Operational Risk Management: Coordinating Capital Investment and Firm Value

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Abstract: In this paper, we consider a stochastic control framework with a jump process to analyze the impact of large shocks caused by operational risk events on a financial firm's value process. We study capital investments in the infrastructure of a financial firm that aims at mitigating the impact of operational risk events by reducing the frequency of the large shocks. We analyze and compare investment strategies in two settings: one to maximize the firm’s expected utility function over a fixed investment horizon, and the other one to minimize the probability of ruin over an infinite horizon. We characterize the analytical solutions of optimal investment strategies as a function of the firm’s growth rate, the jump process parameters, and a loss reduction efficiency factor. We then proceed to discuss the impact of an insurance contract, and determine the regime in which a financial firm, while carrying insurance, still has an incentive to invest in infrastructure and the regime in which it has not. Our modeling framework, combining a typical operational risk process with stochastic control, suggests a future research direction in operations management and operational risk management.

Key words: operational risk, stochastic control, jump process, investment, firm value, utility, ruin probability, insurance.

1 Introduction

Financial services firms are subject to various types of risks, in particular credit risk, market risk and operational risk. Of these three types of risks, operational risk, commonly referred to as OpRisk, is a type that is very difficult to assess and that can be quite devastating (Cruz (2002)). Typically, it is regarded as the risk of losses due to failures of internal processes, people or systems, or due to occurrences of unexpected external events (see the Basel Committee on Banking Supervision (2006)). According to Basel II, there are seven different categories of OpRisk event types, namely (i) internal fraud, (ii) external fraud, (iii) employment practices and workplace safety, (iv) clients, products and business practices, (v) damage to physical assets, (vi) business disruptions and systems failures, and (vii) execution, delivery and process management. However, the literature on the modeling
of the management problems associated with OpRisk is not very extensive, see Xu et al. (2016) for an overview. Xu et al. (2015) study this problem in a static setting assuming that operational losses can be caused by two forms of workload, namely the workload on personnel and the workload on equipment (or systems). They mainly consider asymptotic budget allocation rules. This is one of the issues considered by the Basel Committee on Banking Supervision (2011). In what follows, we provide a more general setting and consider operational risk events that can take the form of major shocks to financial firms and we consider the management problem in a stochastic control framework.

As claimed by Deloitte (2013), the cost of OpRisk management is often perceived by a firm’s management as being more controllable than the cost of the management of other types of risks. We consider investing in the infrastructure of a financial services firm in order to reduce the frequency of the occurrences of large shocks. Mathematically speaking, we focus in our model on the impact of investments on the Poisson arrival process, and more specifically on its frequency distribution. Our model in general is designed to analyze the implications of the large shocks caused by operational risk events on the firm’s value process, and we address the following specific questions:

- How do risk exposures and investments change as we alter the investment horizon?
- To which components of the firm’s value is the optimal investment most sensitive?
- With the option of operational risk insurance, is it still necessary to invest in infrastructure?

Therefore, our objective is to study the optimal investment strategies in two different settings: one is to maximize the firm’s value over a fixed investment horizon, and the other is to minimize the probability of bankruptcy over an infinite horizon. Moreover, with the general framework we propose, this model can probably be generalized to analyze operational risk in other industries as well, e.g., the aviation industry.

In this paper, we propose a dynamic model to study the implementation of effective controls. As discussed in many industry reports, a simple risk model normally considers three types of controls: (i) preventive, (ii) detective, and (iii) corrective controls (see, for example, American Institute of Certified Public Accountants (AICPA) (2006), Kurt et al. (2007), Moeller (2007), Merchant and Van der Stede (2007)). The Information Security Handbook (2009) states:

“Of the three types of controls, preventative controls are clearly the best, since they minimize the possibility of loss by preventing the event from occurring. Corrective controls are next in line, since they minimize the impact of the loss by restoring the system to the point before the event. .... The least effective form of control, but the one most frequently used, is detective controls - identifying events after they have happened.”
Moreover, as stated in Chernobai et al. (2012), the severity of operational losses typically takes several years after the event occurrence date to materialize. Therefore, in what follows, we only focus on the preventative control which is the control over the operational risk loss frequency as described in Chernobai et al. (2012). Mathematically, we model the preventive control as a capital investment in risk frequency reduction.

To model the investment impact on the stochastic nature of the firm’s value process, we consider the impact of investments on the Poisson arrival process of the operational risk events. The investments affect the frequency distribution of the losses by lowering the rate of the Poisson process according to a given frequency decay factor. We first determine the optimal investment strategy over a finite time horizon that maximizes the expected Hyperbolic Absolute Risk Aversion (HARA) utility assuming a constant risk tolerance level. We find that the optimal investment is a constant ratio strategy that is independent of the investment horizon and market volatility, which provides us with an opportunity to compare it with the optimal investment strategy under the ruin probability minimization. In an infinite time horizon setting, we find that the optimal investment for the ruin probability minimization is also a constant ratio strategy.

We conclude this paper with a discussion on insurance contracts. Whether to issue and how to issue operational risk insurance are nowadays important topics of discussion in the financial services industry. In our work, we discuss whether it is still necessary to invest in infrastructure when an insurance contract is available or whether an insurance contract can incentivize firms to invest in infrastructure because of the high premium costs. We consider an insurance contract that pays a fixed amount each time the company’s value falls below a given threshold, which is a type of event that is comparable to ruin. Assuming a constant risk-free rate, we find the expected present value of the insurance payout at the time of ruin when investments in infrastructure are made in order to delay the expected time of ruin. We find that when the risk-free rate is high, the company will still invest in its infrastructure even with an insurance contract in place.

The main contributions of this paper are three-fold. First, we establish a stochastic control framework, which is often used in the operations management literature, for operational risk management in the financial industry and we obtain closed-form solutions within this framework. Second, we introduce a new type of application to the literature on the operations-finance interface. Third, the insights obtained illustrate the impact of stochastic control type models on a relatively new and important area in the financial industry.

This paper is organized as follows. In the next section, we describe the literature related to our work. In Section 3, we describe our general modeling framework that characterizes the firm’s value process. In Section 4, we study the Hyperbolic Absolute Risk Aversion (HARA) utility maximization problem. We present the optimal solutions and discuss specific examples. In Section 5, we study the ruin probability, and analyze the probability minimization problem. In Section 6, we consider the investment problem with an insurance contract.
contract and discuss when and how much to invest in infrastructure when an insurance contract is in effect. Section 7 presents our conclusions.

2 Related Literature

Our paper draws from three streams of work: (i) operational risk, (ii) stochastic control models in service operations, and (iii) the operations-finance interface.

Compared to the extensive literature on market risk and credit risk, OpRisk in financial services has not received that much attention in the literature. However, over the last decade some books have appeared; see, for example, Cruz (2002), Chernobai et al. (2007), Scharfman (2008), Gregoriou (2009), Shevchenko (2011), and Cruz et al. (2015). These books discuss OpRisk modeling, measurements, managerial aspects, and regulations in fairly general terms. Most of the recent papers on OpRisk tend to deal with the statistical aspects. They tend to focus on the measurement of OpRisk, i.e., the statistical modeling of aggregate OpRisk loss distributions, in order to estimate the Operational Value at Risk (Ops-VaR), see Sparrow (2000), Neil et al. (2005), Bocker and Kluppelberg (2010).

It is only recently that papers on the financial aspects of OpRisk have started to appear, see Leippold and Vanini (2003), Cheng et al. (2007), Jarrow (2008), Brown et al. (2008), Chapelle et al. (2008), Brown et al. (2009), Jarrow et al. (2010). However, similarly to the statistics literature, these papers focus also on the quantification of operational losses. Among these papers, Jarrow (2008) and Jarrow et al. (2010) are two of the papers that have directly inspired our model. Jarrow (2008) is the first paper that considers incorporating operational risk events as large shocks into the firm’s valuation process for asset pricing. However, this paper did not consider the investment problem with the objective of mitigating its operational losses. On the other hand, although operational risk, by its nature, should be closely related to operations management, we have seen so far little work in the OM literature on this problem, except for Hora and Klassen (2013), Xu et al. (2016), Xu et al. (2015). Therefore, one goal of this paper is to take an initial step in filling the gap between OpRisk in finance and operations management. Moreover, even though optimization models with stochastic controls are widely used in finance, most of the optimization models in finance deal with the management of investment risks. In particular, numerous papers have considered optimization issues with respect to market risk and credit risk, e.g., Markowitz (1952) and Black and Litterman (1992). We want to take an initial step in modeling stochastic control problems in the management of operational risks.

A second stream of literature considers stochastic control models in service operations. Stochastic control has been applied to different problems in financial services, see for example the book of Øksendal and Sulem (2005) and Geering et al. (2010). There are also many papers that have been written on this topic, i.e., Leung and Ludkovski (2011), Lorig and Sircar (2014), and Leung and Lorig (2016). Moreover, stochastic control models have
also been applied in many other industries, see Sethi and Thompson (2000). Papers in this stream of literature have studied supply chain management, inventory management, and hospital reimbursement problems, see Huggins and Olsen (2003), van Houtum et al. (2007), Olsen and Parker (2008), Besbes and Maglaras (2012), Ata et al. (2013), Chen et al. (2014). We would like to introduce one new application of stochastic control type of problems with operational risk management and capital allocation. Within the stochastic control framework proposed we obtain analytical solutions.

Finally, the last stream of literature is on the operations-finance interface. Tunca and Zhu (2014) analyze the role and efficiency of buyer intermediation in supplier financing and show that buyer intermediated financing can significantly improve supply chain performance. Yang et al. (2015) examine how a firm’s financial distress and the legal environment regarding the ease of bankruptcy reorganization can alter product market competition and supplier-buyer relationships. There are also papers on other types of risks, i.e., Osadchiy et al. (2015). Our work also lies on the interface between operations management and finance, as we illustrate how proper operations management can mitigate operational risk losses in the financial services industry.

3 Firm Value

In this section, we characterize the firm’s value process and study the impact of investment in infrastructure to reduce operational risk losses. We consider a finite horizon, continuous setting with time \( t \in [0, T] \), where \( T \) is the terminal time. A firm that is operating in this setting, trades financial securities or invests in real assets, and generates a value \( V_t \), which we denote as the firm’s value. We write the firm’s value process \( V_t \) as a logarithmic growth process which has a similar form as the standard Black-Scholes stochastic differential equation (SDE):

\[
dV_t = rV_t dt + \sigma V_t dB_t, \quad V_0 = v > 0.
\]

Or equivalently,

\[
V_t = v \exp \left\{ \left( r - \frac{1}{2} \sigma^2 \right) t + \sigma B_t \right\},
\]

where \( r \) denotes the natural logarithmic growth rate of the firm, and \( \sigma \) the value volatility caused by market uncertainty. We assume that \( B_t \) is the standard Brownian motion process.

Following Jarrow (2008), we consider operational risk losses that follow a jump process which will be incorporated to the firm value process. We can then write the value process of a financial services firm that is subject to operational risk as

\[
V_t = v \exp \left\{ \left( r - \frac{1}{2} \sigma^2 \right) t + \sigma B_t - J_t \right\},
\]

where
\[ J_t = \sum_{i=1}^{N_t} Y_i, \]  

(2)

and \( Y_i \) are i.i.d. \( \mathbb{R}^+ \) valued random variables with the same distribution as a \( \mathbb{R}^+ \) valued random variable \( Y \) with probability density function \( f(y), y > 0 \). When there is no impact from the investment, \( N_t \) is a standard Poisson process with intensity rate \( \lambda > 0 \). In practice, people refer to the distribution of \( N_t \) as the frequency distribution of the operational risk process, and the distribution of \( Y_i \) as its severity distribution.

We now focus on the investment in infrastructure \( I_t \geq 0 \) that may reduce operational risk losses. The investment in infrastructure causes a running cost to the company, while it reduces the operational risk losses. Intuitively this investment \( I_t \) at time \( t \) would be proportional to the firm’s value \( V_t \), as are the losses caused by market uncertainty and operational risks. We therefore define our feedback control \( u(t) = I_t/V_t \geq 0 \), here as the investment ratio. The optimal investment strategy we study in this paper is this investment ratio. In what follows, we will therefore use the term “investment ratio” and “investment strategy” interchangeably. The corresponding investment at time \( t \) can then be written as

\[ I_t = u(t)V_t. \]

We model the reduction in operational risk losses through a reduction in the frequency of loss occurrences. We assume that the investments have no impact on the loss severity distribution for two more reasons (besides those already mentioned in the introduction). First, many operational risk events can only be either 0% or 100% hedged. For instance, if we hire additional process auditors, they either may be able to detect a fat finger mistake (data input error) and thus avoid a potential loss completely, or they may overlook the error and the loss severity does not change at all. Second, the reduction in loss severity is often hard to measure. Therefore, in this paper, we only consider a reduction in loss frequency.

With an investment ratio \( u(t) \) we assume that the intensity at which operational risk events occur is reduced to \( \lambda_t \) which takes the form

\[ \lambda_t = \lambda e^{-\delta u(t)}, \]  

(3)

where \( \delta > 0 \) denotes the exponential decay factor of the rate of occurrences. We can see that the larger the \( \delta \) is, the more effective the investment is. Therefore, one interesting question would be to determine how the factor \( \delta \) affects our investment decisions. One advantage of the form in (3) is that we can guarantee that no matter how large the investment in infrastructure is, the operational risk can never be reduced to zero, because the investment ratio \( u(t) \) then has to go to infinity, which is clearly not optimal for the firm’s revenue. This guarantee is important, because in practice operational risk can never be entirely eliminated. Now considering the diminishing returns with the investments in infrastructure
on operational risk reduction, a firm’s value function is

\[ V_t = v \exp \left\{ \left( r - \frac{1}{2} \sigma^2 \right) t - \int_0^t u(s) ds + \sigma B_t - J_t \right\}, \]  

(4)

For mathematical convenience, denoting \( X_t = \log V_t \) and \( x = \log v \), we now consider the linear transformation

\[ X_t = x + \left( r - \frac{1}{2} \sigma^2 \right) t - \int_0^t u(s) ds + \sigma B_t - J_t. \]  

(5)

Therefore, our model framework, in its most general case could be considered as a jump-diffusion model. In the following sections, we will analyze the optimal investments in this model in two different settings. One is to maximize the firm’s value over a finite time horizon \( T \). The other one is to minimize the ruin probability over an infinite time horizon. We know that managers, in the short run, very much care about an increase in the firm’s value; however, in the long run they also need to protect their firm from ruin.

4 Risk-averse Value Maximization

In this section, we consider the Hyperbolic Absolute Risk Aversion (HARA) utility \( U(V_t) \) of the firm’s value \( V_t \). Following the established literature in finance and economics, a HARA utility function typically describes a decision-maker’s degree of satisfaction with the outcome of an asset value. Decision makers are assumed to make their decisions (i.e., asset allocations) in order to maximize the expected value of the utility functions, see Zevelev (2014).

Let \( \beta \in (0,1) \) denote the risk tolerance level coefficient. Following the standard definition, a utility function is said to exhibit a hyperbolic absolute risk aversion if and only if the level of risk tolerance \( T(V) \), where \( V \) is the firm’s value, is a linear function of \( V \), i.e.,

\[ T(V) = \frac{1}{A(V)} = \frac{V}{1 - \beta}, \]

where \( A(V) \) is the reciprocal of absolute risk aversion. The larger the \( \beta \) is, the more risk tolerant the investor is. Let

\[ A(V) = -\frac{U''(V)}{U'(V)}. \]

Therefore, replacing \( V \) with \( V_t \), the HARA utility function with respect to the firm’s value \( V_t \) is here defined as:

\[ U(V_t) = \frac{V_t^\beta}{\beta}. \]

Now consider maximizing the expected HARA utility, i.e.,
\[
\sup_{u(\cdot) \in \mathcal{U}} \mathbb{E}[U(V_T)] = \sup_{u(\cdot) \in \mathcal{U}} \frac{1}{\beta} \mathbb{E}[e^{\beta X_T}],
\]

where \( \mathcal{U} \) is the set of admissible strategies, consisting of all nonnegative processes \( u(t) \) that are càdlàg (i.e., defined on the real numbers (or a subset of them) and being everywhere right-continuous with left limits), progressively measurable and predictable.

Note that if we consider maximizing the expected present value of the utility, with \( r_f \geq 0 \) being the risk-free rate, we have

\[
\sup_{u(\cdot) \in \mathcal{U}} \mathbb{E}[e^{-r_f T} U(V_T)] = e^{-r_f T} \sup_{u(\cdot) \in \mathcal{U}} \mathbb{E}[U(V_T)].
\]

Therefore, it suffices to study the optimization problem \( \sup_{u(\cdot) \in \mathcal{U}} \mathbb{E}[U(V_T)] \).

Recall that \( X_t = \log V_t \) satisfies the dynamics

\[
dX_t = \left( r - \frac{1}{2} \sigma^2 - u(t) \right) dt + \sigma dB_t - dJ_t,
\]

with \( X_0 = x = \log v \). Therefore, we can denote our value function for the Hamilton-Jacobi-Bellman (HJB) equation as

\[
V(t, x) = \sup_{u(\cdot) \in \mathcal{U}} \mathbb{E}[e^{\beta X_T | X_t = x}].
\]

To maximize the expected HARA utility (6) is equivalent to the maximization problem in (7) with boundary condition \( X_0 = x = \log v \); in other words, (6) is equivalent to \( V(0, x) \). In this section we discuss the optimal solution for this utility maximization problem as well as the structural properties of the optimal solution.

### 4.1 Optimal Investment

We first characterize the optimal investment ratio for problem (7) in the following proposition.

**Proposition 1** The optimal investment strategy \( u(t) \) for problem (7) is a threshold strategy with \( u(t) = u^* \) for all \( t \leq T \), where

\[
u^* = \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\beta} (1 - \mathbb{E}[e^{-\beta Y}]) \right)
\]

when \( \lambda \delta (1 - \mathbb{E}[e^{-\beta Y}]) > 0 \), and \( u^* = 0 \) otherwise. When \( \lambda \delta (1 - \mathbb{E}[e^{-\beta Y}]) > 0 \), the maximum expected utility of the investor at the terminal time \( T \) is

\[
\sup_{u(\cdot) \in \mathcal{U}} \mathbb{E} \left[ \frac{1}{\beta} e^{\beta X_T | X_0 = x} \right] = \frac{1}{\beta} V(0, x) = \frac{v^\beta}{\beta} \exp \left( \left( r - u^* - \frac{1}{\delta} + \frac{1}{2} \sigma^2 (\beta - 1) \right) \beta T \right);
\]
otherwise it is

\[
\sup_{u(\cdot) \in \mathcal{U}} \mathbb{E} \left[ \frac{1}{\beta} e^{\beta X_T} | X_0 = x \right] = \frac{u^\beta}{\beta} \exp \left( \left( r + \frac{\lambda}{\beta} (\mathbb{E}[e^{-\beta Y}] - 1) + \frac{1}{2} \sigma^2 (\beta - 1) \right) \beta T \right). \tag{10}
\]

In Proposition 1, we characterize the optimal investment ratio and the maximum expected utility in (8) and (9) respectively. We see that the optimal investment ratio is a threshold policy that is independent of time \( t \), of the market volatility \( \sigma \), and of the growth rate \( r \). It is also shown that the optimal investment ratio is a constant strategy depending on the parameters of the jump process. We can see that it is optimal to invest a positive amount of capital only if the occurrence rate of the operational risk events exceeds the threshold given in Proposition 1, and that the threshold as well as the amount \( u^* \) are determined by the exponential decay factor \( \delta \), the risk tolerance coefficient \( \beta \), the frequency rate \( \lambda \), and the severity distribution of \( Y \). However, note that the actual optimal investment \( I^*_t \) at time \( t \) is a time dependent variable equal to \( u^* V^*_t \), since it is a function of the firm's value \( V^*_t \) at time \( t \).

The following question can now be raised: how large should the investment ratio be if the probability of extreme events (a very large \( Y \)) is very high? From the optimal investment ratio \( u^* \) in (8), we can see that

\[
u^* = \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\beta} (1 - \mathbb{E}[e^{-\beta Y}]) \right) \leq \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\beta} \right),
\]

where equality is achieved when \( Y = \infty \) with probability 1. Therefore, the optimal investment ratio \( u^* \) is bounded from above by

\[
\bar{u} = \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\beta} \right), \tag{11}
\]

regardless of the severity distribution of \( Y \). We can see here that even if the probability of a large \( Y \) is high, the optimal investment ratio will always be bounded from above by a constant value.

We now proceed to analyze the structural properties of the optimal investment ratio \( u^* \). First, it is easy to see that the optimal investment ratio \( u^* \) increases in the Poisson arrival rate \( \lambda \) of operational risk events. The more frequent the occurrences of operational risk events are (before any investment is made), the more we need to invest.

Moreover, intuitively we would expect that a higher decay factor \( \delta \) would result in a lower investment ratio, because each unit of investment will be more efficiently converted into a loss reduction. However, the following corollary shows that the optimal investment ratio is not a monotone function of the decay factor \( \delta \), but a unimodal function.

**Corollary 1** The optimal investment ratio \( u^* \) given in (8) is a unimodal function of the
decay factor $\delta$. It increases over the range $0 < \delta \leq \bar{\delta}$ and it decreases over the range $\delta > \bar{\delta}$, where

$$
\bar{\delta} = \frac{e^\beta}{\lambda(1 - \mathbb{E}[e^{-\beta Y}])}.
$$

(12)

It is shown in the corollary above that when the decay factor $\delta$ is small, we still would like to invest more. Because with a larger investment ratio we will be able to further reduce the operational risk losses through a more effective reduction ratio. However, when this factor is higher than a given threshold, we may find that one unit of capital investment already can bring about a loss reduction that is large enough; it may therefore be optimal to reduce the investment in order to increase the firm’s value.

We now consider how the optimal investment ratio depends on the tolerance level $\beta$.

**Corollary 2** The optimal investment ratio $u^*$ decreases in $\beta$, i.e., the higher the level of risk tolerance of the investor, the smaller the investment ratio.

The above corollary captures the important role of the risk tolerance level $\beta$. However, it is easy to see that if we consider, instead of a HARA utility function, a value function that is the maximum expected terminal value of the firm, i.e.,

$$
V(t, x) = \sup_{u(\cdot) \in \mathcal{U}} \mathbb{E}[e^{\beta T} | X_t = x],
$$

then, it is simply the special case of $\beta = 1$, and $e^{-r T} \sup_{u(\cdot) \in \mathcal{U}} \mathbb{E}[V_T]$ gives the maximum expected present value of the firm. Hence, we directly obtain the following corollary from Proposition 1.

**Corollary 3** The optimal investment strategy $u(t)$ for problem (13) is a threshold strategy with $u(t) = u^*$ for all $t \leq T$, where

$$
u^* = \frac{1}{\delta} \log \left( \lambda \delta (1 - \mathbb{E}[e^{-Y}]) \right)
$$

(14)

when $\lambda \delta (1 - \mathbb{E}[e^{-Y}]) > 0$, and $u^* = 0$ otherwise.

When $\lambda \delta (1 - \mathbb{E}[e^{-Y}]) > 0$, the maximum expected value of the firm at the terminal time $T$ is

$$
\sup_{u(\cdot) \in \mathcal{U}} \mathbb{E}[V_T | V_0 = v] = v \exp \left( \left( r - u^* - \frac{1}{\delta} \right) T \right);
$$

(15)

otherwise it is

$$
\sup_{u(\cdot) \in \mathcal{U}} \mathbb{E}[V_T | V_0 = v] = v \exp \left( \left( r + \lambda (\mathbb{E}[e^{-Y}] - 1) \right) T \right).
$$

(16)
We can see from (14) that all the structural results we discussed before still hold. The HARA utility function indeed generalizes the expected value maximization problem, and we will see later that it also shows similarity with the ruin probability minimization problem. Next, besides these structural properties, we can see from Proposition 1 that the severity distribution of $Y$ has a considerable effect on the investment ratio.

Consider two random variables $Y_1$ and $Y_2$ with one being stochastically larger than the other. The random variable $Y_1$ is stochastically larger than the random variable $Y_2$, written as $Y_1 \geq_{st} Y_2$, if $\mathbb{P}(Y_1 > a) \geq \mathbb{P}(Y_2 > a)$ for all $a$. This form of stochastic dominance is equivalent to $E[g(Y_1)] \geq E[g(Y_2)]$ for all increasing function $g$ (see Ross (1983)). If $Y_1$ and $Y_2$ are positive random variables with severity distributions $f_1$ and $f_2$ and $Y_1 \geq_{st} Y_2$, then it can be shown easily that the corresponding optimal investment ratios satisfy $u_1^* \geq u_2^*$ (a result that is to be expected).

Compare now two severity random variables $Y_1$ and $Y_2$, such that $E[g(Y_1)] \geq E[g(Y_2)]$ for all convex functions $g$. The random variable $Y_1$ is then said to be larger than the random variable $Y_2$ in the convex ordering sense, denoted as $Y_1 \geq_{cx} Y_2$. (At times it is also said that $Y_1$ is then stochastically more variable than $Y_2$.) If the random variable $Y_1$ is larger than the random variable $Y_2$ in the convex ordering sense, then $E(Y_1) = E(Y_2)$ and $\text{Var}(Y_1) > \text{Var}(Y_2)$, see Ross (1983). Based on the optimal investment ratio in Proposition 1, we can now show that if the severity random variable $Y_1$ is larger than the severity random variable $Y_2$ in the convex ordering sense, the following relationship exists between the corresponding optimal investment ratios $u_1^*$ and $u_2^*$.

**Corollary 4** If $Y_1$ and $Y_2$ are random variables from severity distributions $f_1$ and $f_2$ such that $Y_1 \geq_{cx} Y_2$, then the corresponding optimal investment ratios satisfy $u_1^* \leq u_2^*$.

The results above indicate that if we have two severity distributions that are stochastically ordered, the one with the larger mean requires a larger investment. On the other hand, if we have two random variables that are convexly ordered with equal means, the one with the larger variance requires a smaller investment.

This result may not be very intuitive at a first glance, however, it actually shows us the opportunity cost of such controls. Note that here our investment only affects the loss frequency and not the loss severity, and hence it is possible that the event being prevented may be an event with a small loss amount. We need to consider here the trade-off between investment cost and the loss reduction. In the next subsection, we consider several types of severity distributions that are popular in the financial industry and compare their properties.

### 4.2 Numerical Experiments: Examples of Severity Distributions

In this subsection, we first characterize closed form solutions of optimal investment ratios with three types of distribution functions that are frequently used in the finance world to characterize the loss severity $Y$: the Exponential distribution, the Pareto distribution, and
the Folded Normal distribution. It is known in the financial industry that operational risk loss distribution functions tend to have heavy tails. It is of interest to consider some of the potential effects of the right tails of the distribution functions on the optimal investment ratio. We therefore conduct numerical experiments in order to analyze some potential tail effects of the severity distributions.

**Example 1 (Exponential Distribution)** If \( Y_i \) is exponentially distributed, \( f(y) = \nu e^{-\nu y} \), then the optimal investment strategy is given by

\[
u^* = \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\beta + \nu} \right),\]

when \( \beta < \lambda \delta - \nu \); otherwise \( u^* = 0 \).

**Example 2 (Pareto Distribution)** If \( Y_i \) is Pareto distributed, we have \( f(y) = \frac{\alpha y}{y_0^\alpha} y^{-\alpha - 1} \) for \( y \geq y_0 \) and \( f(y) = 0 \) otherwise, where \( y_0 > 0 \) is the scale and \( \alpha > 0 \) is the shape parameter of the Pareto distribution. The optimal investment strategy is then

\[
u^* = \frac{1}{\delta} \left[ \log \left( \frac{\lambda \delta}{\beta} \right) + \log (1 - \alpha (y_0 \beta)^\alpha \Gamma(-\alpha, y_0 \beta)) \right],\]

when \( \beta < \lambda \delta (1 - \alpha (y_0 \beta)^\alpha \Gamma(-\alpha, y_0 \beta)) \); otherwise \( u^* = 0 \). Here \( \Gamma(s, x) = \int_x^\infty t^{s-1}e^{-t}dt \) is the upper incomplete gamma function.

**Example 3 (Folded-Normal Distribution)** If \( Y_i \) is distributed according to the Folded-Normal distribution, that is, \( f(y) = \sqrt{2} \sigma \sqrt{\pi} e^{-\frac{y^2}{2\sigma^2}} \) for \( y > 0 \), then the optimal investment strategy is given by

\[
u^* = \frac{1}{\delta} \left[ \log \left( \frac{\lambda \delta}{\beta} \right) + \log \left( 1 - 2e^{\frac{2\beta^2}{2}} \Phi(-\beta) \right) \right],\]

when \( \beta < \lambda \delta \left( 1 - 2e^{\frac{2\beta^2}{2}} \Phi(-\beta) \right) \); otherwise \( u^* = 0 \). Here \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2}dy \) is the cumulative distribution function of the standard Normal distribution.

Now, to better understand the impact of the right tails on the optimal investment ratio, we conduct several numerical experiments. First, in order to have a clear understanding of the tail behavior, we control the means and variances of the three types of distributions and compare their probability density functions. To compare the three distributions, we first fix their means. The variances will then be fixed as well since the Exponential distribution has only one parameter. We set the mean \( \mu \) of the three distributions equal to 60; the \( \sigma \) of the three distributions have to be then 60 as well. Figures 1 (a) and (b) depict the PDF of the three distribution functions. Since it is hard to distinguish the tail behaviors when
$x$ gets very large, we magnify the tail segments in Figure 1(b). We can see that the tail of the Pareto distribution is clearly heavier than the tails of both the Exponential and the Folded-Normal distribution. Following such a fixed mean approach, we are able to compare some potential tail impacts on the optimal investment ratios.

![PDF Comparison](image1)

![Tail Comparison](image2)

**Figure 1:** PDF comparison of three distributions with the same mean $\mu = 60$ and s.d. $\sigma = 60$.

Next, we focus on the effects of the four key parameters, namely $\beta$, $\lambda$, $\mu$, $\delta$, on the optimal investment ratio $u^*$. In Figure 2(a), we set $\delta = 0.1$, $\lambda = 100$, $\mu = 1$ and let $\beta$ vary from 0.001 to 0.991. It is clear to see that when $\beta$ is large, the investment ratio under the Pareto distribution with the parameters we choose here, always dominates the investment ratios in the other two cases. We refer to this phenomenon as the right tail dominance effect. Furthermore, the right tail has an impact on the optimal investment ratio through the term $\mathbb{E}[e^{-\beta Y}]$, and if $\beta$ is very small, i.e., $\beta \to 0$, then the tail effects will be reduced. Therefore, the risk tolerance level $\beta$ actually determines the right tail dominance regime. Moreover, from this experiment, we can see that the right tail dominance effect may not always hold with different parameters. In Figure 2(b), we set $\beta = 0.5$, and let $\lambda$ vary from 101 to 220, and keep all other parameters the same as in Figure 2(a). We can see that our investment strategy increases in $\lambda$, and again we have a right tail dominance effect.

In Figure 2(c), we let $\beta = 0.5$, and let $\mu$ vary from 0.5 to 60, while keeping all other parameters as in Figure 2(a). We can see here the optimal investment ratio is increasing concavely in the mean of the severity distribution, and the investment ratio under the Pareto distribution, which has the heaviest tail, always dominates the investments in the other two cases, which again exhibits the right tail dominance effect. However, we can see from the same figure that as $\mu$ gets larger and larger, this effect is diminishing. Because if the average risk event size is large, then the probability of large losses is always high regardless of the tail distribution. Furthermore, the investment ratio increases sharply when $\mu$ is small but
it gets flatter and flatter as $\mu$ increases more and more. This is consistent with our previous discussion that regardless of the distribution of $Y_i$, the optimal investment ratio always converges to a constant value when the the probability of incurring large losses is high.

![Figure 2(a) β vs Investment](image)

![Figure 2(b) λ vs Investment](image)

![Figure 2(c) Mean (µ) vs Investment](image)

![Figure 2(d) δ vs Investment](image)

Figure 2: Impact of key parameters on $u^*$. 

In Figure 2(d), we let $\beta = 0.5$ and let $\delta$ vary from 0.01 to 1, while all other parameters remain as in Figure 2(a). We find that the behavior of the optimal investment ratio $u^*$ is consistent with our Corollary 1, which shows that the investment increases first, and after reaching a certain threshold point $\delta$, it decreases. Furthermore, our numerical results show that it actually decreases convexly, which means that when $\delta$ becomes larger and larger the marginal change of investment decreases. As $\delta$ becomes larger and larger, i.e., $\delta \to \infty$, we see that $u^*$ actually converges to 0, which can also be derived from the investment ratio formula in Proposition 1.

It would be of interest to understand the conditions under which the “right tail dominance” effect holds and see how the numerical results presented above relate to Corol-
lary 4. If $Y_1$ follows a Pareto distribution and $Y_2$ follows an exponential distribution and $\mathbb{E}(Y_1) = \mathbb{E}(Y_2)$, then it is clear that the two random variables are not stochastically ordered. However, they may not be convexly ordered either. (Even though the Pareto distribution has a heavier tail than the exponential distribution, it still may not be larger than the exponential distribution in the convex ordering sense). It is also easy to see that the $\beta$ in our model works as a scale parameter to the random variable of the severity distribution, and hence when $\beta$ varies, the effect of the “right tail dominance” also changes (consider in Figure 2-a, for example, the low values of $\beta$). It is clear that the “right tail dominance” effect only plays a role in certain situations.

5 Ruin Probability

The previous section focuses on a finite horizon capital investment problem with the objective of maximizing the expected HARA utility of the firm over a given horizon. Now instead of utility maximization, over an infinite horizon, an important concern of a firm is its probability of ruin. In this section, we consider an investment problem over an infinite investment horizon and focus on ruin probabilities. The results of this section also lay the foundation for our next section which focuses on insurance.

First, we describe how we define the ruin probability. Note that mathematically the firm’s value process $V_t$ is always strictly positive. The problem of interest is to minimize the probability that the value process $V_t$ drops below a given lower bound which is equivalent to the logarithmic net value process $X_t$ hitting a certain lower bound $L < x$, where $x$ is the initial value of the process $X_t$ at $t = 0$. We consider this problem in an infinite horizon setting as $\inf_{0 < t < \infty} X_t < L$.

We define now the ruin probability of the firm as the probability of hitting the lower bound before it reaches a given (higher) net value level denoted by a constant $M$. Note that $M$ may be infinite, and the problem of $M \to \infty$ would be simpler than the problem we discuss next. We define two types of hitting times, namely

$$\tau_M = \inf\{t > 0 : X_t \geq M\}, \quad \tau_L = \inf\{t > 0 : X_t < L\}. \quad (17)$$

The probability $\mathbb{P}(\tau_L < \tau_M)$ is in what follows referred to as the ruin probability and our goal is to minimize this probability. The corresponding value function is then defined as the minimum probability of hitting this lower bound before hitting the upper bound, i.e.,

$$V(x) = \inf_{u(.) \in U} \mathbb{P}(\tau_L < \tau_M | X_0 = x). \quad (18)$$

We have to deal with two technical challenges in computing the analytical solution of this value function. The first challenge is that when crossing the lower bound $L$ from above, an “overshoot” may occur due to a sudden drop in the value $V_t$ caused by an operational risk event. By “overshoot”, we refer to the process crossing the lower bound $L$ from above due to
a sudden drop caused by an operational risk event, i.e., $X_{\tau_L} < L$ has a positive probability. Let $L - X_{\tau_L}$ denote the shortfall. If we include such an overshoot into our problem we would not be able to find an analytical solution to the ruin probability. Therefore, to avoid this problem, we assume in this section that $Y_i$ is exponentially distributed, say $f(y) = \nu e^{-\nu y}$, where $\nu > 0$. The shortfall $L - X_{\tau_L}$ is then also exponentially distributed, and by the memoryless property of exponential random variables independent of $\mathcal{F}_{\tau_L}$. We can then use this memoryless property of an exponentially distributed random variable to keep our problem analytically tractable.

The second challenge is that when $\sigma \neq 0$, the optimal control problem is not analytically tractable because of difficulties in determining the probability of crossing $L$ by either the jump process or the diffusion process, and in determining the optimal control with the two boundary conditions $V(M) = 0$ and $V(L) = 1$. Although the ruin probability with zero investment can be found in Kou and Wang (2003), a closed form solution for our problem cannot be found via the ruin probability expression given in Kou and Wang (2003).

Therefore, following the discussions in the probability and stochastic modeling literature regarding ruin probabilities in jump-diffusion processes and stochastic controls (see Gaier et al. (2003) and Liang and Guo (2007)), we do not include the Brownian Motion in this section in order to keep our problem analytically tractable. Considering $\sigma = 0$, we only need the boundary condition $V(M) = 0$, because the value of $V(L)$ is not known apriori, and we can solve the control problem analytically. Furthermore, since our paper indeed focuses on the jump process rather than on the impact of the Brownian motion, the simplified model still provides meaningful insights into our problem. Discussions in the literature have focused on diffusion approximations as well as on numerical solutions of stochastic controls in jump-diffusion models, but so far it still has proven to be very hard to obtain analytical solutions. A more general model with $\sigma \neq 0$ may be an interesting direction for future research.

So the logarithm of the Net Value Process for the special case with $\sigma = 0$ is

$$X_t = x + rt - \int_0^t u(s)ds - J_t \quad (19)$$

and the value function $V_t$, assuming $X_t = \log V_t$ and $x = \log v$, is

$$V_t = v \exp \left\{ rt - \int_0^t u(s)ds - J_t \right\}. \quad (20)$$

Recall that we define our feedback control $u(t) > 0$, here as the investment ratio. The corresponding investment at time $t$ is again $I_t = u(t)V_t$.

To start our discussion with regard to the optimal investment strategy, we first characterize the ruin probability when no investments are made, i.e., $u(t) = 0$ for all $t$. Note that the initial condition is still $X_0 = x$.

**Lemma 1** If $u(t) = 0$ for all $t$, then the probability that the logarithmic NPV process $X_t$
will hit the lower threshold \( L \) before it hits the upper threshold \( M \) is

\[
\mathbb{P}(\tau_L < \tau_M) = \frac{\lambda}{r e^{\alpha_0 L} - \lambda e^{\alpha_0 M}} (e^{\alpha_0 x} - e^{\alpha_0 M}),
\]

(21)

where \( \alpha_0 = \lambda/r - \nu \) is defined as the endogeneous risk tolerance level with zero investment.

Note that if \( \alpha_0 = 0 \), then \( \mathbb{P}(\tau_L < \tau_M) \) will be zero divided by zero which turns out to be a finite value using L'Hôpital's rule. We can also see from Lemma 1 that if \( M \to \infty \) and \( \alpha_0 > 0 \), i.e., \( \nu r < \lambda \), then the ruin probability \( \mathbb{P}(\tau_L < \infty) = 1 \). We now characterize the optimal investment strategy assuming \( f(y) = \nu e^{-\nu y} \). Our objective is to minimize the probability

\[
\inf_{u(\cdot) \in U} \mathbb{P}(\tau_L < \tau_M). \tag{22}
\]

The corresponding value function \( V(x) = \inf_{u(\cdot) \in U} \mathbb{P}(\tau_M|X_0 = x) \) satisfies the HJB equation

\[
\inf_{u(t) \geq 0} \left\{ (r - u(t)) V'(x) + \lambda e^{-\delta u(t)} \int_0^{x-L} V(x-y) f(y) dy + \lambda e^{-\delta u(t)} \int_{x-L}^{\infty} f(y) dy - \lambda e^{-\delta u(t)} V(x) \right\} = 0,
\]

with the boundary condition \( V(M) = 0 \). In this case, the process can only cross the lower threshold through a jump. We characterize the optimal investment strategy as follows.

**Proposition 2** The optimal investment is a threshold strategy given by

\[
u^* = \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\alpha_u + \nu} \right) = r - \frac{1}{\delta},
\]

(23)

where \( \alpha_u = \lambda \delta e^{-\nu} - \nu \) when \( r \delta > 1 \). Otherwise, when \( r \delta \leq 1 \), \( \nu^* = 0 \) with \( \alpha_u = \lambda/r - \nu \).

The minimum ruin probability is given by

\[
\inf_{u(\cdot) \in U} \mathbb{P}(\tau_L < \tau_M) = \frac{\alpha_u + \nu}{\nu e^{\alpha_u L} - \lambda e^{\alpha_u M}} (e^{\alpha_u x} - e^{\alpha_u M}).
\]

(24)

From Proposition 2 we see that the optimal investment ratio \( \nu^* \) here is again a constant independent of time, which is a result that is due to the assumptions regarding the parameters. Moreover, we see that if \( M \to \infty \) and \( \alpha_u < 0 \), we have

\[
\inf_{u(\cdot) \in U} \mathbb{P}(\tau_L < \infty|X_0 = x) = \frac{\alpha_u + \nu}{\nu} e^{\alpha_u (x-L)} \leq 1,
\]

(25)

which equals to 1 only if \( \alpha_u = 0 \), i.e., \( \lambda = r \nu \). On the other hand, if \( M \to \infty \) and \( \alpha_u > 0 \), we would always have \( \inf_{u(\cdot) \in U} \mathbb{P}(\tau_L < \infty|X_0 = x) = 1 \).

Furthermore, note that the optimal \( \nu^* \) in Proposition 1 with the \( Y_i \)s being exponentially distributed (see Example 1 in Subsection 4.2) and the optimal \( \nu^* \) in Proposition 2 are similar functions of the parameters \( \beta \) and \( \alpha_u \). In Example 1 of Section 4.2 the \( \nu^* \) turns
positive when the risk tolerance level $\beta$ drops below $\lambda\delta - \nu$. In Proposition 2 the $u^*$ turns positive when $r\delta > 1$ and $\alpha_u$ drops below $\lambda\delta - \nu$. So the $\alpha_u$ in Proposition 2 plays a role that is very similar to the role of the risk tolerance level $\beta$ in Proposition 1. We refer to the $\alpha_u$ as the endogeneous risk tolerance level with respect to investment ratio $u$. This risk tolerance level is referred to as endogeneous since its value is determined by the parameters that characterize the investment environment itself. The higher this risk tolerance level, the lower the $u^*$.

However, there are still some differences between Propositions 1 and 2 because of the $\alpha_u$. In contrast to Proposition 1, we find that the optimal investment ratio in Proposition 2 can now also be written as $r - 1/\delta$, which does not depend on the parameters of the jump process (the $\lambda$ and the $\nu$), but does depend on the natural growth rate $r$. For certain combinations of parameters, we can have $\beta = \alpha_u$.

Note that from Proposition 1 we know that the optimal investment ratio $u^*$ with HARA utility maximization is independent of the investment horizon; we can, therefore, compare the investment ratio directly with the strategy in the ruin probability problem. Now, let $u^*_1$ and $u^*_2$ denote the optimal investment strategies considering the HARA utility maximization over the ruin time and the ruin probability minimization, respectively. We have the following corollary corresponding to three investment regimes w.r.t. different $\beta$ levels and different $r$ levels. (Note that the endogeneous risk tolerance level $\alpha_u = \lambda\delta e^{-r\delta+1} - \nu$ is a function of the growth rate $r$.)

**Corollary 5** Based on the risk tolerance level $\beta$ and the growth rate $r$, we can characterize three investment regimes comparing the two objectives as follows:

(i) (Ruin-sensitive Regime) If $r > \frac{1}{\delta} + \frac{1}{\delta} \log(\frac{\lambda\delta}{\nu})$, then $u^*_1 < u^*_2$;

(ii) (Risk-tolerant Regime) If $r \leq \frac{1}{\delta} + \frac{1}{\delta} \log(\frac{\lambda\delta}{\nu})$, and $\beta \geq \lambda\delta e^{-r\delta+1} - \nu$, then $u^*_1 \leq u^*_2$;

(iii) (Risk-averse Regime) If $r \leq \frac{1}{\delta} + \frac{1}{\delta} \log(\frac{\lambda\delta}{\nu})$, and $\beta < \lambda\delta e^{-r\delta+1} - \nu$, then $u^*_1 > u^*_2$.

We illustrate the relationship in Corollary 5 between $\alpha_u$ and $\beta$ as a function of $r$ in Figure 3. In this figure, we set $\lambda = 1$, $\nu = 0.1$, $\delta = 0.5$, and $r$ goes from 0 to 10. We depict in this figure the curve $\beta = \alpha_u = \lambda\delta e^{-r\delta+1} - \nu$ in order to compare the values of $u^*_1$ and $u^*_2$, and we refer to this curve as the $(\beta = \alpha_u)$-curve. We find that when the growth rate is low, the investment ratio to maximize HARA utility could be either larger or smaller than the investment ratio to minimize the ruin probability depending on the relative value of the risk tolerance level. Also, the two problems can be equivalent for some risk tolerance levels. However, when the growth rate is high, the investment ratio to minimize the ruin probability will always be larger.

We also find that both investment strategies depend on the exponential decay factor $\delta$; in Figure 4 we compare the structural properties of the optimal results for the two problems. We set $r = 60$, $\lambda = 100$, $\nu = 0.1$, $\beta = 0.5$, $L = 0$, $M = 2$, $x = 1$, $T = 1$. We then characterize
the optimal investment ratio curve in Figure 4(a), and the maximum objective function in Figure 4(b). For a fair comparison, instead of minimizing the ruin probability, we use the maximization of the survival probability (i.e., 1 minus the ruin probability) in Figure 4(b).

From Figure 4, we find that for the ruin probability problem, the optimal investment ratio would always increase in the exponential decay factor $\delta$ and approaches the limit; however, for the HARA utility maximization problem, the investment would increase first and then decrease to the limit. It shows that as long as the loss reduction becomes more efficient, increasing the investment can always reduce the ruin probability, but the speed of the increment is decreasing. As for the maximum objective function, we find that the HARA utility function increases slower than the survival probability at the beginning when $\delta$ is small, which is due to the unimodal function of the investment ratio. The turning point in the investment ratio function delays the increase of the utility in $\delta$. 
Finally, we study properties of the optimal solution when $\sigma \neq 0$. As stated before, when $\sigma > 0$ we cannot find a closed form solution for the optimal control. However, we can show that when $\sigma > 0$ the optimal $u^*$ is not a constant anymore.

**Proposition 3** When $\sigma > 0$, the optimal $u^*$ is not constant.

Furthermore, although it is hard to find a closed form solution for the optimal strategy and the corresponding minimized ruin probability when $\sigma > 0$, we can establish upper and lower bounds for the minimized ruin probability $\inf_{u(\cdot) \in \mathcal{U}} \mathbb{P}(\tau_L < \infty)$.

**Proposition 4** The minimized ruin probability is bounded from below by $\mathbb{L}$ and from above by $\mathbb{U}$, where

$$\mathbb{L} = \exp\left(-\frac{2(r - \frac{1}{2} \sigma^2)}{\sigma^2} (x - L)\right);$$ (26)

and

$$\mathbb{U} = \frac{\nu + \alpha_{1,u}}{\nu} \frac{\alpha_{2,u} - \alpha_{1,u}}{\alpha_{2,u} - \alpha_{1,u}} e^{\alpha_{1,u}(x-L)} + \frac{\nu + \alpha_{2,u}}{\nu} \frac{\alpha_{1,u}}{\alpha_{1,u} - \alpha_{2,u}} e^{\alpha_{2,u}(x-L)};$$ (27)

where $\alpha_{1,u} > \alpha_{2,u}$ are the two negative roots of

$$r - u^* - \frac{1}{2} \sigma^2 + \frac{1}{2} \sigma^2 \alpha_u - \frac{\lambda e^{-\delta u^*}}{\alpha_u + \nu} = 0,$$ (28)

and $u^*$ is the constant optimal investment strategy given by the corresponding HJB equation.

### 6 Insurance Contract

Issues concerning financial firms taking out operational risk insurance and the design of such insurance contracts are nowadays important topics of discussion in the finance world. One motivation for a financial firm to take out operational risk insurance is, on one hand, to guarantee liquidity in case operational risk losses do occur; and on the other hand, to improve the risk management of a financial firm given the high insurance premium costs.

We now consider a firm that has an insurance contract in place to cover itself against major shocks; the decision of taking out the insurance has already been taken. The problem we are considering in this section is how the insurance contract affects the firm’s investments in infrastructure in order to reduce the frequency of operational risk events. We consider the simplest form of an insurance contract: a firm receives a payment of a fixed amount, say 1 million USD, at the time when the firm’s value drops below a given threshold. We may assume that such a payment takes place at the time of ruin. If we assume that the risk-free interest rate of such a contract is common knowledge to both sides (the financial firm as well as the insurance company), then the objective of the financial firm is to minimize the premium (face value) of such a contract.
We assume the fixed payment takes place when the firm’s value drops below $L$. To simplify our notation, we assume that the fixed amount to be paid out is 1 million USD. If $r_f \geq 0$ denotes the risk-free rate that is known to both parties, then the expected face value of the insurance payout is $\mathbb{E}[e^{-r_f \tau_L}]$, where $\tau_L$ is the hitting time of the lower threshold $L$ as defined in the previous section. For the same reason as in the previous section, we consider here the problem with $\sigma = 0$ and an exponentially distributed severity variable $Y$.

First, we derive in the following lemma on the expected face value of a contract without any investment in infrastructure.

**Lemma 2** If $u(t) = 0$, then the expected face value of an operational risk insurance contract, given a risk-free interest rate $r_f$ and hitting time $\tau_L$, is

$$
\mathbb{E}[e^{-r_f \tau_L}] = \frac{\gamma_0 + \nu}{\nu} e^{\gamma_0 (z-L)},
$$

where $\gamma_0$ is the unique value in $(-\nu, 0)$, referred to as the risk sensitivity level given by

$$
\gamma_0 = \frac{\lambda - r \nu - \sqrt{(\lambda - r \nu)^2 + 4 r_f r}}{2 r}.
$$

Note that we need to assume $\gamma_0 < 0$ since we need $V(x) \to 0$ as $x \to \infty$. Also, if $\lambda/r < \nu$, then it is easy to see that when $r_f \to 0$, we have $\gamma_0 \to \alpha_0$. It is consistent with the well-known property in applied probability that when the underlying parameter goes to 0, the Laplace transform converges to the ruin probability. Next, as we have discussed before, the financial firm would have an incentive to lower the cost of the insurance contract. One way to achieve this is by investing in infrastructure in order to postpone the expected time of ruin, and hence lower the cost of the insurance contract. However, note that this problem is essentially different from the problem of maximizing the expected ruin time, because the maximized expected ruin time would always be infinity unless one sets the condition $\mathbb{P}(\tau_L < \infty) = 1$.

The value function of this problem can be written as:

$$
V(x) := \inf_{u(\cdot) \in \mathcal{U}} \mathbb{E}[e^{-r_f \tau_L}|X_0 = x].
$$

We again consider the case that the severity distribution is exponential and $\sigma = 0$. We can solve the stochastic control problem and find the optimal $u^*$ explicitly, which again turns out to be a constant. If $\sigma > 0$, then the optimal $u^*$ is not constant.

**Proposition 5** If $f(y) = \nu e^{-\nu y}$, then the optimal investment ratio is given by

$$
u^* = \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\gamma_u + \nu} \right),
$$

when $\lambda \delta > \gamma_u + \nu$ and $u^* = 0$ otherwise. When $u^* > 0$ the $\gamma_u$ is determined by
\[ (r - \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\gamma u + \nu} \right) - \frac{1}{\delta}) \gamma u - r_f = 0. \]  

(32)

The corresponding minimum expected present value of the insurance payout is

\[ \inf_{u(\cdot) \in U} \mathbb{E}[e^{-r_f \tau L} | X_0 = x] = \frac{\gamma u + \nu}{\nu} e^{\gamma u (x-L)}, \]  

and when \( u^* = 0 \), the \( \gamma_u \) in (33) is the unique value in \((-\nu, 0)\) that satisfies

\[ r \gamma_u - \frac{\lambda \gamma_u}{\gamma_u + \nu} - r_f = 0. \]  

(34)

From this proposition, we can see that the optimal investment strategy \( u^* \) is determined by the growth rate \( r \) of the firm and the risk free rate \( r_f \). It is interesting to see how the investment ratio \( u^* \) and the expected face value depend on the risk free rate \( r_f \). In a deterministic setting, the face value is always decreasing in \( r_f \). In the stochastic setting, however, it is not so straightforward to see the property. We now show the structural results in the following corollary.

**Corollary 6** The optimal investment ratio \( u^* \) increases in the natural growth rate \( r \) and in the risk free rate \( r_f \). The expected face value of the insurance contract \( \inf_{u(\cdot) \in U} \mathbb{E}[e^{-r_f \tau L}] \) decreases in the risk free rate \( r_f \).

That the optimal investment ratio \( u^* \) is increasing in both the growth rate \( r \) and the risk free rate \( r_f \) is intuitive. A high growth rate \( r \) of the firm implies that the scale of the firm increases steadily. The value of the firm tends to increase, and the magnitudes of the losses caused by operational risk events increase also. However, the insurance payout (in case the lower threshold \( L \) is reached) remains fixed. So the impact of any insurance payout diminishes over time if the natural growth rate of the firm is high. There is therefore an incentive to invest more in infrastructure in order to reduce the probability of reaching \( L \). A high risk free rate \( r_f \) has a similar effect. The present value of a potential payout by the insurance company diminishes if the risk free rate is high, implying that the effect of a payout on the valuation of the company diminishes. So there is an incentive for the company to invest more in infrastructure.

To better understand the structural pattern of the optimal investment ratio and the face value of the insurance contract in Corollary 6, we conduct numerical experiments varying the risk-free interest rate \( r_f \). We first set the natural growth rate of the firm \( r = 0.1 \), the exponential decay factor \( \delta = 1 \), the risk events arrival rate \( \lambda = 3 \), and the risk severity \( \nu = 4 \), the initial firm value \( x = 1 \), and the lower bound \( L = 0 \). We then vary the risk-free interest rate \( r_f \) from 0.05 to 5. From figure 5(a) we can see that if the interest rate is low, the firm will not have any incentive to invest in the infrastructure anymore, and would just purchase the insurance. However, as the interest rate increases, the firm would invest more
and more in infrastructure to prolong the expected time till ruin and lower the face value of the insurance contract as shown in Figure 5(b).

![Optimal investment](image1.png)

![Face value](image2.png)

(a) Optimal investment
(b) Face value

Figure 5: The impact of $r_f$ on investment ratio and face value

Next, based on our numerical experiments, we characterize in the following corollary the range of $r_f$, where the company would have an incentive to still invest in infrastructure.

**Corollary 7** If a company takes out an insurance contract under a risk free rate $r_f$ to cover operational risk losses, then the company would still have an incentive to invest in infrastructure if

$$r_f > (r - 1/\delta)(\lambda\delta - \nu).$$

### 7 Concluding Remarks

In this paper, we characterize the effects of large shocks caused by operational risk events on a financial firm’s value process. We consider investing in the infrastructure of a financial firm in order to change the stochastic nature of the occurrences of large shocks. More specifically, we analyze the impact of investments on the Poisson process underlying the operational risk events. The investments affect the loss frequency distribution through a decrease in the rate of the Poisson process according to a fixed loss frequency reduction ratio.

To analyze the investment strategies, we first determine the optimal investment over a finite time horizon that maximizes the expected Hyperbolic Absolute Risk Aversion (HARA) utility assuming a constant risk tolerance level. Our paper uses a HARA utility function to measure the risk attitudes, because by carefully choosing the tolerance level, this functional form can cover a broad class of risk measurements. The HARA utility function has been widely used in economics and finance, but so far, we have not seen many applications in
operations management. Based on the closed form solution of the optimal investment strategy, we analyze how the optimal investment ratio depends on the different value parameters of the firm. We then focus on a similar problem considering the minimization of the ruin probability of the firm over an infinite time horizon. We again obtain a closed form solution for the optimal investment ratio, but now under assumptions that are somewhat tighter than the assumptions in our HARA utility maximization problem. A comparison between the closed form solutions to the optimal investment ratios for the two problems shows some interesting similarities and differences. We conclude this paper with a discussion on insurance contracts.

Our work takes an initial step towards modeling operational risk management in a stochastic control framework, with a feedback control to change the dynamics of the jump process. However, there are still several limitations, some of which may serve as interesting areas for future research. In addition to the several directions mentioned above, we mention three other potential research directions. First, our model considers time homogeneous processes, in other words we only consider constant parameters for the drift, diffusion and jumping processes. However, in most cases, as have been observed in the economics and finance literature, the firm value process is a time inhomogeneous process, and hence parameters should be dependent on time and the instantaneous firm value process. Future research may generalize our model and consider a time inhomogeneous process. However, some additional assumptions on the severity distributions may be needed to guarantee a closed form solution. Second, it would be of interest to study how the parameters of the severity distribution affect the optimal investment ratio. From Corollary 4 it follows that a more variable distribution (with a higher second moment) has a lower optimal investment ratio. On the other hand, from the numerical experiments in Section 4 it follows that distributions with heavier tails (with higher third and fourth moments) tend to have higher optimal investment ratios (but note that the risk tolerance level $\beta$ has an effect here as well). Third, we studied operational investments with a fixed insurance contract being in effect; however, we have only considered the simplest possible insurance contract. One could possibly consider more complicated forms of insurance and conduct a more thorough analysis on which contracts to use. That is, deciding whether or not to take out insurance and if an insurance is taken out, the size of the coverage.

Appendix A: Proofs

Proof of Proposition 1. First, we have $V(t, x)$ satisfy the Hamilton-Jacobi-Bellman (HJB) equation:

$$\sup_{u(t) \geq 0} \left\{ \frac{\partial V}{\partial t} + \left( r - \frac{1}{2} \sigma^2 - u(t) \right) \frac{\partial V}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} - \lambda e^{-\delta u(t)} \int_0^\infty [V(t, x) - V(t, x - y)] f(y) dy \right\} = 0,$$

\[24\]
with boundary condition \( V(T, x) = e^{\beta x} \). Solving the above HJB, the optimal \( u^*(t) \) is given by
\[
u^*(t) = \frac{1}{\delta} \log \left( \frac{\lambda \delta \int_0^\infty [V(t, x) - V(t, x - y)] f(y) dy}{\partial V/\partial x} \right).
\]

It is also straightforward to see that if
\[
\frac{\lambda \delta \int_0^\infty [V(t, x) - V(t, x - y)] f(y) dy}{\partial V/\partial x} \leq 1,
\]
then \( u^*(t) = 0 \). From now on, we only consider the case when \( u^*(t) > 0 \). Applying this \( u^*(t) \), we rewrite the HJB equation as:
\[
\frac{\partial V}{\partial t} + \left( r - \frac{1}{2} \sigma^2 - \frac{1}{\delta} \log \left( \frac{\lambda \delta \int_0^\infty [V(t, x) - V(t, x - y)] f(y) dy}{\partial V/\partial x} \right) - \frac{1}{\delta} \right) \frac{\partial V}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} = 0,
\]
with boundary condition \( V(T, x) = e^{\beta x} \). In order to get \( V(0, x) \), we need to find first a solution to the above equation, and then prove that this solution is equal to \( V(t, x) \).

Let \( w(t, x) \in C_b^{1,1} \) denote a solution to the HJB equation. Note here that \( C_b^{1,1} \) implies continuity and differentiability in both time and space as well as being bounded. We first verify that \( w(t, x) = A(t)e^{\beta x} \) is a feasible solution. It is easy to see that \( A(t) \) has to satisfy the ODE
\[
A'(t) + \left[ \beta \left( r - \frac{1}{2} \sigma^2 \right) - \frac{\beta}{\delta} \log \left( \frac{\lambda \gamma}{\beta} \int_0^\infty [1 - e^{-\beta y}] f(y) dy \right) - \frac{\beta}{\delta} + \frac{1}{2} \sigma^2 \beta^2 \right] A(t) = 0
\]
with boundary condition \( A(T) = 1 \). We can solve this ODE and get
\[
A(t) = \exp \left\{ - \left[ \beta \left( r - \frac{1}{2} \sigma^2 \right) - \frac{\beta}{\delta} \log \left( \frac{\lambda \gamma}{\beta} \int_0^\infty [1 - e^{-\beta y}] f(y) dy \right) - \frac{\beta}{\delta} + \frac{1}{2} \sigma^2 \beta^2 \right] (t - T) \right\}.
\]
Therefore, \( w(t, x) \) is indeed a solution of the HJB equation. Next, we need to verify that \( V(t, x) = w(t, x) \). We first show that \( w(t, x) \geq V(t, x) \). Note that for any \( u(t) > 0 \), by Itô's formula, we have
\[
\mathbb{E}[e^{\beta X_T}] = \mathbb{E}[w(T, X_T)]
\]
\[
= w(t, x) + \mathbb{E} \left[ \int_t^T \left[ \frac{\partial w}{\partial s}(s, X_s) - \left( r - \frac{1}{2} \sigma^2 - u(s) \right) \frac{\partial w}{\partial x}(s, X_s) + \frac{1}{2} \sigma^2 \frac{\partial^2 w}{\partial x^2}(s, X_s) \right] ds \right]
\]
\[
- \lambda e^{-\delta u(s)} \int_s^\infty [w(s, X_s) - w(s, X_s - y)] f(y) dy ds \right] \leq w(t, x).
\]
The first equality holds due to the fact that $A(T) = 1$. The inequality holds because

$$
\mathbb{E} \left[ \int_t^T \left( \frac{\partial w}{\partial s}(s, X_s) - \left( r - \frac{1}{2} \sigma^2 - u(s) \right) \frac{\partial w}{\partial x}(s, X_s) + \frac{1}{2} \sigma^2 \frac{\partial^2 w}{\partial x^2}(s, X_s) \right. \right.
\left. - \lambda e^{-\delta u(s)} \int_0^\infty [w(s, X_s) - w(s, X_s - y)] f(y) dy \right] ds
\right.
\left. \leq \mathbb{E} \left[ \int_t^T \sup_{u(s) \geq 0} \left[ \frac{\partial w}{\partial s}(s, X_s) - \left( r - \frac{1}{2} \sigma^2 - u(s) \right) \frac{\partial w}{\partial x}(s, X_s) + \frac{1}{2} \sigma^2 \frac{\partial^2 w}{\partial x^2}(s, X_s) \right. \right.
\left. - \lambda e^{-\delta u(s)} \int_0^\infty [w(s, X_s) - w(s, X_s - y)] f(y) dy \right] ds \right] = 0,
$$

following from the HJB equation. Taking the supremum over $u(t) \in \mathcal{U}$, we obtain $V(t, x) \leq w(t, x)$. Next, let $w(t, x) = A(t)e^{\beta x}$, where $A(t)$ is defined as above; we want to show $V(t, x) = w(t, x)$. We have already shown that $w(t, x)$ is a solution to the HJB equation. Moreover, it is easy to see that $w(t, x) \in C_b^{1,1}$. On the other hand, denoting $X_T^* = \log V_T^*$, we have $V(t, x) \geq \mathbb{E}[e^{\beta X_T^*}]$. If $A$ denotes the infinitesimal generator of the stochastic process, then for any compactly-supported (twice differentiable with continuous second derivative) functions $f(x) : \mathbb{R} \to \mathbb{R}$, we have

$$
Af(x) = (r - \frac{1}{2} \sigma^2) f'(x) + \frac{1}{2} \sigma^2 f''(x) + \lambda \int_0^\infty [f(x - y) - f(x)] p(y) dy.
$$

Then by Dynkin’s formula we have

$$
\mathbb{E}[e^{\beta X_T^*}] = e^{\beta x} + \mathbb{E} \left[ \int_t^T A e^{\beta X_s^*} ds \right]
$$

$$
= e^{\beta x} + \left( \left( r - \frac{1}{2} \sigma^2 - u^*(s) \right) \beta + \frac{1}{2} \sigma^2 \beta^2 - \lambda e^{-\delta u^*(s)} \int_0^\infty [1 - e^{-\beta y}] f(y) dy \right) \int_t^T \mathbb{E} e^{\beta X_s^*} ds,
$$

which implies that $\mathbb{E}[e^{\beta X_T^*}] = A(t)e^{\beta x}$.

Hence, the optimal strategy $u^*(t)$ is a constant and is given by

$$
u^* = \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\beta} \int_0^\infty [1 - e^{-\beta y}] f(y) dy \right) = \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\beta} (1 - \mathbb{E}[e^{-\beta Y}]) \right),
$$

and the value function is given by

$$
V(0, x) = v^\beta \exp \left\{ \left[ \beta \left( r - \frac{1}{2} \sigma^2 \right) - \beta u^* - \frac{\beta^2}{2} + \frac{1}{2} \beta^2 \beta^2 \right] T \right\},
$$

26
provided \(\frac{1}{\beta} \log \left( \frac{\lambda \delta}{\beta} (1 - E[e^{-\beta Y}]) \right) > 0\). Otherwise, \(u^* = 0\) and

\[
V(0, x) = v^\beta E \left[ \exp \left( \beta (r - \frac{1}{2} \sigma^2) T + \beta \sigma B_T - \beta J_T \right) \right] \\
= v^\beta \exp \left( \left[ r + \frac{\lambda}{\beta} (E[e^{-\beta Y}] - 1) + \frac{1}{2} \sigma^2 (\beta - 1) \right] \beta T \right).
\]

This completes the proof of Proposition 1. \(\square\)

**Proof of Corollary 1.** Taking the first order derivative of \(u^*\) in (8) w.r.t. \(\delta\), we have

\[
\frac{\partial u^*}{\partial \delta} = \frac{1}{\delta^2} \left( 1 - \log \left( \frac{\lambda \delta}{\beta} (1 - E[e^{-\beta Y}]) \right) \right).
\]

It is obvious that when \(\delta\) is small, the above formula is nonnegative; when \(\delta\) increases, it becomes negative. Furthermore, it goes to zero when

\[
\delta = \bar{\delta} = \frac{e^\beta}{\lambda (1 - E[e^{-\beta Y}])},
\]

completing the proof of the corollary. \(\square\)

**Proof of Corollary 2.** We can compute that

\[
\frac{\partial u^*}{\partial \beta} = \frac{1}{\delta} \left( -\frac{1}{\beta} + \frac{\int_0^\infty ye^{-\beta y} f(y) dy}{\int_0^\infty [1 - e^{-\beta y}] f(y) dy} \right).
\]

We rewrite it as

\[
\frac{\partial u^*}{\partial \beta} = \frac{-1}{\delta} \frac{\int_0^\infty [1 - e^{-\beta y} - \beta ye^{-\beta y}] f(y) dy}{\int_0^\infty [1 - e^{-\beta y}] f(y) dy}.
\]

For any \(x \geq 0\), define \(F(x) = 1 - e^{-x} - xe^{-x}\). Then \(F(0) = 0\) and \(F(\infty) = 1\). Moreover, \(F'(x) = xe^{-x} > 0\) for any \(x > 0\). Thus, \(F(x) > 0\) for any \(x > 0\). Hence, we have \(\frac{\partial u^*}{\partial \beta} < 0\). \(\square\)

**Proof of Corollary 4.** If \(Y_1 \geq_{cx} Y_2\), then \(E[g(Y_1)] \geq E[g(Y_2)]\) for all convex functions \(g\). We have in (8) that \(e^{-\beta Y}\) is convex in \(Y\). It immediately follows that \(E[e^{-\beta Y_1}] \geq E[e^{-\beta Y_2}]\) and \(u_1^* \leq u_2^*\). \(\square\)

To illustrate the result in Corollary 4, we consider two types of severity distributions: a hyperexponential \(Y_1\) and an exponential \(Y_2\). We have a hyperexponential distribution that has a rate \(\nu' = \infty\) with probability \(1/2\) and a rate \(\nu'' = 1/2\) with probability \(1/2\). And we have an exponential distribution with rate \(\nu = 1\). It is clear that the hyperexponential
distribution is more variable than the exponential distribution. We can show that

\[ E[e^{-\beta Y}] = \frac{1}{2} \frac{\nu'}{\nu' + \beta} + \frac{1}{2} \frac{\nu''}{\nu'' + \beta} = \frac{1 + \beta}{1 + 2\beta} > \frac{1}{1 + \beta} = E[e^{-\beta Y}], \]

and therefore, \( u_1^* < u_2^* \).

**Proof of Lemma 1.** For any \( x \geq L \), if \( V(x) \) is a bounded classical solution to the equation

\[ rV'(x) + \lambda \int_0^{x-L} V(x-y) \nu e^{-\nu y} dy + \lambda \int_{x-L}^{\infty} \nu e^{-\nu y} dy - \lambda V(x) = 0, \]

with boundary condition \( V(M) = 0 \) and also define \( V(x) = 1 \) for any \( x < L \). Then, from Dynkin’s formula, for any \( n \in \mathbb{N} \),

\[ E[V(X_{\tau_L \wedge \tau_M \wedge n})] = V(x) + E \left[ \int_0^{\tau_L \wedge \tau_M \wedge n} AV(X_s) ds \right], \quad (35) \]

where

\[ AV(x) = rV'(x) + \lambda \int_0^{x-L} [V(x-y) - V(x)] \nu e^{-\nu y} dy \]

\[ = rV'(x) + \lambda \int_0^{x-L} V(x-y) \nu e^{-\nu y} dy + \lambda \int_{x-L}^{\infty} \nu e^{-\nu y} dy - \lambda V(x), \]

if \( x \geq L \) and \( AV(x) = 0 \) automatically for \( x < L \) and \( x \geq M \) since \( V(x) = 1 \) for \( x < L \) and \( V(x) = 0 \) for \( x \geq M \). Therefore, \( AV(x) = 0 \) for any \( x \), and

\[ V(x) = E[V(X_{\tau_L \wedge \tau_M \wedge n})]. \quad (37) \]

Since \( V \) is bounded, by bounded convergence theorem, we can let \( n \to \infty \), and we get

\[ V(x) = E[V(X_{\tau_L \wedge \tau_M})] = P(\tau_L < \tau_M | X_0 = x), \quad (38) \]

where we have used fact that \( X_{\tau_L} < L \) and hence \( V(X_{\tau_L}) = 1 \) and \( X_{\tau_M} = M \) and hence \( V(X_{\tau_M}) = 0 \).

We try the ansatz \( V(x) = c(e^{\alpha_0 x} - e^{\alpha_0 M}) \) for the value function \( V(x) \) so that

\[ r c e^{\alpha_0 x} + \lambda c \frac{\nu}{\alpha_0 + \nu} e^{\alpha_0 x} \left[ 1 - e^{-(\alpha_0 + \nu)(x-L)} \right] \]

\[ - \lambda c e^{\alpha_0 K} (1 - e^{-\nu(x-L)}) + \lambda e^{-\nu(x-L)} - \lambda c e^{\alpha_0 x} + \lambda c e^{\alpha_0 K} = 0, \]

which implies that

\[ c = \frac{\alpha_0 + \nu}{\nu e^{\alpha_0 L} - (\alpha_0 + \nu)e^{\alpha_0 M}}, \]

where \( \alpha_0 \) satisfies the equation \( r - \lambda/(\alpha_0 + \nu) = 0 \), which implies \( \alpha_0 = \lambda/r - \nu \). This indeed
gives us the classical solution \( V(x) \) for \( x > L \). Since \( \alpha_0 < 0 \), \( V(x) \) is indeed bounded. \( \Box \)

**Proof of Proposition 2.** The HJB equation for \( V(x) = \inf_{u(\cdot) \in \mathcal{U}} \mathbb{P}(\tau_L < \tau_M | X_0 = x) \) is given by

\[
\inf_{u(t) \geq 0} \left\{ (r - u(t)) V'(x) + \lambda e^{-\delta u(t)} \int_0^{x-L} V(x-y) f(y) dy + \lambda e^{-\delta u(t)} \int_{x-L}^\infty f(y) dy - \lambda e^{-\delta u(t)} V(x) \right\} = 0,
\]

with the boundary condition \( V(M) = 0 \). It can be simplified as

\[
\left( r - \frac{1}{\delta} \log \left( \frac{\lambda \delta \int_0^{x-L} V(x-y) f(y) dy + \lambda \delta \int_{x-L}^\infty f(y) dy - \lambda \delta V(x)}{-V'(x)} \right) - \frac{1}{\delta} \right) V'(x) = 0,
\]

with boundary condition \( V(M) = 0 \). Recall that \( f(y) = \nu e^{-\nu y} \). Try \( V(x) = c(e^{\alpha_u x} - e^{\alpha_u M}) \). We get

\[
\int_0^{x-L} V(x-y) f(y) dy + \int_{x-L}^\infty f(y) dy - V(x) = c \int_0^{x-L} e^{\alpha_u (x-y)} \nu e^{-\nu y} dy + e^{-\nu (x-L)} - ce^{\alpha_u x}
\]

\[
= c \left( \frac{\nu}{\alpha_u + \nu} - 1 \right) e^{\alpha_u x} + \left( 1 - \frac{c \nu e^{\alpha_u L}}{\alpha_u + \nu} + ce^{\alpha_u M} \right) e^{-\nu (x-L)}.
\]

If we set

\[
c = \frac{\alpha_u + \nu}{\nu e^{\alpha_u L} - (\alpha_u + \nu) e^{\alpha_u M}},
\]

then the HJB equation can be reduced to

\[
\left( r - \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\alpha_u + \nu} \right) - \frac{1}{\delta} \right) = 0,
\]

and therefore

\[
V(x) = \frac{\alpha_u + \nu}{\nu e^{\alpha_u L} - (\alpha_u + \nu) e^{\alpha_u M}} (e^{\alpha_u x} - e^{\alpha_u M}),
\]

is a solution to the HJB equation, where we can get \( \alpha_u = \lambda \delta e^{-r \delta + 1} - \nu \). We then need to verify that this indeed gives us the value function \( V(x) = \inf_{u(\cdot) \in \mathcal{U}} \mathbb{P}(\tau_L < \tau_M | X_0 = x) \). Let \( w(x) = 0 \) for \( x > M \) and \( w(x) = 1 \) for \( x \leq L \) and

\[
w(x) = \frac{\alpha_u + \nu}{\nu e^{\alpha_u L} - (\alpha_u + \nu) e^{\alpha_u M}} (e^{\alpha_u x} - e^{\alpha_u M})
\]

for \( L \leq x \leq M \).

We already proved that \( w(x) \) is a solution to the HJB equation. We first show that \( w(x) \leq V(x) \). Note that for any \( u(\cdot) \in \mathcal{U} \) we have, by Itô’s formula and the Optional
Stopping Theorem,

\[ P(\tau_L < \tau_M) = E[w(X_{\tau_M \wedge \tau_L})] = w(x) + E \left[ \int_0^{\tau_M \wedge \tau_L} (r - u(t))w'(X_t) + \lambda e^{-\delta u(t)} \int_0^\infty [w(X_t - y) - w(X_t)]f(y)dy \, dt \right] \geq w(x). \]

Since it holds for any \( u(\cdot) \in U \), we have \( V(x) \geq w(x) \).

On the other hand, \( V(x) \leq P(\tau_L < \tau_M) \), where \( \tau_L \) and \( \tau_M \) denote the hitting times under the admissible strategy \( u^* \). Since \( u^* \) is constant, it is easy to check that indeed \( P(\tau_L < \tau_M) = w(x) \). Hence, we conclude that \( V(x) = w(x) \). Furthermore, we have the optimal \( u^* \) given by

\[ u^* = \frac{1}{\delta} \log \left( \frac{\lambda \delta \int_0^{x-L} V(x-y)f(y)dy + \lambda \delta \int_{x-L}^{\infty} f(y)dy - \lambda \delta V(x)}{-V'(x)} \right), \]

where \( f(y) = \nu e^{-\nu y} \) and the optimal \( V(x) \) is given by

\[ V(x) = \frac{\alpha_u + \nu}{\nu e^{\alpha_u L} - (\alpha_u + \nu)e^{\alpha_u M}}(e^{\alpha_u x} - e^{\alpha_u M}). \]

Therefore,

\[ u^* = \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\alpha_u + \nu} \right) = r - \frac{1}{\delta}, \]

completing the proof of Proposition 2.

\[ \square \]

**Proof of Proposition 3.** Note that for the case \( u(t) = 0 \) and \( \sigma > 0 \) Kou and Wang (2003) determined that

\[ P(\tau_L < \infty | X_0 = x) = \frac{\nu + \alpha_1}{\nu} \frac{\alpha_2}{\alpha_2 - \alpha_1} e^{\alpha_1(x-L)} + \frac{\nu + \alpha_2}{\nu} \frac{\alpha_1}{\alpha_1 - \alpha_2} e^{\alpha_2(x-L)}, \]

where \( \alpha_1 > \alpha_2 \) are the two negative roots of the equation

\[ r - \frac{1}{2} \sigma^2 + \frac{1}{2} \sigma^2 \alpha - \frac{\lambda}{\alpha + \nu} = 0. \]

We can prove by contradiction that \( u^* \) is not constant. If \( u^* \) would have been a constant, then

\[ V(x) = \frac{\nu + \alpha_{1,u}}{\nu} \frac{\alpha_{2,u}}{\alpha_{2,u} - \alpha_{1,u}} e^{\alpha_{1,u}(x-L)} + \frac{\nu + \alpha_{2,u}}{\nu} \frac{\alpha_{1,u}}{\alpha_{1,u} - \alpha_{2,u}} e^{\alpha_{2,u}(x-L)}, \]

where \( \alpha_{1,u} > \alpha_{2,u} \) are the two negative roots of the equation

\[ r - u^* - \frac{1}{2} \sigma^2 + \frac{1}{2} \sigma^2 \alpha_u - \frac{\lambda e^{-\delta u^*}}{\alpha_u + \nu} = 0, \]

30
Therefore, $V(x)$ is of the form $V(x) = c_1e^{\alpha_{1,u}x} + c_2e^{\alpha_{2,u}x}$, and we can then compute that

$$
\frac{1}{\delta} \log \left( \frac{\lambda \delta \int_{x-L}^{x} V(x-y) e^{-\nu y} dy + \lambda \delta \int_{x-L}^{\infty} \nu e^{-\nu y} dy - \lambda \delta V(x)}{-V'(x)} \right) = \frac{1}{\delta} \log \left( \frac{-c_1 \alpha_{1,u} e^{\alpha_{1,u} x} - c_2 \alpha_{2,u} e^{\alpha_{2,u} x} + \left(1 - \frac{c_2 \nu}{\alpha_{2,u} + \nu} e^{\alpha_{2,u} L} - \frac{c_1 \nu}{\alpha_{1,u} + \nu} e^{\alpha_{1,u} L} \right) e^{-\nu (x-L)}}{c_1 \alpha_{1,u} e^{\alpha_{1,u} x} + c_2 \alpha_{2,u} e^{\alpha_{2,u} x}} \right).
$$

If the above quantity was constant, we must have

$$
1 - \frac{c_2 \nu}{\alpha_{2,u} + \nu} e^{\alpha_{2,u} L} - \frac{c_1 \nu}{\alpha_{1,u} + \nu} e^{\alpha_{1,u} L} = 0,
$$

and furthermore

$$
\frac{-c_1 \alpha_{1,u}}{\alpha_{1,u} + \nu} / \frac{-c_2 \alpha_{2,u}}{\alpha_{2,u} + \nu} = \frac{c_1 \alpha_{1,u}}{c_2 \alpha_{2,u}},
$$

which leads to a contradiction since $\alpha_{1,u} \neq \alpha_{2,u}$. Hence we conclude that the optimal $u^*$ is not constant when $\sigma > 0$.

Proof of Proposition 4. Notice that for any $u \in \mathcal{U}$, the best outcome one can achieve is zero operational risk, and the minimum amount of investment on infrastructure is always bounded below by zero, therefore, for any $u \in \mathcal{U}$, the $X_t$ process stochastically dominates the process $\hat{X}_t$, defined as:

$$
\frac{d\hat{X}_t}{dt} = \left(r - \frac{1}{2} \sigma^2\right) dt + \sigma dB_t,
$$

with $\hat{X}_0 = x$. That is, $\hat{X}_t = x + \left(r - \frac{1}{2} \sigma^2\right) t + \sigma B_t$ is a Brownian motion with a positive constant drift if $r > \frac{1}{2} \sigma^2$, which is consistent with the assumption regarding the firm’s value process in our model. Therefore,

$$
\inf_{u(\cdot) \in \mathcal{U}} \mathbb{P}(\tau_L < \infty) \geq \mathbb{P}(\hat{\tau}_L < \infty),
$$

where $\hat{\tau}_L := \inf\{t > 0 : \hat{X}_t \leq L\}$.

The ruin probability for a Brownian motion with a constant drift is well known, see e.g., Karatzas and Shreve (1988):

$$
\mathbb{P}(\hat{\tau}_L < \infty) = \exp \left( - \frac{2(r - \frac{1}{2} \sigma^2)}{\sigma^2} (x - L) \right).
$$

Note that if $r \leq \frac{1}{2} \sigma^2$, then $\mathbb{P}(\hat{\tau}_L < \infty) = 1$.

Next, we show the upper bound. One upper bound is the minimum ruin probability
optimized over all the constant strategies. That is,

\[ \inf_{u(\cdot) \in U} \mathbb{P}(\tau_L < \infty) \leq \inf_{u(\cdot) \equiv u \in \mathbb{R}} \mathbb{P}(\tau_L < \infty). \]

Given \( u(\cdot) \equiv u \), where \( u \geq 0 \) is a constant, we have

\[ dX_t = \left( r - \frac{1}{2} \sigma^2 - u \right) dt + \sigma dB_t - dJ_t, \]

where \( J_t = \sum_{i=1}^{N_t} Y_i \) and \( N_t \) is a Poisson process with intensity \( \lambda e^{-\delta u} \). From Kou and Wang (2003), given \( u(\cdot) \equiv u \), \( u \geq 0 \), we have

\[ \mathbb{P}(\tau_L < \infty) = \nu + \frac{\alpha_{1,u}}{\nu} \frac{\alpha_{2,u}}{\alpha_{2,u} - \alpha_{1,u}} e^{\alpha_{1,u}(x-L)} + \frac{\nu + \alpha_{2,u}}{\nu} \frac{\alpha_{1,u}}{\alpha_{1,u} - \alpha_{2,u}} e^{\alpha_{2,u}(x-L)}, \]

where \( \alpha_{1,u} > \alpha_{2,u} \) are the two negative roots of the equation:

\[ r - u - \frac{1}{2} \sigma^2 + \frac{1}{2} \sigma^2 \alpha - \frac{\lambda e^{-\delta u}}{\alpha + \nu} = 0. \]

If \( u^* \) denotes the optimal \( u \), then

\[ \inf_{u(\cdot) \in U} \mathbb{P}(\tau_L < \infty) \leq \frac{\nu + \alpha_{1,u}}{\nu} \frac{\alpha_{2,u}}{\alpha_{2,u} - \alpha_{1,u}} e^{\alpha_{1,u}(x-L)} + \frac{\nu + \alpha_{2,u}}{\nu} \frac{\alpha_{1,u}}{\alpha_{1,u} - \alpha_{2,u}} e^{\alpha_{2,u}(x-L)}, \]

where \( \alpha_{1,u} > \alpha_{2,u} \) are the two negative roots of the equation:

\[ r - u^* - \frac{1}{2} \sigma^2 + \frac{1}{2} \sigma^2 \alpha_u - \frac{\lambda e^{-\delta u^*}}{\alpha_u + \nu} = 0. \]

Note that finding \( u^* \) by solving the PDE with respect to \( u \) is analytically very complicated and does not provide any additional insight. However, one can easily set up an HJB equation similar as before, and solve it numerically. \( \square \)

**Proof of Lemma 2.** For any \( x \geq L \), if \( V(x) \) is a bounded classical solution to the equation

\[ r V'(x) + \lambda \int_0^{x-L} V(x - y) \nu e^{-\nu y} dy + \lambda \int_{x-L}^{\infty} \nu e^{-\nu y} dy - \lambda V(x) - r f V(x) = 0, \]

and if \( V(x) = 1 \) for any \( x < L \), then we have \( V(x) = \mathbb{E}[e^{-r f \tau_L}] \) for \( x \geq L \). Let us define \( A_t = \frac{\partial}{\partial t} + \mathcal{A} \), where \( \mathcal{A} \) is defined in (36). Then, for such a function \( V(x) \), it is easy to check that we have

\[ A_t(e^{-r f t} V(x)) = -r f e^{-r f t} V(x) + e^{-r f t} \mathcal{A} V(x) = e^{-r f t}[\mathcal{A} V(x) - r f V(x)] = 0. \]
Thus, \( e^{-r_f t} V(X_t) \) is a martingale. Since \( e^{-r_f t} V(X_t) \) is bounded, by optional stopping theorem, we have

\[
\mathbb{E} \left[ e^{-r_f \tau_L} V(X_{\tau_L}) \right] = \mathbb{E} \left[ e^{-r_f \tau_L} \right] = V(x),
\]

where we used fact that \( X_{\tau_L} < L \) and hence \( V(X_{\tau_L}) = 1 \).

We try the ansatz \( V(x) = ce^{\gamma_0 x} \) for the value function \( V(x) \), such that,

\[
rc_0 e^{\gamma_0 x} + \lambda c \frac{\nu}{\gamma_0 + \nu} e^{\gamma_0 x} \left[ 1 - e^{-(\gamma_0 + \nu)(x-L)} \right] + \lambda e^{-\nu(x-L)} - \lambda c e^{\gamma_0 x} - r_c e^{\gamma_0 x} = 0,
\]

which implies that \( c = \frac{\gamma_0 + \nu}{\nu} e^{-\gamma_0 L} \), where \( \gamma_0 \) satisfies the equation

\[
r\gamma_0 - \lambda \frac{\gamma_0}{\gamma_0 + \nu} - r_f = 0.
\]

Since \( V(x) \rightarrow 0 \) as \( x \rightarrow \infty \) and \( V(x) > 0 \), we must have \(-\nu < \gamma_0 < 0\). Indeed, we will show that such a \( \gamma_0 \) exists and is unique. Define

\[
F(\gamma_0) = r\gamma_0 - \lambda \frac{\gamma_0}{\gamma_0 + \nu} - r_f.
\]

Then, \( F(0) = -r_f < 0 \) and also for \( \gamma_0 \downarrow -\nu \), \( \gamma_0 \) is negative and \( \frac{\gamma_0}{\gamma_0 + \nu} \downarrow -\infty \) and \( F(\gamma_0) \uparrow +\infty \). Thus, by continuity of \( F(\gamma_0) \) in \( \gamma_0 \), there exists a \( \gamma_0 \) in \((-\nu, 0)\), such that \( F(\gamma_0) = 0 \). Moreover,

\[
F'(\gamma_0) = r - \frac{\lambda \nu}{(\gamma_0 + \nu)^2};
\]

and we have

\[
F''(\gamma_0) = \frac{2\lambda \nu}{(\gamma_0 + \nu)^3} > 0,
\]

for \( \gamma_0 > -\nu \). Therefore \( F(\gamma_0) \) is convex in \( \gamma_0 \) for \(-\nu < \gamma_0 < 0\), which implies that there exists a unique \( \gamma_0 \in (-\nu, 0) \) so that \( F(\gamma_0) = 0 \). And indeed \( F(\gamma_0) = 0 \) can be reduced to a quadratic equation and can be solved. This indeed gives us the classical solution \( V(x) \). \( \square \)

**Proof of Proposition 5.** The HJB equation for \( V(x) := \inf_{u(.) \in \mathcal{U}} \mathbb{E}[e^{-r_f \tau_L} | X_0 = x] \) is given by

\[
\inf_{u(.) \geq 0} \left\{ (r - u(t)) V'(x) + \lambda e^{-\delta u(t)} \int_0^{x-L} V(x-y) f(y) dy \right. \\
+ \left. \lambda e^{-\delta u(t)} \int_{x-L}^{\infty} f(y) dy - \lambda e^{-\delta u(t)} V(x) - r_f V(x) \right\} = 0,
\]

33
with the boundary condition $V(\infty) = 0$, which can be simplified as

$$
\left( r - \frac{1}{\delta} \log \left( \frac{\lambda \delta \int_{x-L}^{x} V(x-y)f(y)dy + \lambda \delta \int_{x-L}^{\infty} f(y)dy - \lambda \delta V(x)}{-V'(x)} \right) - \frac{1}{\delta} \right) V'(x) - r_f V(x) = 0,
$$

with the boundary condition $V(\infty) = 0$. Recall that $f(y) = \nu e^{-\nu y}$. Try now $V(x) = ce^{\gamma_u x}$. We obtain

$$
\int_{0}^{x-L} V(x-y)f(y)dy + \int_{x-L}^{\infty} f(y)dy - V(x) = c \left( \nu \frac{\gamma_u}{\gamma_u + \nu} - 1 \right) e^{\gamma_u x} + \left( 1 - \frac{c \nu \gamma_u L}{\gamma_u + \nu} \right).
$$

Choose $c = (\gamma_u + \nu)/\nu e^{\gamma_u L}$. The HJB equation can now be reduced to

$$
\left( r - \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\gamma_u + \nu} \right) - \frac{1}{\delta} \right) \gamma_u - r_f = 0,
$$

and therefore

$$
V(x) = \frac{\gamma_u + \nu}{\nu} e^{\gamma_u (x-L)},
$$

is a solution to the HJB equation, where $\gamma_u$ solves (32). The verification argument is similar as before and is omitted here. □

**Proof of Corollary 6.** First note that $u^*$ is decreasing in $\gamma_u$; we only need to study the relationship between $\gamma_u$ and $r$. Taking the first order derivative of (32) with respect to $r$, we get

$$
\gamma_u + \frac{\gamma_u}{\delta(\gamma_u + \nu)} \frac{\partial \gamma_u}{\partial r} + \left( r - \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\gamma_u + \nu} \right) - \frac{1}{\delta} \right) \frac{\partial \gamma_u}{\partial r} = 0.
$$

Since we have

$$
\gamma_u < 0, \quad \frac{\gamma_u}{\delta(\gamma_u + \nu)} < 0, \quad \text{and} \quad \left( r - \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\gamma_u + \nu} \right) - \frac{1}{\delta} \right) < 0,
$$

we get $\partial \gamma_u / \partial r < 0$, otherwise the left hand side of the above equation would be negative. Therefore, we conclude that $u^*$ increases in $r$.

It is easy to see that $u^* = \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\gamma_u + \nu} \right)$ decreases as $\gamma_u$ increases. Next, taking the first order derivative of (32) with respect to $r_f$, we get

$$
\frac{\partial \gamma_u}{\partial r_f} = \frac{r_f}{\gamma_u + \frac{1}{\delta(\gamma_u + \nu)}} < 0,
$$

since $-\nu < \gamma_u < 0$. Hence, $u^*$ is increasing in the risk free rate $r_f$. It is then also easy to
see that

\[
\inf_{u(\cdot) \in \mathcal{U}} \mathbb{E}[e^{-rf \tau_L}] = \frac{\gamma_u + \nu}{\nu} e^{\gamma_u (x - L)}
\]

is increasing in \(\gamma_u\) and thus decreasing in \(rf\). \(\Box\)

**Proof of Corollary 7.** We can rewrite the optimal investment ratio \(u^*\) presented in Proposition 5 as

\[
u^* = r - \frac{1}{\delta} - \frac{rf}{\gamma_u} = \frac{1}{\delta} \log \left( \frac{\lambda \delta}{\gamma_u + \nu} \right).
\]

Therefore, the condition \(u^* = 0\) is equivalent to

\[
0 \leq \frac{rf}{\gamma_u} + \frac{1}{\delta} - r,
\]

and \(\gamma_u \leq \lambda \delta - \nu\). Combining these two conditions and the inequality \(\gamma_u < 0\), we have the condition stated in the corollary. \(\Box\)

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