Proposition 1 relate to each other and to elicit more insights about the effects of internal financing on production and investment policies. One way to do this is to characterize the optimal policy under specific types of exogenous processes. This paper considers an i.i.d. process, and it would be interesting to study others such as random walk and mean-reverting processes.

This paper studies a model in which two facilities make different products, serve potentially different markets, and have different depreciation features. Another worthwhile future direction would be to compare the facilities’ optimal production policies in a model where the two facilities make identical products with identical profit margins but have different depreciation parameters.

References


Myers, Stewart C., Nicholas S. Majluf. 1984. Corporate financing and investment decisions when firms have information that investors do not have. *Journal of financial economics* 13(2) 187–221.


A  Proofs

The following change of variable is made throughout the appendix. In place of $d_{jt}$, which is the fraction of post-depreciation capacity $j$ ($j = 1, 2$) that is divested in period $t$, the appendix uses $\delta_{jt}$, the amount of post-depreciation capacity $j$ that is divested in period $t$. Thus,

$$\delta_{jt} = d_{jt}(\theta_j K_{jt} - \lambda_1 q_{1t} - \lambda_2 q_{2t})$$ (29)

Therefore, (4), (5), and (6) correspond to

$$0 \leq \delta_{jt} \leq \theta_j K_{jt} - \lambda_1 q_{1t} - \lambda_2 q_{2t},$$ (30)

$$K_{j,t+1} = y_{jt} i_{jt} + \theta_j K_{jt} - \lambda_1 q_{1t} - \lambda_2 q_{2t} - \delta_{jt},$$ (31)

$$W_{t+1} = W_t - x_t - \sum_{j=1}^{2} i_{jt} + \sum_{j=1}^{2} r_{jt} \delta_{jt} + \sum_{j=1}^{2} \sum_{k=1}^{2} p_{jkt} q_{jkt}.$$ (32)

Dynamic program (10) corresponds to $V_0(\cdot, \cdot, \cdot, \cdot) \equiv 0$ and for $n \in \mathbb{N} = \{1, 2, \ldots, N\}$, with $a = (q_{jk}, \delta_j, i_j, x : j, k = 1, 2)$,

$$V_n(K_1, K_2, W, e) = \max_{0 \leq a} \left\{ J_n(a; K_1, K_2, W, e) : \delta_j + \sum_{k=1}^{2} \lambda_{jk} q_{jk} \leq \theta_j K_j, \right. \right.$$

$$\left. \sum_{k=1}^{2} q_{jk} \leq K_j, \ i_1 + i_2 + x \leq W : j = 1, 2 \right\}, \ (K_1, K_2, W, e) \in \mathbb{R}_+^3 \times \Omega$$ (33a)

$$J_n(a; K_1, K_2, W, e) = x + \beta \mathbb{E}[V_{n-1}(K'_1, K'_2, W', \zeta_e)],$$ (33b)

$$K'_j = y_{j}^e i_{j} + \theta_j K_j - \sum_{k=1}^{2} \lambda_{jk} q_{jk} - \delta_j \quad (j = 1, 2),$$ (33c)

$$W' = W - x - \sum_{j=1}^{2} i_{jt} + \sum_{j=1}^{2} r_{jt} \delta_{jt} + \sum_{j=1}^{2} \sum_{k=1}^{2} p_{jkt} q_{jkt}.$$ (33d)

A.1  Proof of Lemma 1 in §3.3

Proof of Lemma 1. The contrapositive assertion is that a policy is strictly sub-optimal if it specifies $q_{jkt} > 0$ for some $j, k, t, e$ such that $p_{jkt}^e < 0$. Let $p_{jkt}^e < 0$, and let $\pi$ be a policy that specifies $q_{jkt} > 0$ for some $t$ if $e_t = e$. Let $\pi'$ be another policy with primed action variables (production
Proof of Theorem 1. Initiate an inductive proof of (11) at $A.2$ Proof of statements in §4 for the actions under $q$ quantities $q_{jt}$ that are the same as the action variables under $\pi$ except that $q'_{jt} = 0$ and $x_{t+1} = x_t + \sum |p'_{jt}| q_{jt} > x_t$. The feasibility of the actions under $\pi$ implies feasibility for the actions under $\pi'$, but the latter has a strictly larger EPV of dividends (conditional on any partial history $H_t$ that ends with $e_t = e$). Therefore, $\pi$ is strictly sub-optimal.

A.2 Proof of statements in §4

Define $F_{jn}(e) = \mathbb{E}[f_{j,n-1}(\zeta_e)]$ ($j = 1, 2$) and $G_n(e) = \mathbb{E}[g_{n-1}(\zeta_e)]$.

Proof of Theorem 1. Initiate an inductive proof of (11) at $n = 0$ with $V_0(\cdot, \cdot) \equiv 0$, $f_{1,0}(\cdot) \equiv 0$, $f_{2,0}(\cdot) \equiv 0$, and $g_0(\cdot) \equiv 0$. If for some $n \in \mathbb{N}$, $V_{n-1}(K_1, K_2, W, e) = f_{1,n-1}(e)K_1 + f_{2,n-1}(e)K_2 + g_{n-1}(e)W$ for all $(K_1, K_2, W, e) \in \mathbb{R}_+^3 \times \Omega$, then (33) implies

$$V_n(K_1, K_2, W, e) = \max_{0 \leq a} \left\{ x + \beta \mathbb{E} \left[ V_{n-1} \left( K'_1, K'_2, W', \zeta_e \right) \right] : \right.$$ \[ \begin{align*} &\delta_j + \sum_{k=1}^2 \lambda_{jk}q_{jk} \leq \theta_j K_j, \quad \sum_{k=1}^2 q_{jk} \leq K_j, \quad i_1 + i_2 + x \leq W : j = 1, 2 \right\} \\
&= \max_{0 \leq a} \left\{ x + \beta \mathbb{E} \left[ f_{1,n-1}(\zeta_e)K'_1 + f_{2,n-1}(\zeta_e)K'_2 + g_{n-1}(\zeta_e)W' \right] : \right. \\
&\left. \delta_j + \sum_{k=1}^2 \lambda_{jk}q_{jk} \leq \theta_j K_j, \quad \sum_{k=1}^2 q_{jk} \leq K_j, \quad i_1 + i_2 + x \leq W : j = 1, 2 \right\} \\
&= \max_{0 \leq a} \left\{ x + \beta \mathbb{E} \left[ f_{1,n-1}(\zeta_e) \left( g_{1}'i_1 + \theta_1 K_1 - \sum_{k=1}^2 \lambda_{1k} q_{1k} - \delta_1 \right) + f_{2,n-1}(\zeta_e) \left( g_{2}'i_2 + \theta_2 K_2 - \sum_{k=1}^2 \lambda_{2k} q_{2k} - \delta_2 \right) \\
&+ g_{n-1}(\zeta_e) \left( W - x - \sum_{j=1}^2 i_j + \sum_{j=1}^2 r_{j}' \delta_j + \sum_{j=1}^2 \sum_{k=1}^2 r_{j}' \delta_{jk} q_{jk} \right) \right] : \right. \\
&\left. \delta_j + \sum_{k=1}^2 \lambda_{jk}q_{jk} \leq \theta_j K_j, \quad \sum_{k=1}^2 q_{jk} \leq K_j, \quad i_1 + i_2 + x \leq W : j = 1, 2 \right\} \\
&= \beta \theta_1 F_{1n}(e)K_1 + \beta \theta_2 F_{2n}(e)K_2 + \beta G_n(e)W \quad (34a) \\
&+ \beta \sum_{j=1}^2 \max_{0 \leq q_{jk}, \delta_j} \left\{ \sum_{k=1}^2 \left( p_{jk}^e G_n(e) - \lambda_{jk} F_{j,n}(e) \right) q_{jk} + \left( r_{j}' G_n(e) - \bar{F}_{j,n}(e) \right) \delta_j : \right. \\
&\left. \sum_{k=1}^2 q_{jk} \leq K_j, \quad \delta_j + \sum_{k=1}^2 \lambda_{jk}q_{jk} \leq \theta_j K_j \right\} \quad (34b) \\
&+ \max_{0 \leq \bar{r}_{j}, \delta_j} \left\{ [1 - \beta G_n(e)] x + \sum_{j=1}^2 \beta \left( y_j^e F_{j,n}(e) - G_n(e) \right) i_j : i_1 + i_2 + x \leq W \right\} \quad (34c)
Expression (34b) sums the values of two linear programs with $j = 1$ and $j = 2$. The decision variables in the $j^{th}$ are $q_{j1}, q_{j2}$ and $\delta_j$, and the extreme points of its feasibility set are \{$(q_{j1}, q_{j2}, \delta_j) = (0, 0, 0), (q_{j1}, q_{j2}, \delta_j) = (K_j, 0, 0), (q_{j1}, q_{j2}, \delta_j) = (0, K_j, 0), (q_{j1}, q_{j2}, \delta_j) = (0, 0, \theta_j K_j), (q_{j1}, q_{j2}, \delta_j) = (K_j, 0, (\theta_j - \lambda_j) K_j))$, $(q_{j1}, q_{j2}, \delta_j) = (0, K_j, (\theta_j - \lambda_j) K_j))$\}. Prospective extreme point $(q_{j1}, q_{j2}, \delta_j) = (\theta_j K_j/\lambda_j, 0, 0)$ is not included in this set because it is either infeasible or redundant. Recall the assumption $0 \leq \lambda_{jk} \leq \theta_j$. If $\lambda_{j1} < \theta_j$, then $q_{j1} = \theta_j K_j/\lambda_{j1} > K_j$ with $q_{j2} = 0$ violates $\sum_{k=1}^2 q_{jk} \leq K_j$. If $\lambda_{jk} = \theta_j$, then $(q_{j1}, q_{j2}, \delta_j) = (\theta_j K_j/\lambda_{j1}, 0, 0) = (K_j, 0, 0)$ which is already in the set of six extreme points in the previous paragraph. Similar comments apply to prospective extreme point $(q_{j1}, q_{j2}, \delta_j) = (0, \theta_j K_j/\lambda_j, 0)$. The linear program (34b) is not infeasible or unbounded because $K_j, q_{j1}, q_{j2}$, and $\delta_j$ are non-negative. Therefore, at least one of the six extreme points is optimal, and the value of the linear program is the maximum of the values of the six extreme points. Pricing the six, the value of the $j^{th}$ linear program is

$$\beta \max \left\{ 0, \ p_{j1}^e G_n(e) - \lambda_{j1} F_{jn}(e), \ p_{j2}^e G_n(e) - \lambda_{j2} F_{jn}(e), \ [r_j^e G_n(e) - F_{jn}(e)] \theta_j, \ p_{j1}^e G_n(e) - \lambda_{j1} F_{jn}(e) + [r_j^e G_n(e) - F_{jn}(e)] (\theta_j - \lambda_{j1}), \ p_{j2}^e G_n(e) - \lambda_{j2} F_{jn}(e) + [r_j^e G_n(e) - F_{jn}(e)] (\theta_j - \lambda_{j2}) \right\} K_j$$

$$\beta \max \left\{ 0, \ p_{j1}^e G_n(e) - \lambda_{j1} F_{jn}(e), \ p_{j2}^e G_n(e) - \lambda_{j2} F_{jn}(e), \ [r_j^e G_n(e) - F_{jn}(e)] \theta_j, \ [p_{j1}^e + (\theta_j - \lambda_{j1}) r_j^e] G_n(e) - \theta_j F_{jn}(e), \ [p_{j2}^e + (\theta_j - \lambda_{j2}) r_j^e] G_n(e) - \theta_j F_{jn}(e) \right\} K_j$$

From (34a) and (35), the sum of the terms involving $K_j$ in $V_n(K_1, K_2, W, e)$ is linear in $K_j$ with the coefficient

$$\beta \theta_j F_{jn}(e) + \beta \max \left\{ 0, \ p_{j1}^e G_n(e) - \lambda_{j1} F_{jn}(e), \ p_{j2}^e G_n(e) - \lambda_{j2} F_{jn}(e), \ [r_j^e G_n(e) - F_{jn}(e)] \theta_j, \ [p_{j1}^e + (\theta_j - \lambda_{j1}) r_j^e] G_n(e) - \theta_j F_{jn}(e), \ [p_{j2}^e + (\theta_j - \lambda_{j2}) r_j^e] G_n(e) - \theta_j F_{jn}(e) \right\}$$

$$= \beta \max \left\{ \theta_j F_{jn}(e), \ (\theta_j - \lambda_{j1}) F_{jn}(e) + p_{j1}^e G_n(e), \ (\theta_j - \lambda_{j2}) F_{jn}(e) + p_{j2}^e G_n(e), \ \theta_j r_j^e G_n(e), \ [p_{j1}^e + (\theta_j - \lambda_{j1}) r_j^e] G_n(e), \ [p_{j2}^e + (\theta_j - \lambda_{j2}) r_j^e] G_n(e) \right\} = f_n(j)$$

34
which confirms that $V_n(K_1, K_2, W, e) = f_{1n}(e)K_1 + f_{2n}(e)K_2 + \text{ terms involving } W$.

Optimization (34c) is a linear program with decision variables $x, i_1$, and $i_2$, and the set of its extreme points is $\{(x, i_1, i_2) = (0, 0, 0), (x, i_1, i_2) = (W, 0, 0), (x, i_1, i_2) = (0, W, 0), (x, i_1, i_2) = (0, 0, W)\}$. It is neither infeasible nor unbounded because $W, i_1, i_2$ and $x$ are nonnegative. Thus, at least one of these extreme points is optimal in the linear program. So the value of (34c) is

$$
\max \left\{ 0, 1 - \beta G_n(e), \beta [y_1^e F_{1n}(e) - G_n(e)], \beta [y_2^e F_{2n}(e) - G_n(e)] \right\} W.
$$

(36)

From (34) and (36), the sum of the terms in $V_n(K_1, K_2, W, e)$ that involve $W$ is linear with coefficient

$$
\beta G_n(e) + \max \left\{ 0, 1 - \beta G_n(e), \beta [y_1^e F_{1n}(e) - G_n(e)], \beta [y_2^e F_{2n}(e) - G_n(e)] \right\}
$$

$$
= \max \left\{ \beta G_n(e), 1, \beta y_1^e F_{1n}(e), \beta y_2^e F_{2n}(e) \right\} = g_n(j).
$$

Proof of Theorem 2. Variables $\{q_{jk}, \delta_j\}$ and $\{x, i_j\}$ are optimal in (33) if and only if they are optimal in linear programs (34b) and (34c), respectively. Therefore, the optimal extremal points in (34b) and (34c) correspond to an optimal policy.

From the proof of Theorem 1, the extreme points of linear program (34b) are: $(q_{j1}, q_{j2}, \delta_j) \in \{(0, 0, 0), (K_j, 0, 0), (0, K_j, 0), (0, 0, \theta_j K_j), (K_j, 0, (\theta_j - \lambda_j) K_j), (0, K_j, (\theta_j - \lambda_j) K_j)\}$. Replacing $\delta_j$ with $d_j$ using (29), these six extreme points can be written as $(q_{j1}, q_{j2}, d_j) \in \{(0, 0, 0), (K_j, 0, 0), (0, K_j, 0), (0, 0, 1), (K_j, 0, 1), (0, K_j, 1)\}$. From (35), their respective objective values are: $0, p_{j1}^e G_n(e) - \lambda_{j1} F_{jn}(e), p_{j2}^e G_n(e) - \lambda_{j2} F_{jn}(e), [r_j^e G_n(e) - F_{jn}(e)] \theta_j, [p_{j1}^e + (\theta_j - \lambda_{j1}) r_j^e] G_n(e) - \theta_j F_{jn}(e),$ and $[p_{j2}^e + (\theta_j - \lambda_{j2}) r_j^e] G_n(e) - \theta_j F_{jn}(e)$, which can be written as $B_{j1} / \beta - \theta_j F_{jn}(e), B_{j2} / \beta - \theta_j F_{jn}(e), B_{j3} / \beta - \theta_j F_{jn}(e)$, $B_{j4} / \beta - \theta_j F_{jn}(e), B_{j5} / \beta - \theta_j F_{jn}(e),$ and $B_{j6} / \beta - \theta_j F_{jn}(e)$ using (14). Therefore, if $B_{j1}(e) > \max\{B_{jl} : l = 2, \ldots, 6\}$, then the extreme point $(q_{j1}, q_{j2}, d_j) = (0, 0, 0)$ is optimal. Similar arguments apply to $B_{j2}, \ldots, B_{j6}$. This confirms (15a).

Similarly, the extreme points of linear program (34c) are: $(i_1, i_2, x) \in \{(0, 0, 0), (W, 0, 0), (0, W, 0), (0, 0, W)\}$. Their respective objective values are: $0, 1 - \beta G_n(e), \beta [y_1^e F_{1n}(e) - G_n(e)], \beta [y_2^e F_{2n}(e) - G_n(e)]$. Comparing these values with Table 4 yields (15b).

Proof of Corollary 1. For $n = 0$, $f_{j0}(\cdot) \equiv g_0(\cdot) \equiv 0$. Thus, $F_{j1}(\cdot) = G_1(\cdot) = 0$. From the recursion...
(12)–(13) in Theorem 1, \( f_{j1}(\cdot) \equiv 0 \) and \( g_1(\cdot) \equiv 1 \). Thus, \( f_{jn}(e) \geq f_{jn-1}(e) \) and \( g_n(e) \geq g_{n-1}(e) \) at \( n = 1 \), initializing an induction.

The inductive assumption that \( f_{jn}(e) \geq f_{jn-1}(e) \) and \( g_n(e) \geq g_{n-1}(e) \) for \( n > 1 \) implies \( \mathcal{F}_{jn+1}(e) \geq F_{jn}(e) \) and \( \mathcal{G}_{n+1}(e) \geq \mathcal{G}_n(e) \). Thus, \( \theta_j, r_j^e \geq 0 \) and (16) imply \( B_{j,n+1,1} \geq B_{jn1} \) and \( B_{j,n+1,4} \geq B_{jn4} \). If \( p_{jk}^e \geq 0 \), then \( B_{j,n+1,l} \geq B_{jn} \) for \( l = 2, 3, 5, 6 \) as well. Because \( f_{jn+1}(e) = \max\{B_{j,n+1,l}(e) : l = 1, 2, \ldots, 6\} \) and \( B_{j,n+1,l}(e) (l = 2, 3, 5, 6) \) never achieves the maximum when \( p_{jk}^e < 0 \) (Lemma 1 and Theorem 2), \( f_{jn+1}(e) \geq \max\{B_{jn+1}(e) : l = 1, 2, \ldots, 6\} = f_{jn}(e) \).

Similarly, because \( y_j^e \geq 0 \) (\( j = 1, 2 \)), from (13), \( g_{n+1}(e) = \max\{\beta G_{n+1}(e), 1, \beta y_1^e F_{1,n+1}(e), \beta y_2^e \mathcal{F}_{2,n+1}(e)\} \geq \max\{\beta G_n(e), 1, \beta y_1^e F_{1,n}(e), \beta y_2^e \mathcal{F}_{2,n}(e)\} = g_n(e) \), completing the induction. \( \square \)

### A.3 Proofs of statements in §5

**Proof of Proposition 1.** This proof is similar to that of Theorem 2 and uses optimal solutions of linear programs (34b) and (34c). In (34b), \( 0 \leq \delta_j \leq \theta_j K_j - \lambda_j q_{j1} - \lambda_j q_{j2} \); the upper bound is nonnegative due to (2). Therefore, if \( r_j^e G_n(e) - F_{jn}(e) > 0 \) (the left side is the coefficient of \( \delta_j \) in the objective of (34b)), then optimality implies \( \delta_j = \theta_j K_j - \sum_{k=1}^{2} \lambda_j q_{jk} \). If \( r_j^e G_n(e) - F_{jn}(e) \leq 0 \), then \( \delta_j = 0 \). By definition, \( d_j = \delta_j / (\theta_j K_j - \sum_{k=1}^{2} \lambda_j q_{jk}) \). Thus, \( D_{jn} = 1(r_j^e G_n(e) - F_{jn}(e) > 0) \) is optimal, which confirms (18).

If \( r_j^e G_n(e) - F_{jn}(e) > 0 \), i.e., if \( e \in D_{jn} \), then \( \delta_j = \theta_j K_j - \sum_{k=1}^{2} \lambda_j q_{jk} \) causes (34b) to become

\[
\max_{q_{j1}, q_{j2} \geq 0, q_{j1} + q_{j2} \leq K_j} \sum_{k=1}^{2} \left( p_{jk}^e G_n(e) - \lambda_j K_j \right) q_{jk} + \left( r_j^e G_n(e) - F_{jn}(e) \right) \left( \theta_j K_j - \sum_{k=1}^{2} \lambda_j q_{jk} \right)
\]

which reduces to

\[
\max_{q_{j1}, q_{j2} \geq 0, q_{j1} + q_{j2} \leq K_j} \sum_{k=1}^{2} \left( p_{jk}^e - \lambda_j r_j^e \right) G_n(e) q_{jk} + \theta_j [r_j^e G_n(e) - F_{jn}(e)] K_j.
\]

(38)

Therefore, if \( p_{j1} - \lambda_j r_j^e > 0 \) and \( p_{j1} - \lambda_j r_j^e > i_{j1} r_j^e - \lambda_j r_j^e \), then \( (q_{j1}, q_{j2}) = (K_j, 0) \) is optimal. If \( p_{j2} - \lambda_j r_j^e > 0 \) and \( p_{j1} - \lambda_j r_j^e < i_{j2} r_j^e - \lambda_j r_j^e \), then \( (q_{j1}, q_{j2}) = (0, K_j) \) is optimal. Therefore, if \( e \in D_{jn} \cap \mathcal{C}_j^e \), then \( \hat{Q}_{jkn} = 1 \).
If \( r^e_j G_n(e) - F_{j,n}(e) \leq 0 \), i.e., if \( e \in D^c_{jn} \), then \( \delta_j = 0 \) is optimal and (34b) becomes

\[
\max_{q_{j1}, q_{j2} \geq 0} \sum_{k=1}^{2} \left( p_{jk}^e G_n(e) - \lambda_{jk} F_{j,n}(e) \right) q_{jk}.
\]  

(39)

If \( p_{j1}^e G_n(e) - \lambda_{j1} F_{j,n}(e) > 0 \) and \( p_{j2}^e G_n(e) - \lambda_{j1} F_{j,n}(e) \geq p_{j2}^e G_n(e) - \lambda_{j2} F_{j,n}(e) \), then \((q_{j1}, q_{j2}) = (K_j, 0)\) is optimal. If \( p_{j2}^e G_n(e) - \lambda_{j2} F_{j,n}(e) > 0 \) and \( p_{j1}^e G_n(e) - \lambda_{j1} F_{j,n}(e) < p_{j2}^e G_n(e) - \lambda_{j2} F_{j,n}(e) \), then \((q_{j1}, q_{j2}) = (0, K_j)\) is optimal. Therefore, if \( e \in D^c_{jn} \cap P^k_j \), then \( \tilde{Q}_{jkn} = 1 \). Combining the results from \( e \in D_{jn} \) and \( e \in D^c_{jn} \) yields (17).

In linear program (34c), it is optimal to assign the value \( W \) to the variable with the largest positive coefficient (if any). Thus, an optimal solution consists of

\[
I_{1n}(e) = W \times 1 \left( y_1^e F_{1,n}(e) - G_n(e) > 0, \quad y_1^e F_{1,n}(e) \geq y_2^e F_{2,n}(e), \quad y_1^e F_{1,n}(e) \geq 1 + (1 - \beta) G_n(e) \right),
\]

(40a)

\[
I_{2n}(e) = W \times 1 \left( y_2^e F_{2,n}(e) - G_n(e) > 0, \quad y_1^e F_{1,n}(e) < y_2^e F_{2,n}(e), \quad y_2^e F_{2,n}(e) \geq 1 + (1 - \beta) G_n(e) \right),
\]

(40b)

\[
X_n(e) = W \times 1 \left( 1 - \beta G_n(e) > 0, \quad 1 > y_j^e F_{j,n}(e) + G_n(e) : j = 1, 2 \right).
\]

(40c)

Combining (40) with (15) and (16f-h) yields (19)–(20).

**Proof of Corollary 2.** The assertion follows from (16a), (16g-h), and (18)–(19) in Proposition 1.

**Proof of Corollary 3.** In the i.i.d. case, (16a) is \( D_{jn} = \{ e \in \Omega : r^e_j G_n - F_{j,n} > 0 \} \). Thus, (18) in Proposition 1 becomes \( D_{jn} = 1(r^e_j > F_{jn}/G_n) \).

**Proof of Corollary 4.** Replace \( F_{j,n}(e) \) and \( G_n(e) \) in (16a–e) with \( F_{jn} \) and \( G_n \) and use the subsequent \( D_{jn}, C^k_j \) and \( P^k_{jn} \) in (17) to obtain (21a) and (21b).

**Proof of Corollary 5.** Replace \( F_{j,n}(e) \) and \( G_n(e) \) in (16f–h) with \( F_{jn} \) and \( G_n \) and use the subsequent \( \mathcal{W}^d_n \ (j = 1, 2) \) and \( \mathcal{W}^c_n \) in (19) and (20) to obtain (22).
A.4 Proofs of statements in §6

Proof of Proposition 2. First, use Theorem 1 and nonnegativity of $f_{j,n}(e)$ and $g_{n}(e)$ (for each $j,n,e$) to prove by induction on $n$ that $f_{j,n}(e)$ and $g_{n}(e)$ are nondecreasing in each of the eight components of $e = (p_{j,k}^{e}, r_{j}^{e}, y_{j}^{e} (j, k = 1, 2))$ (for each $j,n$). Second, use the following property of stochastic dominance (Shaked and Shanthikumar 1994, p. 4): $\zeta'$ is stochastically greater than $\zeta$ if and only if $E[\phi(\zeta)] \leq E[\phi(\zeta')]$ for every nondecreasing function $\phi$ for which the expectations exist. Therefore, 

$$f_{j,n}(e) \leq f'_{j,n}(e) \quad \text{and} \quad g_{n}(e) \leq g'_{n}(e) \quad \text{for all } e \in \Omega, j = 1, 2, \text{and } n \in \mathbb{N}.$$  

(41)

The proof of Proposition 3 uses the following lemma.

Lemma 2. If $\Omega$ is a convex set, then $V_{n}(K_{1}, K_{2}, W, \cdot), f_{jn}(\cdot)$, and $g_{n}(\cdot)$ are convex functions on $\Omega$ for each $(K_{1}, K_{2}, W) \in \mathbb{R}_{+}^{3}, j = 1, 2$, and $n \in \mathbb{N}$.

Proof. Mimicking the proof of Theorem 1, initiate an inductive proof that $f_{j,n}(\cdot)$ and $g_{n}(\cdot)$ are convex functions on $\Omega$ ($j = 1, 2$) with $f_{1,0}(\cdot) \equiv 0$ and $g_{0}(\cdot) \equiv 0$ which are convex functions. If $f_{j,n-1}(\cdot)$ and $g_{n-1}(\cdot)$ are convex functions on $\Omega$ ($j = 1, 2$), then all but two of the maximands in (12) and (13) are obviously convex functions. The exceptions are $B_{jn5}(e)$ and $B_{jn6}(e)$ in (14e,f); their convexity appears to require the additional assumption that $p_{jk}^{e}$ is a convex function of $e \in \Omega$ for each $j, k \in \{1, 2\}$. However, $\{p_{jk}^{e} : j, k \in \{1, 2\}\}$ are four of the eight elements of $e$. That is, $p_{jk}^{e}$ is the identity mapping on $\Omega$ which singles out element $p_{jk}^{e}$ in $e$. Therefore, all the maxima are convex functions on $\Omega$. Since point-wise maxima of convex functions are themselves convex, $f_{j,n}(\cdot)$ and $g_{n}(\cdot)$ are convex functions on $\Omega$ ($j = 1, 2$). Therefore, $f_{j,n}(\cdot)$ and $g_{n}(\cdot)$ are convex functions on $\Omega$ ($j = 1, 2$) for all $n \in \mathbb{N}$.

Proof of Proposition 3. The definition of comparative riskiness in §6 is equivalent (Rothschild and Stiglitz (1970)) to $E[\phi(\zeta)] \leq E[\phi(\zeta')]$ for all convex functions $\phi$ for which the expectations exist. Therefore, the proposition is implied by Lemma 2.

Proof of Corollary 6. Let $F'_{jn}$ and $G'_{n}$ denote the expected unit values of capacity and cash after
replacement. Using (22c),

\[
W_n^x = \{(y_1^x, y_2^x): y_1^x < 1/(\beta F_{1n}), y_2^x < 1/(\beta F_{2n}), \beta G_n < 1\},
\]

(42)

\[
W'_n = \{(y_1', y_2'): y_1' < 1/(\beta F'_{1n}), y_2' < 1/(\beta F'_{2n}), \beta G'_n < 1\}.
\]

(43)

Propositions 2 and 3 imply \(F_{jn} \leq F'_{jn}\) \((j = 1, 2)\) and \(G_n \leq G'_n\). Therefore, the inequalities in (43) become more stringent as \(\zeta'\) becomes stochastically larger or riskier.

\[\square\]

A.5 Proofs of statements in §7

Proof of Theorem 3. The proof is essentially the same as that of Theorem 1. The only difference is that in the linear programs in (34b) for \(j = 1, 2\), under the assumption \(\lambda_{j1} > \theta_j > \lambda_{j2}\), the set of extreme points becomes \((q_{j1}, q_{j2}, \delta_j) \in \{(0, 0, 0), (K_j, 0, 0), (0, K_j, 0), (0, 0, \theta_j K_j), (K_j, 0, (\theta_j - \lambda_{j1}) K_j), (0, K_j, (\theta_j - \lambda_{j2}) K_j), (\frac{\theta_j - \lambda_{j2}}{\lambda_{j1} - \lambda_{j2}} K_j, \frac{\lambda_{j1} - \theta_j}{\lambda_{j1} - \lambda_{j2}} K_j, 0)\}\}. Using this new set of extreme points for the linear programs in (34b) and repeating all the steps yields the theorem.

\[\square\]

Proof of Theorem 4. The proof is essentially the same as that of Theorem 2. The seven possible values of \((\bar{Q}_{jkn}, \bar{D}_{jn}: j, k = 1, 2)\) correspond to the seven extreme points of (34b). Their respective objective values correspond to the seven maximands of \(\bar{f}_{jn}\) in (25). Therefore, (28a) is valid.

Similarly, the four possible values of \((\bar{I}_{jn}, \bar{X}_n: j = 1, 2)\) correspond to the four extreme points of (34c). Their respective objective values correspond to the four maximands of \(\bar{g}_{jn}\) in (26). Therefore, (28b) is valid.

\[\square\]