Uncertainty, Investor Sentiment, and Innovation*

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Abstract

We develop a theory of investor sentiment based on uncertainty aversion. In our paper, sentiment is not based on erroneous beliefs disjoint from economic fundamentals – in contrast, it depends on agents’ uncertainty on fundamentals. We develop our approach in the context of innovation. Investors must typically decide whether to fund an innovative project with very limited knowledge of the odds of success, a situation best described as “Knightian uncertainty.” We show that uncertainty-averse investors are more optimistic on an innovation if they can also make contemporaneous investments in other innovative ventures, resulting in innovation waves. Innovation waves occur when there is a critical mass of innovative companies and are characterized by stronger investor sentiment, higher equity valuation in the technology sector, and “hot” IPO and M&A markets. Our model can explain the emergence of sector-specific booms that are uncorrelated with aggregate economic factors such as the overall stock market or the general level of economic activity.

Keywords: Investor Sentiment, Ambiguity Aversion, Innovation, Hot IPO Markets

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“Most, probably, of our decisions to do something positive, the full consequences of which will be drawn out over many days to come, can only be taken as the result of animal spirits—a spontaneous urge to action rather than inaction, and not as the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities.”


Innovation is arguably one of the most important value drivers in modern corporations and a key source of economic growth (Solow, 1957). There are times when innovation is stagnant, but other times when technology leaps forward in innovation waves. These spurts of innovation activity are often associated with higher stock market valuations for technology firms, with strong (or “optimistic”) investor sentiment (Shiller 2000, Perez, 2002, and Baker and Wurgler, 2007), and an active market for mergers and acquisitions (Bena and Li, 2014). Interestingly, such flare-ups in stock market valuations are often confined to specific technology sectors, with relatively little correlation with the broader equity markets.1

Time-varying investor sentiment is often identified as an important cause of the alternating periods of booms and busts in an economy. For example, Shiller (2000) describe the boom in the technology sector of 1999-2000 as a temporary episode of “irrational exuberance” by investors. While investor sentiment seemingly plays such an important role in the economy and financial markets, we still have a limited understanding of its economic drivers.2

In this paper we develop a theory of investor sentiment in the context of innovation. Innovation, by its very nature, is characterized by a very limited knowledge of the probability distributions relevant for the innovation process, a situation best described as “Knightian uncertainty” (Knight, 1921). In this situation, investors must typically decide whether to fund an innovative project with very limited knowledge of the odds of success.

We show that uncertainty aversion can cause investor sentiment to fluctuate (endogenously) between periods of pessimism toward innovative ventures (“cold markets”) and periods of relatively greater optimism (“hot markets”). These alternating phases of investor sentiment spur innovation

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1For example, in the boom years of 1998-2000, the NASDAQ index, which is dominated by technology companies, more than doubled while the general market, as measured for example by the S&P500 index, remained stable.

waves associated with higher stock market valuations, greater volumes of Initial Public Offerings (IPOs), and an active M&A market for technology companies. In addition, our approach explains the emergence of periods of sector-specific hot and cold sentiment uncorrelated with aggregate economic factors, such as overall economic activity or stock market performance.

There are many reasons why innovation develops in waves. These include fundamentals such as random scientific breakthroughs with externalities and technological spillovers. In this paper, we focus on the interaction between financial markets and the incentives to innovate. We argue that innovation waves can be the product of changing investor sentiment due to uncertainty aversion.

We study an economy with multiple entrepreneurs endowed with project-ideas. Project-ideas are risky and, if successful, may lead to innovations. The innovation process consists of two stages. In the first stage, entrepreneurs must decide whether or not to invest personal resources, such as effort, to innovate. If the first stage is successful, further development of the innovation requires additional investment in the second stage. Entrepreneurs raise funds for the second-stage investment by selling shares of their firms to uncertainty-averse investors. The second stage of the innovation process is uncertain: outside investors are uncertain of the exact distribution of the residual success probability of the innovation process. We model uncertainty aversion by assuming outside investors maximize Minimum Expected Utility (MEU), as in Gilboa and Schmeidler (1989).

An important implication of uncertainty aversion, which plays a key role in our paper, is that probabilistic assessments (or “beliefs” in the sense of de Finetti, 1974) held by an uncertainty-averse investor on future returns are not uniquely determined by a single prior but, rather, are determined endogenously as the solution of a minimization problem. In addition, and importantly, uncertainty-averse investors prefer to hold an uncertain asset if they can also hold other uncertain assets in their portfolios. This happens because, by holding a combination of uncertain assets in a portfolio, investors can lower their overall exposure to the sources of uncertainty in the economy, a property that is known as “uncertainty hedging” (see Epstein and Schneider, 2007, and 2010). In addition, because of uncertainty hedging, an investor will also hold (weakly) more favorable probabilistic assessments toward an innovation – and thus be relatively more “optimistic” – if he/she is able to

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3 Uncertainty hedging is a direct consequence of the “uncertainty aversion axiom” of Gilboa and Schmeidler (1989). Uncertainty hedging is the analog to traditional risk diversification in standard portfolio theory, but in the context of uncertainty on the true probability distributions that govern the random variables relevant to the decision maker.
invest in other innovations as well. We will refer to the (endogenous) probabilistic assessments held by investors on the success of innovations as characterizing their “sentiment.” Importantly, in our model, investors’ sentiment is not based on erroneous beliefs disjoint from economic fundamentals but, in contrast, it depends on their uncertainty on fundamentals.

In our economy, uncertainty-averse investors are (relatively) more optimistic and are willing to pay more for equity in a given entrepreneurial firm when other entrepreneurs innovate as well. This happens because, by investing in a portfolio of (possibly independent) R&D processes, uncertainty-averse investors reduce their exposure to joint (uncertain) event that all such R&D efforts will fail. Thus, investments in different innovative companies, by reducing exposure to the underlying uncertainty, are effectively complements and are associated with more optimistic probability assessments (i.e., stronger investor sentiment) and higher valuations.

The key feature of our model is that uncertainty aversion creates a strategic complementarity between innovative activities which can generate innovation waves. An innovation wave occurs when the number of innovators reaches critical mass. Arrival of innovation opportunities in the economy may be random and due to exogenous technological progress. We argue that such technological advances, while seeding the ground for an innovation wave, may not be sufficient to ignite one. Rather, an innovation wave will start when a critical mass of innovators is attained, which spurs a “hot” market for innovative companies. Thus, innovation waves are characterized by strong investor sentiment and a wave of “rational exuberance” with higher equity market valuations. Interestingly, equity market “booms” in technology markets materialize in our model even in an otherwise stationary environment: these booms are beneficial since they spur valuable innovation.

The channel we propose, based on uncertainty aversion, differs substantially from traditional “neoclassical” explanations. Shleifer (1986) argues that innovations in one sector have a positive externality on innovators in other sectors, because of their positive effect on aggregate demand.

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4 For example, there is currently considerable uncertainty on the technical difficulties related to the development of self-driving cars, an area in which several companies are engaged in substantial R&D effort. Clearly, there is very little information on the true odds of discovery that are relevant for each producer. Most likely, however, one of such innovators will generate a workable technology that will become the industry standard. By investing in a portfolio of companies, investors (such as VCs) limit their exposure to the state of the world where all projects fail, and increase the exposure to the possibility of having a successful project in their portfolio.

5 Note that, by design, we ignore the potential adverse effect on innovation incentives due to competitive pressure of the product market. The effect of competition (and patent races) on innovations incentives has been extensively examined in the literature (see, for example, Aghion et al, 2005, among many others).
Similar to our paper, innovators prefer to postpone their innovation to periods of time when other innovators undertake theirs, generating self-fulfilling boom-and-bust cycles. Different from our paper, the boom-and-bust cycle occurs through the effect that a favorable aggregate macroeconomic environment has on the value of innovation, while in our model waves may be localized in a specific sector, even if the overall economy is not booming. Thus, our model can explain the boom in the biotech market in 1989-1992, which occurred around the economic recession of early 1990’s and within a relatively calmer overall stock market (see Booth, 2016).

Acemoglu and Zilibotti (1997) argue that at the early stages of economic development, when capital is critically limited, the presence of project indivisibilities caps the range of risky investment projects that will be implemented in an economy, reducing the benefits of risk sharing, thus discouraging investment in risky assets. Our paper differs from Acemoglu and Zilibotti in many important dimensions. First, our results do not rely on the limited supply of capital but, rather, are driven by the random arrival of innovative ideas in the economy. In our model capital is abundant and, thus, it is better suited to explain innovation waves in more mature economies, while Acemoglu and Zilibotti is better suited to explain the random growth rates of economies at the earlier stages of their development. Second, in Acemoglu and Zilibotti a “wave” (or, perhaps a “crash”) may occur as the outcome of negative production shocks that reduce capital available in the economy and, thus, restricts its diversification opportunities, setting back its growth path. In contrast, in our paper, a wave ends when the pipeline of innovations that were initiated in that wave are completed, and a new wave starts when a new critical mass is achieved.

More generally, traditional portfolio-diversification arguments can only generate innovation waves and high stock market valuations as the outcome of a reduction of the economy-wide market price of risk. In this case, innovation waves will necessarily be associated with economy-wide equity market booms. Our approach, in contrast, can explain the apparent “boom and bust” behavior concentrated in technology sectors, such as the Life Sciences and the Information Technology, where hot periods alternate with cold periods in innovation rates, merger activity and asset valuations. This divergent behavior between a technology sector and the general market would be difficult to reconcile on the basis of risk aversion alone.

Observational equivalence between models based on uncertainty aversion and those based on standard risk-averse models is an issue discussed in the literature (see, for example, Maenhout, 2004, and Skiadas, 2003, among others).
Our paper also has implications for the impact of M&A activity and, more generally, of corporate ownership structure on innovation. In the new channel we propose, mergers of innovative firms create synergies and spur innovation. Positive synergies in an acquisition are endogenous, and are the direct outcome of the beneficial spillover on the probabilistic assessments of future returns on innovation due to uncertainty aversion. Our model also predicts that merger activities involving innovative firms will be associated with strong investor sentiment and, thus, greater valuations.

**Relation to Existing Literature.** Our paper contributes insights from uncertainty aversion to three strands of literature. First, and foremost, our paper belongs to the rapidly expanding literature on determinants of innovation and innovation waves (see Fagerberg, Mowery and Nelson, 2005, Chemmanur and Fulghieri, 2014, and He and Tian, 2017). The critical role of innovation and innovation waves in modern economies has been extensively studied at least since Schumpeter (1939) and (1942), Kuznets (1940), Schmookler (1966), and, more recently, Kleinknecht (1987), Aghion and Howitt (1992), and Klepper (1996). In early research, which focused mostly on the technological “fundamentals” behind innovation, innovation waves are driven by a technological breakthrough that affects an entire sector, such as positive spillover effects across different technologies.

Recent research focuses on the link between innovation waves, availability of financing, and stock market booms. Scharfstein and Stein (1990) suggest reputation considerations by investment managers may induce them to herd, facilitating the financing of technology firms. Gompers et al (2008) show that periods of high stock market valuations are also associated with greater fund raising by VCs. Nanda and Rhodes-Kropf (2013) find that in “hot markets” VCs invest in riskier and more innovative firms. Nanda and Rhodes-Kropf (2016) argue favorable financial market conditions reduce refinancing risk for VCs, promoting investment in more innovative projects. A positive effect of investor sentiment on innovation is documented in Aramonte and Carl (2018).

Our work also contributes to the emerging literature on uncertainty aversion in financial decision making and asset pricing. Uncertainty aversion has been proposed as an alternative to Subjective

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7 Hart and Holmstrom (2010) develop a model where mergers create value by internalizing externalities, such as coordinating on a technological standard.

8 Gompers and Lerner (2000) find higher venture capital valuations are not linked to better success rates of portfolio companies. Perez (2002) shows technological revolutions are associated with “overheated” financial markets.

Expected Utility (SEU) to describe decision making in cases where agents have limited information on probability distributions. This stream of research was motivated by a large body of work documenting important deviations from SEU and the classic Bayesian paradigm (see Etner, Jeleva, and Tallon, 2012). While the degree of ambiguity aversion may vary across treatments and subjects, the presence of ambiguity aversion appears to be a robust experimental regularity. Interestingly, Chew, Ratchford, and Sagi (2018) document that ambiguity-averse behavior is particularly relevant among more educated (and analytically sophisticated) subjects.\(^{10}\)

The second stream of literature is the recent debate on the links between technological innovation and stock market prices. Nicholas (2008) shows that an important driver of the stock market run-up experienced in the American economy in the late 1920’s was the strong innovative activity by industrial companies which affected the market valuation of “knowledge assets.”

Two closely related papers are Pastor and Veronesi (2005) and (2009). The first paper argues that IPO waves can be the outcome of a change in the “fundamentals” characterizing a firm and its environment, such as an exogenous decrease in the market expected return. In our paper, in contrast, IPO waves can also occur in a stationary environment, and endogenously occur when stock market valuations are high. The second paper argues that technological revolutions can generate dynamics in asset prices in innovative firms observationally similar to assets bubbles followed by a valuation crash. Their paper argues that this “bubble-like” behavior of stock prices is the rational outcome of learning about the productivity of new technologies, where the risk is essentially idiosyncratic, followed by the adoption of the new technologies on large scale, where the risk becomes systematic. Our paper proposes a new explanation for the link between innovative activity and stock market booms. In Pastor and Veronesi (2009) stock market booms (and subsequent crashes) are the outcome of the changing nature of risk that characterizes technological revolutions, from idiosyncratic to systematic, and its impact on discount rates. In our model, periods of strong inno-

\(^{10}\)Uncertainty aversion has also been shown to be an important driver of asset pricing, providing an explanation for observed behavior that would otherwise be puzzling in the context of SEU. Anderson, Ghysels, and Juergens (2009) find stronger empirical evidence for uncertainty than for traditional risk aversion as a driver of cross-sectional expected returns. Jeong, Kim, and Park (2015) estimate that ambiguity aversion explains up to 45% of the observed equity premium. Boyarchenko (2012) shows that the sudden increase in credit spreads during the financial crisis can be explained by a surge in uncertainty faced by uncertainty-averse market participants. Dimmock et al. (2016) show that ambiguity aversion helps explain several household portfolio choice puzzles, such as low stock market participation, low foreign stock ownership, and high own-company stock ownership.
ative activity are accompanied by high valuations because innovation waves are, in equilibrium, associated with more optimistic expectations on cash flows from innovative firms. Thus, our model, which focuses on expected cash flows, complements theirs, that focus on discount rates.

The third stream of literature focuses on the drivers of merger waves and the impact of M&A activity – and, more generally, of ownership structure – on incentives to innovate. High stock market valuations are associated with strong M&A activity in merger waves (Maksimovic and Phillips, 2001, and Jovanovic and Rousseau, 2001). Rhodes-Kropf and Viswanathan (2004) argue that such correlation is the outcome of misvaluation of the true synergies created in a merger when the overall market is overvalued. The impact of M&A activity on corporate innovative activity has been documented by several empirical studies. Phillips and Zhdanov (2013) show that a firm’s R&D expenditures increase in periods of strong M&A activity in the same industry. Bena and Li (2014) argue that the presence of technological overlap between two firms innovative activities is a predictor of the probability of a merger between firms.\textsuperscript{11}

In our model we are able to jointly generate the observed positive correlations between stock market valuations, the level of M&A activity, and innovation rates. Specifically, our paper creates a novel direct link between stock price valuations, M&A activity, and greater innovation rates based on investors’ uncertainty aversion. Endogeneity of probabilistic assessments creates an externality between innovations that is at the heart of synergy creation in mergers of innovative companies. This externality results in greater innovation rates and innovation waves that are characterized by strong investor sentiment and greater stock market valuations.

Finally our paper is linked to the recent literature on investor sentiment and stock market valuations. Baker and Wurgler (2007) suggest that investor sentiment, in the form of “optimism or pessimism about stocks,” is likely to affect more those stocks that are harder to evaluate, that is, stocks that are surrounded by more uncertainty.\textsuperscript{12} These include stocks of companies that are

\textsuperscript{11} Bernstein (2015) documents that in the three years after their IPO, firms engage in strong M&A activity, acquiring a substantial number of patents. Sevilir and Tian (2012) show that acquiring innovative target firms is positively related to acquirer abnormal announcement returns and long-term stock return performance. The importance of the presence of technological overlaps between acquiring firms and targets is confirmed by Seru (2014), which finds that innovation rates are lower in diversifying mergers, where the technological benefits of a merger are likely to be absent. Entezarkheir and Moshiri (2016) show mergers are more likely among innovative firms.

\textsuperscript{12} Shefrin (2008) considers investor sentiment as nonfundamental distortions in the stochastic discount factor. Jouini and Napp (2006) and (2007) show that individual attitudes toward the economy’s prospects aggregate up. Dumas, Kurshev, and Uppal (2009), considering how rational investors should behave when other traders overreact to new information, show that the sentiment is not traded away and that overconfident traders survive a long time. See also
younger, smaller, or with extreme growth potential, such as highly innovative companies.

The paper is organized as follows. In Section 1, we introduce the basic model of our paper. In Section 2, we derive the paper’s main results. Section 3 examines the impact of mergers on the incentives to innovate. Section 4 develops the dynamic version of our model. Section 5 discusses in more detail certain critical assumptions of our model and possible extensions, as well as the main empirical implications of our model. Section 6 concludes. All proofs are in the appendix.

1 The Basic Model

In the basic model, we study a two-period economy with three dates, \( t \in \{1, 2, 3\} \). The economy has two classes of agents: investors and entrepreneurs. Entrepreneurs are endowed with unique project-ideas that may lead to an innovation. Project-ideas are risky and require an investment both at the beginning, \( t = 1 \), and at the interim date, \( t = 2 \). If successful, project-ideas generate a valuable innovation at \( t = 3 \). If a project-idea is unsuccessful, it has zero payoff. For simplicity, we assume initially that there are only two entrepreneurs, each endowed with a unique project-idea, denominated by \( \tau \), with \( \tau \in \{A, B\} \).

Entrepreneurs are penniless and require financing from investors. There is a unit mass of investors, with \( w_0 \) units of an initial endowment. The initial endowment can either be invested in one (or both) of the two types of project-ideas or it can invested in another (risky) asset. Investment in the other asset, which can be interpreted as the market portfolio, can be made at either \( t = 1 \) and \( t = 2 \), and yields a unit gross expected return per period (a normalization).

We assume that project-ideas are specific to each entrepreneur: an entrepreneur can invest in only her own type of project-idea. This assumption captures the notion that project-ideas are creative innovations that can be successfully pursued only by the entrepreneur who generated them.

The innovation process is structured in two stages. To implement a project-idea, and thus to “innovate,” an entrepreneur must first make at \( t = 1 \) a fixed, non-pecuniary investment \( k_\tau \). We interpret the initial investment as representing all the preliminary personal effort that the entrepreneur must exert in order to generate the idea. We will often refer to the initial personal

Barone-Adesi, Mancini, and Shefrin (2017). All of these involve sentiment about the aggregate state, which could result in innovation waves from a decreased market price of risk, as in Pastor and Veronessi (2005).
investment, \( k_r \), as the “discovery cost” necessary for the innovation. We will denote the decision made by entrepreneur \( \tau \) of whether or not to incur such cost with \( d_\tau \in \{0, 1\} \), where \( d_\tau = 1 \) indicates that the personal investment is made, and \( d_\tau = 0 \) otherwise.

The innovation process is inherently risky, and we denote with \( q_\tau \) the success probability of the first stage of the process. We allow the first-stage success probabilities of the two project-ideas to be correlated. Specifically, we assume that the probability that both entrepreneurs are successful in the first stage is \( q_A q_B + r \), while the probability that only entrepreneur \( \tau \) is successful is \( q_\tau (1 - q_\tau') - r \), with \( \tau', \tau \in \{A, B\} \), \( \tau' \neq \tau \), where \( r \) is restricted so all probabilities are between 0 and 1.\(^{13}\) The parameter \( r \) captures similarities between entrepreneurial project-ideas, and thus characterizes the degree of “relatedness” of the innovations.

If the first stage is successful, at \( t = 2 \) the innovation process enters the second stage. In this second stage, the entrepreneur must decide the level of intensity (or scale) of the innovation process, \( y_\tau \). Innovation intensity reflects, for example, the level of R&D expenditures committed to the innovation which affect the ultimate value of the innovation at \( t = 3 \). For notational simplicity, we normalize the final payoff from the innovation to \( y_\tau \). Innovation intensity is costly: entrepreneur \( \tau \) choosing an innovation intensity \( y_\tau \) must sustain a cost \( c_\tau (y_\tau) = \frac{1}{Z_\tau (1+\gamma)} y_\tau^{1+\gamma} \), with \( \gamma > 0 \), where \( Z_\tau \) represents the productivity of entrepreneur \( \tau \)’s project-idea. For tractability, we assume the productivity of the two potential innovations are not too far from each other: \( \frac{Z_A}{Z_B} \in \left(\frac{1}{\psi}, \psi\right) \), where \( \psi = \left[\frac{1+2e^2}{4}\right]^{\gamma+1}.\(^{14}\) Entrepreneurs pay for the cost \( c_\tau (y_\tau) \) by selling equity to a large number of investors, for example, in an IPO. We assume that both innovation intensity \( y_\tau \) and related costs \( c_\tau (y_\tau) \) are contractible, eliminating moral hazard concerns (which are not the focus of our paper). Finally, for simplicity, we assume that entrepreneurs are impatient and that they sell at the interim period, \( t = 2 \), their entire firm to outside investors, at a value \( V_\tau \).

The second stage of the innovation process is also risky and, if successful, the innovation generates at the end of the last period, \( t = 3 \), the payoff \( y_\tau \) with probability \( p \), and zero otherwise. We

\(^{13}\)Formally, \( r \in \left(-\min \{q_A q_B, (1 - q_A) (1 - q_B)\}_\tau, \min_q q_\tau (1 - q_\tau)\right) \). Note that the correlation of the first-stage projects is \( r [q_A (1 - q_A) q_B (1 - q_B)]^{\frac{1}{2}} \).

\(^{14}\)If this assumption is violated, there may be asymmetric equilibria, with leader-follower dynamics. Specifically, one innovator will lead and decide whether to initiate the innovation process even under lukewarm investor sentiment, while the second innovator follows and receives a positive sentiment if the first innovates. We leave the analysis of this possibility to future research.
assume, for simplicity, that the success probabilities of the second stage are independent.

1.1 Modeling uncertainty

A critical feature of our model is that outside investors are uncertain about the success probability of the second stage of project-ideas, \( p \). We model uncertainty (or “ambiguity”) aversion by adopting the minimum expected utility (MEU) approach developed in Gilboa and Schmeidler (1989).\(^{15}\) In this framework, economic agents do not have a single prior on future events but, rather, they believe that the probability distribution of future events belongs to a given set \( \mathcal{M} \), denoted as the investor’s “core beliefs set.” Thus, uncertainty-averse agents maximize \( U \), where

\[
U = \min_{\mu \in \mathcal{M}} E_{\mu} [u (\cdot)],
\]

where \( \mu \) is a probability distribution over future events, and \( u (\cdot) \) is a von-Neumann Morgenstern (vNM) utility function.\(^{16}\) When \( u \) is a linear (or affine) function, the investor will be risk neutral but uncertainty averse agent. In addition, we assume uncertainty-averse agents are sophisticated in that they correctly anticipate the impact of uncertainty aversion on their portfolio choices.

The critical feature of uncertainty aversion is that uncertainty-averse agents weakly prefer randomizations over random variables (more precisely, over acts described in Anscombe and Aumann, 1963) rather than each individual variable in isolation.\(^{17}\) In the context of MEU, this feature may be seen immediately as follows. Given two random variables, \( y_k, k \in \{1, 2\} \), with joint distribution \( \mu \in \mathcal{M} \), by the property of the minimum operator, we have that for all \( q \in [0, 1] \)

\[
q \min_{\mu \in \mathcal{M}} E_{\mu} [u (y_1)] + (1 - q) \min_{\mu \in \mathcal{M}} E_{\mu} [u (y_2)] \leq \min_{\mu \in \mathcal{M}} \{ qE_{\mu} [u (y_1)] + (1 - q)E_{\mu} [u (y_2)] \}. \tag{2}
\]

We model investor uncertainty aversion by assuming that investors are uncertain on the success

\(^{15}\)An alternative approach is “smooth ambiguity” developed by Klibanoff, Marinacci, and Mukerji (2005). In their model, agents maximize expected felicity of expected utility, and agents are uncertainty averse if the felicity function is concave. The main results of our paper will hold in this approach (if the felicity function is sufficiently concave), but at the cost of requiring a substantially greater analytical complexity. Similarly, our results also hold under variational preferences of Maccheroni, Marinacci, and Rustichini (2006) if the ambiguity index \( c (p) \) has a positive cross-partial. See also Siniscalchi (2011).

\(^{16}\)In the traditional framework, players have a single prior \( \mu \) and maximize expected utility \( E_{\mu} [u (\cdot)] \).

\(^{17}\)This is the “uncertainty-aversion axiom” of Gilboa and Schmeidler (1989); formally, for any two uncertain acts that the investor is indifferent between, \( f \sim g \), the investor (weakly) prefers the mixture: \( \alpha f + (1 - \alpha) g \succeq f \). Note that for SEU agents, the latter always holds with indifference.
probability of the second stage of the innovation process, \( p \). Following Hansen and Sargent (2001) and (2008), we characterize the core beliefs set \( \mathcal{M} \) in (1) by using the notion of relative entropy.\(^{18}\) For a given pair of (discrete) probability distributions \((p, \hat{p})\), the relative entropy of \( p \) with respect to \( \hat{p} \) is the Kullback-Leibler divergence of \( p \) from \( \hat{p} \):

\[
R(p|\hat{p}) = \sum_i p^i \log \frac{p^i}{\hat{p}^i}.
\]

Thus, the core beliefs set for the uncertainty-averse investors in our economy is

\[
\mathcal{M} \equiv \{p : R(p|\hat{p}) \leq \eta\},
\]

where \( p \) is the joint distribution of the success probability of the second stage of the two projects, and \( \hat{p} \) is an exogenously given “reference” probability distribution of such success probabilities. From (3), it is easy to see that the relative entropy of \( p \) with respect to \( \hat{p} \) represents the (expected) log-likelihood ratio of the pairs of distributions \((p, \hat{p})\), when the “true” probability distribution is \( p \). Thus, interestingly, the core beliefs set \( \mathcal{M} \) can be interpreted as the set of probability distributions, \( p \), with the property that, if true, the investor would expect not to reject the (“null”) hypothesis \( \hat{p} \) in a likelihood-ratio test.

Intuitively, the core belief set \( \mathcal{M} \) includes probability distributions that are not “too unlikely” to be the true (joint) probability distribution that characterizes the two technologies, given the reference distribution \( \hat{p} \). Note that a small value of \( \eta \) represents situations where agents have more confidence that the probability distribution \( \hat{p} \) is a good representation of the success probability of the two technologies, while a large value of \( \eta \) corresponds to situations where there is great uncertainty on the true probabilities underlying the two technological processes.\(^{19}\)

An important effect of restricting investors’ beliefs to the core beliefs set (4) is to rule out probability distributions that are “too far” from the reference probability \( \hat{p} \). In other words, the maximum entropy criterion implied by (4) has the effect of excluding from the core-belief set

\(^{18}\)This specification of ambiguity aversion, which is often referred to as the “constrained preferences” approach, is a particular case of the larger class of “variational preferences.” Strzalecki (2011) provides a general characterization of different approaches to modeling ambiguity aversion.

\(^{19}\)As in Hansen and Sargent (2001), (2007), (2008), and Epstein and Schneider (2010), relative entropy can be interpreted as characterizing the extent of “misspecification error” that affects investors.
probability distributions that give too much weight to extreme events. Because uncertainty-averse investors are essentially concerned about “left-tail” events, we interpret this property as “trimming pessimism.” The following lemma provides a simple characterization of the core beliefs set \( \mathcal{M} \) that will play a critical role in our paper.

**Lemma 1** Let \( \eta < \eta(\hat{p}) \), defined in the appendix. The core beliefs set \( \mathcal{M} \) is a strictly convex set with smooth boundary. If investors have nonnegative investments in both innovations, the solution to (1) is on the lower left-hand boundary of \( \mathcal{M} \).

Lemma 1 is a direct implication of the fact that relative entropy \( R(p|\hat{p}) \) is a strictly convex function. Uncertainty-averse investors with positive investment in both project ideas select probability assessments that lie in the “lower-left” boundary of the core beliefs set. Thus, the relevant part of the core beliefs set \( \mathcal{M} \) is a smooth, decreasing, and convex function (see Figure 1).

Because there is no closed-form solution for the level set of relative entropy for binomial distributions in (4), for ease of exposition, we model the relevant portion of the core beliefs set (namely, the decreasing and convex “lower-left” boundary) by using a lower-dimensional parametrization, as follows. We assume that the success probability of project idea \( \tau \) depends on the value of an underlying parameter \( \theta_\tau \), and is denoted by \( p(\theta_\tau) \), with \( \theta_\tau \in [\theta_L, \theta_H] \subseteq [\theta_m, \theta_M] \). For analytical tractability, we assume that \( p(\theta_\tau) = e^{\theta_\tau - \theta_M} \), with \( \tau \in \{A, B\} \).

Uncertainty-averse agents treat the vector \( \bar{\theta} \equiv (\theta_A, \theta_B) \) as ambiguous and assess that \( \bar{\theta} \in C \subseteq \{(\theta_A, \theta_B) : (\theta_A, \theta_B) \in [\theta_L, \theta_H]^2 \} \). We interpret the parameter combination \( \bar{\theta} \) as describing the state of the economy at \( t = 3 \) and we denote \( C \) as the set of “core beliefs” of our uncertainty-averse investors. In light of Lemma 1 and subsequent discussion, we assume that for \( \bar{\theta} \in C \) we have that \( (\theta_A + \theta_B)/2 = \theta_T \), where \( \theta_T \equiv (\theta_H + \theta_L)/2 \). Importantly, given \( \bar{\theta} \), the success probabilities of the second-stage of project-ideas are independent. We will characterize the extent of technological uncertainty as \( \alpha \equiv \theta_T - \theta_L \). To simplify the exposition, we assume \( \alpha \geq \ln(2) \); this assumption guarantees equilibrium interior beliefs when investors face multiple innovations.

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20 Referring back to our example on self-driving cars, the relative entropy criterion eliminates from \( \mathcal{M} \) probability distributions that give too much weight to the extreme event that all technologies under development will fail.

21 For a general discussion, see Theorem 2.5.3 and 2.7.2 of Cover and Thomas (2006). Our results hold, generally, when the core belief set \( \mathcal{M} \) is a strictly convex set with smooth boundaries (note that “rectangular” core-belief sets do not satisfy such condition); see Section 5 for more discussion.
Finally, we assume that, while outside investors are uncertain on the parameter combination \( \bar{\theta} \), there are no other sources of uncertainty (as opposed to “risk”) in the economy. In addition, we assume that investors are otherwise risk-neutral. This means that investors may have access to other (risky) investment opportunities without affecting our results. We make this risk-neutrality assumption to isolate the effect of uncertainty aversion from the traditional (and well-understood) risk-aversion channel. For simplicity, we also assume that entrepreneurs are both uncertainty and risk neutral. In Section 5 we discuss more explicitly the impact of these assumptions on our analysis.

We will at times benchmark the behavior of uncertainty-averse agents with the behavior of an uncertainty-neutral SEU agent, and we will assume that an uncertainty-neutral investor has \( \theta_L = \theta_H \), so that he assesses \( \theta_\tau = \theta_T \). This assumption guarantees that the uncertainty-neutral investor has the same probability assessment on the success probability of each project-idea as a well-diversified uncertainty-averse investor (and thus there is no “hard-wired” difference between the two types of investors).

The game unfolds as follows. At the beginning of the period, \( t = 1 \), entrepreneurs simultaneously decide whether or not to innovate, and select \( d_\tau \in \{0, 1\} \). Investors invest their endowment in the other asset. At the interim date, \( t = 2 \), entrepreneurs with successful first stage select the innovation intensity of the second stage of their innovation, \( y_\tau \), and then sell their entire firm to the outside investors for value \( V_\tau \), which represents their payoff from the innovation. Outside investors purchase a fraction \( \omega_\tau \) of firm \( \tau \) and invest the residual value \( \omega_0 - \omega_A V_A - \omega_B V_B \) in the other asset. At the last stage, \( t = 3 \), residual uncertainty on the success or failure of the second stage of each innovation is resolved and payoff realized. Investors’ final payoff depends on their investments in each innovation, \( \omega_\tau \), and on the return from their investments in innovation.

1.2 Investor Sentiment

An important implication of uncertainty aversion is that the investor’s probabilistic assessment at the interim date on the parameter \( \theta_\tau \) depends on their overall exposure to the source of risk and, thus, on the structure of their portfolios. This means that the probability assessment (i.e., the “beliefs”) held by an uncertainty-averse investor (that is, their assessment of the parameter
combination $\bar{\theta}$) are endogenous, and depend on the agent’s overall exposure to uncertainty.

Endogeneity of beliefs is the outcome of the fact that the minimization operator in (1), which determines the probability assessment held by an investor on the success probability of the second stage of the project-ideas, in general depends on the composition of the investor’s overall portfolio. Note that this property, which plays a critical role in our paper, implies that uncertainty-averse agents are more willing to hold uncertain assets if they can hold such assets in a portfolio rather than in isolation. By holding uncertain assets in a portfolio, because of uncertainty hedging, investors can lower their overall exposure to the sources of uncertainty in the economy. In our setting, by investing in both project-ideas, investors will limit their exposure to the “tail event” that both project-ideas have a very low success probability in the second stage.

The effect of uncertainty hedging in our model is that investors hold more favorable probability assessments on the success probability of project-ideas if they invest in both projects, rather than in just one project.\(^{23}\) Specifically, if an investor decides to purchase a proportion $\omega_\tau$ of entrepreneur $\tau$’s firm, with innovation intensity $y_\tau$, the investor will hold a risky portfolio $\Pi = \{\omega_A y_A, \omega_B y_B, \omega_0 - \omega_A \nu_A - \omega_B \nu_B\}$. Because investors are uncertainty averse but otherwise risk neutral, portfolio $\Pi$ provides the investor with utility $U(\Pi) = \min_{\theta \in C} u(\Pi, \bar{\theta})$, where

$$u(\Pi, \bar{\theta}) = e^{\theta_A - \theta_M} \omega_A y_A + e^{\theta_B - \theta_M} \omega_B y_B + \omega_0 - \omega_A \nu_A - \omega_B \nu_B.$$  \hspace{1cm} (5)

Because of uncertainty aversion, the investor’s assessment at $t = 2$ on the state of the economy, $\bar{\theta}_a$, is the solution to the minimization problem

$$\bar{\theta}_a(\Pi) = \arg \min_{\theta \in C} u(\Pi, \theta).$$  \hspace{1cm} (6)

and is characterized in the following lemma.

**Lemma 2** Increasing an investor’s exposure to one innovation risk induces a more favorable assessment of the other innovation risk. Formally, given a portfolio $\Pi$, and letting

$$\bar{\theta}_a(\Pi) = \theta_T + \frac{1}{2} \ln \frac{\omega_\tau' y_\tau}{\omega_\tau y_\tau},$$  \hspace{1cm} (7)

\(^{23}\)This happens when (2) holds as a strict inequality.
an uncertainty-averse investor holds an assessment \( \theta^a_{\tau} \) on the uncertain parameter \( \theta_{\tau} \) equal to

\[
\theta^a_{\tau} (\Pi) = \begin{cases} 
\theta_L & \tilde{\theta}^0_{\tau} (\Pi) \leq \theta_L \\
\tilde{\theta}^0_{\tau} (\Pi) & \tilde{\theta}^0_{\tau} (\Pi) \in (\theta_L, \theta_H) \\
\theta_H & \tilde{\theta}^0_{\tau} (\Pi) \geq \theta_H 
\end{cases}
\]

Lemma 2 shows that an investor’s assessment on \( \tilde{\theta} \) is endogenous, and it depends crucially on the composition of portfolio, \( \Pi \). We will say that the agent has “interior assessments” (or beliefs) when \( \tilde{\theta}^a_{\tau} \in (\theta_L, \theta_H) \), in which case, the agent’s assessments are equal to \( \tilde{\theta}_{\tau} (\Pi) \) as in (7). Otherwise, we will say that the investor holds “corner assessments.” Further, an uncertainty-averse investor’s assessment of \( \tilde{\theta} \) determines the views held by the investor on the future state of the economy. Thus, we will refer to the assessment \( \tilde{\theta}^a \equiv (\tilde{\theta}^a_A, \tilde{\theta}^a_B) \) as characterizing “investor sentiment.”

Lemma 2 shows that when an investor has a relatively greater proportion of her portfolio invested in innovation \( \tau \), \( \omega_\tau y_\tau > \omega_\tau' y_{\tau'} \), he will be relatively more pessimistic about the return on that innovation. This happens because a greater exposure to the risk generated by a given innovation, relative to another innovation, will make an uncertainty-averse investor relatively more concerned about priors that are less favorable to that innovation. Correspondingly, the investor will give more weight to the states of nature that are more favorable to the other innovation. In other words, the investor will be more “optimistic” on the success probability of that innovation (i.e., will have a “stronger sentiment”).

A key implication of uncertainty hedging is to create a positive externality among investment projects through its effect on investors beliefs. Suppose entrepreneur A decides to innovate, but entrepreneur B does not. Because \( y_B = 0 \), by Lemma 2, \( \tilde{\theta}^a_A (\Pi) = \theta_L \) for any \( \omega_A y_A > 0 \). Correspondingly, if entrepreneur B decides to innovate, but entrepreneur A does not, \( \tilde{\theta}^a_B (\Pi) = \theta_L \). Similar situations emerge if only one entrepreneur has a successful first-stage project-idea, while the other entrepreneur fails. In this case, at the interim date, \( t = 2 \), investors hold more pessimistic assessments about the successful innovation than if both entrepreneurs have a successful first-stage project-idea. This means investors, when facing only one innovation, are more pessimistic than when facing both innovations. This happens because, by investing in only one project-idea, investors forego the benefits of uncertainty hedging and hold a portfolio with greater exposure to the
possibility that the second-stage success probability is very low. In contrast, by investing in both
technologies, the investor protects herself from the situation that both technologies have very low
success probability, a hypothesis rejected by the relative entropy criterion (4).

In our model, probability assessments (8) determine investors’ expectations on the ultimate
success probability of the innovation processes in the economy, and thus characterize their “sent-
timent” toward innovations. An important implication of Lemma 2 that will play a key role in our
analysis is that investor sentiment about one innovation will crucially depend on the availability
of other innovations in the economy, and their innovation intensity. An investor will be more opti-
mistic about an innovation success probability (he values it more) if he will also be able to make
greater investments in other innovations. Thus, investors’ uncertainty aversion creates a strategic
complementarity among entrepreneurs: an entrepreneur’s innovation will be more valuable if other
entrepreneurs have successful innovations as well. In other words, if both entrepreneurs innovate
and are successful at the first stage, investor sentiment toward both innovations improves making
both innovations more valuable. This positive spillover from one innovation to another is driven
by investor sentiment due to uncertainty aversion, resulting in innovation waves.

2 The Innovation Decision

As a benchmark, we start the analysis by characterizing innovation decisions when investors are
uncertainty-neutral, then we consider the case where investors are uncertainty-averse.

2.1 The Uncertainty-Neutral Case

When investors are uncertainty neutral, under risk neutrality equity prices depend only on their
prior $\theta_T$ and on the level of innovation intensity, $y_\tau$, chosen by the firm, giving

$$V_\tau^n \equiv p(\theta_T) y_\tau, \quad \text{for } \tau \in \{A, B\}. \quad (9)$$

Equation (9) shows that equity value for innovation $\tau$ depends only on the investor assessments
of the success probability of the second stage of the innovation process, $p(\theta_T)$, and its level of
innovation intensity, $y_\tau$. Specifically, absent uncertainty aversion, there are no interactions between
the choice of the innovation intensities by the two entrepreneurs. In this case, if the first stage of the project-idea was successful, entrepreneur \( \tau \) chooses the level of innovation intensity for the second stage, \( y_{\tau} \), by solving

\[
\max_{y_{\tau}} \mathcal{U}_{\tau}^n \equiv V^n_{\tau} - c_{\tau}(y_{\tau}) = p(\theta_T) y_{\tau} - \frac{1}{(1 + \gamma)} Z_{\tau} y_{\tau}^{1+\gamma}.
\] (10)

From (10) it follows that the optimal innovation intensity, \( y_{\tau} \), is

\[
y_{\tau}^* = \left[ p(\theta_T) Z_{\tau} \right]^{\frac{\gamma}{1+\gamma}},
\] (11)

By direct substitution of \( y_{\tau}^* \) into (10), we obtain the ex-ante expected payoff for entrepreneur \( \tau \) from initiating the innovation process, and thus incurring discovery cost \( k_{\tau} \), is equal to

\[
E\mathcal{U}_{\tau}^n = q_{\tau} \frac{\gamma}{1 + \gamma} \left[ p(\theta_T) \right]^{\frac{1+\gamma}{\gamma}} Z_{\tau}^{\frac{1}{\gamma}} - k_{\tau}.
\]

Thus, entrepreneur \( \tau \) innovates at \( t = 0 \) if \( E\mathcal{U}_{\tau}^n \geq 0 \), leading to the following theorem.

**Theorem 1** When investors are uncertainty-neutral, entrepreneurs of type \( \tau \) innovate iff

\[
k_{\tau} \leq k_{\tau}^n \equiv q_{\tau} \frac{\gamma}{1 + \gamma} \left[ p(\theta_T) \right]^{\frac{1+\gamma}{\gamma}} Z_{\tau}^{\frac{1}{\gamma}}, \quad \tau \in \{A, B\},
\]

and the innovation processes of the two entrepreneurs are independent.

Theorem 1 shows that when investors are uncertainty neutral, the investment decisions by the two entrepreneurs are effectively independent from each other. When investors are uncertainty averse, however, the innovation processes of the two firms become interconnected, creating the possibility of innovation waves.

### 2.2 Uncertainty Aversion and Innovation

We solve the model recursively. First, we determine the value \( V_{\tau} \) that investors are willing to pay at the interim date, \( t = 2 \), to entrepreneurs, given their choice of innovation intensity, \( y_{\tau} \). Next, we find the optimal level of innovation intensity, \( y_{\tau} \), chosen by entrepreneurs that are successful in
the first stage of the innovation process. Finally, we solve for the initial choice by entrepreneurs on whether or not to initiate the innovation process by incurring the initial discovery cost $k$. The implementation of the second stage of the innovation process requires entrepreneurs to raise capital from investors by selling equity in the capital markets at $t = 1$. For simplicity, entrepreneurs sell their entire firm to investors, use the proceeds from the equity sale to pay for the cost $c(y)$, and then pocket the difference. In this case, the choice of innovation intensity $y$ by entrepreneur $\tau$ depends on the price that outside investors are willing to pay for her firm, that is, on the market value of the equity of the firm. This, in turn, depends on the assessments held by investors on the success probability of the innovation, $p(\theta)$, as follows.

**Lemma 3** Uncertainty-averse (and risk-neutral) investors will price entrepreneurial firms at their expected value, $V^a = p(\theta^a) y$, given their beliefs $\overline{\theta}^a$. Furthermore, it is (weakly) optimal for investors to hold a balanced portfolio: $\omega_A^* = \omega_B^*$. Lemma 3 shows that uncertainty-averse, risk-neutral investors price equity at its expected value, given their belief assessments, and will hold a balanced portfolio by optimally making equal investments in both entrepreneurial firms. Desirability of equal investments is a direct consequence of uncertainty hedging and it represents the analog of the benefit of diversification for uncertainty-averse investors. For notational simplicity, we normalize investor portfolio holding and set $\omega_A^* = \omega_B^* = 1$.

Under uncertainty aversion, investor beliefs $\overline{\theta}^a$ and, thus, their assessments on the success probability of the second stage of each innovation, $p(\theta^a)$, are endogenous, and depend on investors’ overall exposure to uncertainty. From Lemma 3, each investors exposure is driven by the relative level of the innovation intensities of each firm, $y$. This implies the following market valuation.

**Lemma 4** If investors are uncertainty averse, the market value of entrepreneur $\tau$’s firm is

$$
V^a = \begin{cases} 
    p(\theta_H) y & y \leq e^{-2\alpha} y' \\
    p(\theta_T) (y, y')^{1/2} & y' \in (e^{-2\alpha} y', e^{2\alpha} y') \\
    p(\theta_L) y & y' \geq e^{2\alpha} y' 
\end{cases}
$$

where $y$ is the innovation intensity selected by entrepreneur $\tau$, with $\tau, \tau' \in \{A, B\}, \tau \neq \tau'$. 18
Lemma 4 shows that, when investors are uncertainty averse, the market value of one firm depends on the level of innovation intensity chosen by its entrepreneur as well as on the level chosen by the other firm. The linkage between the market value of the two firms occurs through endogenous investor sentiment. From Lemma 2 an increase of the innovation intensity of one firm will increase the relative exposure of investors to that firm’s risk relative to the other firm’s risk, making (all else equal) investors relatively more conservative (or pessimistic) about that firm’s success probability and, correspondingly, relatively more ebullient (or optimistic) about the other firm’s success probability. This interaction between equity market values of the two firms creates the strategic externality between the two entrepreneurs.

Lemma 4 also implies that an increase of the level of innovation intensity in one firm, $y_T$, has two opposing effects on its value $V_T^a$. First is the positive direct effect that greater innovation intensity has on the ultimate value of the innovation. This positive effect can however be mitigated by a second negative effect that an increase in innovation intensity has on investor sentiment, that is, on their assessment of the ultimate success probability, $p(\theta_T)$. The net effect, however, is positive, and firm equity value is an increasing function of the innovation intensities of both firms.

If one of the two firms does not innovate, or if the innovation is unsuccessful in the first stage, the level of innovation intensity for that firm is necessarily zero. Lemma 4 implies that in this case the market value of that one firm is given by the worst-case scenario value, that is, $V_T^a = p(\theta_L) y_T$.

We can now determine the optimal level of innovation intensity, $y_T$, for each entrepreneur. From Lemma 4, we know that the market value of equity depends on the number of firms seeking financing from investors, which in turn depends on whether only one or both firms decided to initiate the innovation process, and whether the first stage of the process was successful or not. Thus, there are four possible states of the world that we need to analyze: (i) when both entrepreneurs had a successful first stage, state $SS$; (ii) when only one entrepreneur has a successful first-stage, state $SF$ with the symmetric $FS$ state, (iii) when both entrepreneur fail in the first stage and no innovation can take place, state $FF$. Since the last state $FF$ is trivial, we focus on the first two.
2.2.1 Only One Firm Has Successful First-Stage Project, State SF

Consider first the case in which only one entrepreneur had a successful first-stage project-idea, state SF. This state may obtain either because the other entrepreneur has not initiated the innovation process (she did not sustain the discovery cost \(k_T\)), or because the first stage was unsuccessful. In this case, the entrepreneur chooses the level of innovation intensity, \(y_T\), anticipating that investors will value equity under the worst case scenario, \(p(\theta_L)\). Thus, she solves

\[
\max_{y_T} U^{a, SF}_\tau \equiv p(\theta_L) y_T - \frac{1}{Z_\tau(1 + \gamma)^{y_T + 1}},
\]

giving the following.

**Lemma 5** If only one entrepreneur has a successful first stage project-idea (state SF), she selects innovation intensity equal to

\[
y_T^{a, SF} = [p(\theta_L) Z_\tau]^{\frac{1}{\gamma}}.
\]

The market value of the entrepreneur’s firm is equal to

\[
V^{a, SF}_\tau = [p(\theta_L)]^{\frac{1 + \gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}},
\]

and the corresponding continuation utility for the entrepreneur is

\[
U^{a, SF}_\tau \equiv V^{a, SF}_\tau \frac{\gamma}{1 + \gamma}.
\]

If only one entrepreneur successfully develops a first-stage project, there will only be one innovation available to investors. In this case, investors value equity on the basis of the worst-case scenario about that innovation, resulting in negative investor sentiment and low equity valuations. Therefore, the lone entrepreneur chooses a low level of innovation intensity, consistent with negative sentiment.

2.2.2 Both Firms Have Successful First-Stage Projects, State SS

If both entrepreneurs innovate and have a successful first-stage project-idea (state SS), equity market valuation is given in Lemma 4. As assumed, we focus on the case in which the two projects are not too dissimilar (formally, \(Z_A \) and \(Z_B \) are not too far away from each other). This approach
allows us to simplify the exposition by ensuring that, if both firms have successful first-stage projects, state $SS$, they find it optimal to chose levels of innovation intensity $\{y_A, y_B\}$ which, from Lemma 2, result in interior assessments for investors, $\hat{\theta}_\tau^a$ (II).

If both entrepreneurs have a successful first-stage project-ideas, they choose (simultaneously) the innovation intensities $y_\tau$ of their firms by solving

$$\max_{y_\tau} U_\tau^a,SS = V_\tau^a,SS(y_\tau, y_{\tau'}) - c_\tau(y_\tau) = p(\hat{\theta}_\tau^a)y_\tau - \frac{1}{Z_\tau (1 + \gamma)}y_\tau^{1+\gamma},$$

(17)

where $V_\tau^a,SS$ is given in (12). The choice of innovation intensity by entrepreneur $\tau$ is determined by three factors. The first two factors are the direct impact of innovation intensity on firm value, for given beliefs, and its impact on cost, $c(y_\tau)$; these factors are in common with the entrepreneur maximization problem in the $SF$ state, (13). Uncertainty aversion introduces a third new factor: by increasing the innovation intensity, $y_\tau$, an entrepreneur induces investors to be more pessimistic about the ultimate success probability for her firm, $p(\theta_\tau^a)$, decreasing its value, $V_\tau^a,SS$. This adverse effect on firm value has a negative impact on the choice of innovation intensity by entrepreneur $\tau$.

**Theorem 2** If both entrepreneurs have a successful first-stage project-idea (state $SS$), they select innovation intensity according to

$$Y_\tau^a,SS(y_{\tau'}) = \left[ \frac{1}{2} p(\theta_T) Z_\tau(y_{\tau'})^{1/2} \right]^{\frac{1+\gamma}{2}} , \text{ with } \tau \neq \tau', \text{ and } \tau, \tau' \in \{A, B\};$$

(18)

which is locally increasing in the other entrepreneur’s innovation intensity, $y_{\tau'}$. Nash-Equilibrium innovation intensities, $\{y_A^a,SS, y_B^a,SS\}$, in the subgame where both entrepreneurs have a successful first-stage project-ideas (state $SS$), are

$$y_\tau^a,SS = \left[ \frac{1}{2} p(\theta_T) Z_\tau Z_{\tau'}^{1-\chi} \right]^{\frac{1}{2}},$$

(19)

where $\chi = \frac{2\gamma + 1}{2(1+\gamma)}$. The equilibrium market value, $V_\tau^a,SS$, of the entrepreneurial firms is

$$V_\tau^a,SS = \frac{1}{2^\gamma} \left[ p(\theta_T) \right]^{\frac{1+\gamma}{\gamma}} (Z_\tau Z_{\tau'})^{\frac{1}{2\gamma}},$$

(20)
and the corresponding continuation utility $\mathcal{U}_{\tau}^{a,SS}$ for each entrepreneur is

$$\mathcal{U}_{\tau}^{a,SS} = \chi V_{\tau}^{a,SS}. \quad (21)$$

From Lemma 2 and Lemma 4, investor assessment of the success probability of the second stage of an innovation process (and, thus, its market valuation) depends on the innovation intensities chosen by both entrepreneurs. In particular, from (7), greater innovation intensity by entrepreneur $\tau$, makes investors more optimistic on innovation $\tau'$. In turn, the improved investor sentiment (and potential valuation) on innovation $\tau'$ leads its entrepreneur to choose a greater innovation intensity $y_{\tau'}$ as well. This means that uncertainty aversion makes investors perceive innovations effectively as complements, leading them to more optimistic assessments. In turn, stronger investor sentiment generates greater equity market valuations and higher levels of innovation intensity by both entrepreneurs. The following corollary compares the equilibria in the subgames where one or both entrepreneurs have successful innovations.

**Corollary 1** When both entrepreneurs have a successful first-stage project, they implement more innovation intensity, $y_{\tau}^{a,SS} > y_{\tau}^{a,SF}$, receive higher equity valuation, $V_{\tau}^{a,SS} > V_{\tau}^{a,SF}$, and are better off, $U_{\tau}^{a,SS} > U_{\tau}^{a,SF}$.

An important implication of Corollary 1 is that, because of the complementarity of innovations generated by uncertainty aversion, investors value an innovation more when they can also invest in the other innovation: $V_{\tau}^{a,SS} > V_{\tau}^{a,SF}$, leading to greater innovation intensity $y_{\tau}^{a,SS} > y_{\tau}^{a,SF}$.

### 2.3 The Initial Innovation Decision

We now proceed to discuss the ex-ante entrepreneurial decision, at $t = 1$, of whether or not to sustain the (non-pecuniary) discover cost $k_{\tau}$ and, thus, to initiate the innovation process. In Section 2.2, we have shown that investor uncertainty aversion affects equity valuation $V_{\tau}^{a}$ and generates strategic complementarity in the interim choice of innovation intensity, $y_{\tau}$. In turn, this interim complementarity generates a strategic complementarity also in the entrepreneurs’ decisions to innovate at the beginning of the innovation process, $t = 1$. 

22
If entrepreneur $\tau'$ chooses to innovate, $d^a_{\tau'} = 1$, the expected utility for entrepreneur $\tau$ from sustaining at $t = 1$ the initial discover cost $k_\tau$ is

$$EU^a_{\tau} = (q_\tau q_{\tau'}) + (q_\tau(1 - q_{\tau'})) - r)U^{a,SS}_\tau - k_\tau$$

for $\tau, \tau' \in \{A, B\}$ and $\tau \neq \tau'$. Conversely, if entrepreneur $\tau'$ does not innovate at $t = 0$, $d^a_{\tau'} = 0$, the expected utility for entrepreneur $\tau$ from choosing to innovate at $t = 1$ is

$$EU^a_{\tau} = q_\tau U^{a,SF}_\tau - k_\tau.$$ 

Thus, entrepreneur $\tau$ objective function at $t = 1$ is given by:

$$EU^a_{\tau} = d^a_{\tau} EU^a_{\tau} + (1 - d^a_{\tau}) EU^a_{\tau}.$$  \hspace{1cm} (22)

Note that entrepreneur $\tau$ earns $EU^a_{\tau}$ if they set $d^a_{\tau} = 1$, but she earns zero if she selects $d^a_{\tau} = 0$. We adopt the notion of subgame-perfect Nash Equilibrium, as follows.

**Definition 1** A subgame-perfect Nash Equilibrium is a strategy combination $\{d^a_{\tau}, y^{a,SF}_{\tau}, y^{a,SS}_{\tau}\}$ and investor equity valuation $V^a_{\tau}(y_\tau, y_{\tau'})$, for $\tau \in \{A, B\}$, such that: (i) each entrepreneur $\tau \in \{A, B\}$ at $t = 1$ maximizes (22), and in state SS and state SF maximizes (17) and (13), respectively, given the other entrepreneur’s optimal strategy and investor equity valuation $V^a_{\tau}(y_\tau, y_{\tau'})$; (ii) investor equity valuation, $V^a_{\tau}(y_\tau, y_{\tau'})$, for $\tau \in \{A, B\}$, is given by (12), given entrepreneurs’ optimal strategies.

We can now characterize equilibrium innovation decisions at $t = 1$.

**Theorem 3** There are critical thresholds $\{k_\tau, \bar{k}_\tau\}$, with $k_\tau < \bar{k}_\tau$, such that: (i) for low levels of discover cost, $k_\tau \leq k_\tau$, an entrepreneur always innovates, $d^a_{\tau} = 1$; (ii) for high levels of discovery cost, $k_\tau \geq \bar{k}_\tau$, an entrepreneur never innovates, $d^a_{\tau} = 0$; (iii) for intermediate levels of the discovery cost, $k_\tau \in (k_\tau, \bar{k}_\tau)$, an entrepreneur innovates only if the other entrepreneur innovates as well, $d^a_{\tau} = d^a_{\tau'}$. If both entrepreneurs have intermediate levels of discovery costs, there are two subgame perfect equilibria, namely one where both entrepreneurs innovate, $d^a_A = d^a_B = 1$, and one where neither innovate, $d^a_A = d^a_B = 0$. Furthermore, the equilibrium where both entrepreneurs innovate Pareto-dominates the no-innovation equilibrium.
For very small levels of discovery costs, $k_\tau \leq \bar{k}_\tau$, it is a dominant strategy for an entrepreneur to innovate. For very large levels of discovery costs, $k_\tau \geq \bar{k}_\tau$, it is a dominant strategy for an entrepreneur not to innovate. For intermediate levels of discovery costs, $k_\tau \in (\underline{k}_\tau, \bar{k}_\tau)$, entrepreneur $\tau$ wishes to innovate only if the other entrepreneur innovates as well.

The possibility of multiple equilibria is the direct effect of the strategic complementarity created by uncertainty aversion. When both entrepreneurs have intermediate levels of the discovery cost, there are equilibria with and without innovation. In this case, entrepreneurs face a classic “assurance game,” in which there is a Pareto-dominant equilibrium, where both entrepreneurs innovate, yet there is also an inefficient, Pareto-inferior equilibrium, where neither entrepreneur innovates. Multiplicity of equilibria depends on the fact that it is profitable for one entrepreneur to innovate only if she expects the other entrepreneur to innovate as well.

**Corollary 2** The threshold levels $\{\bar{k}_\tau\}_{\tau \in \{A,B\}}$ are increasing functions of $q_\tau, q_{\tau'}, Z_\tau, Z_{\tau'}$ and $r$, and the threshold levels $\{\underline{k}_\tau\}_{\tau \in \{A,B\}}$ are increasing functions of $q_\tau$ and $Z_\tau$.

Corollary 2 has the interesting implication that an increase in one entrepreneur’s probability of success, $q_\tau$, makes not only that entrepreneur, but also other entrepreneurs, more willing to attempt first-stage discovery of a product-idea. This follows from the strategic complementarity induced by uncertainty aversion. In the absence of uncertainty aversion, an increase in the probability of discovery affects only that entrepreneur. Corollary 2 also shows entrepreneurs are more willing to innovate if her innovation is more related to other entrepreneurs’ innovations, that is, $r$ is greater. A greater degree of relatedness increases the probability that both project-ideas are simultaneously successful in the first-stage, increasing the market value of innovation. Finally, Corollary 2 also shows that an increase in productivity of an entrepreneur increases not only that entrepreneur’s willingness to innovate, but also makes other entrepreneurs more willing to innovate as well.

### 3 Acquiring Innovation

We have shown that investor uncertainty aversion creates externalities across innovations. These externalities create the possibility of a value dissipation due to coordination failure. This means there may be gains from internalizing such externalities via acquisitions.
There are two externalities at work in our model. The first externality is the valuation spillover discussed in Lemma 2: for any level of innovation intensities, \( \{y_r, y_{r'}\} \), the two firms are more valuable to uncertainty-averse investors when they are held in the same portfolio than when they are owned separately. The second externality is due to the strategic complementarity between the choices of innovation intensity \( y_r \), discussed in Lemma 4: the market value of an individual firm, \( V^o_r \), is an increasing function of the innovation intensity chosen by both firms, \( \{y_r, y_{r'}\} \), through its effect on investor sentiment. When firms choose their own optimal level of innovation intensity, however, they ignore the positive externality on the other firm’s valuation.

We modify our model as follows. If both entrepreneurs are successful in the first stage, state \( SS \), we allow for the possibility that at the interim date, \( t = 2 \), both entrepreneurs merge their firms in a new firm. After the merger, the entrepreneurs jointly determine the innovation intensity, \( y_r \), for both innovation processes \( \tau \in \{A, B\} \). After the selection of the innovation intensities \( y_r \), the merged firm will again sell its equity in the public equity market.\(^{24}\)

After the merger of the first-stage innovations, the problem of the merged firm is to maximize the combined value of the two innovation projects. By identical reasoning to that leading to Lemma 3, the merged firm will value the projects at \( V_r = p(\theta^m_\tau) y_r \), for \( \tau \in \{A, B\} \), where \( \theta^m_\tau \) is the investors’ assessment when the merged firm is sold. Thus, the merged firm’s objective is

\[
\max_{\{y_A, y_B\}} U^m = p(\theta^m_A) y_A + p(\theta^m_B) y_B - c_A(y_A) - c_B(y_B).
\]

If investors are uncertainty neutral, \( \theta^m_\tau = \theta_\tau \), so the choice of \( y_A \) and \( y_B \) are again independent: the merged firm solves the same problem as the original entrepreneurs, (10), giving \( U^m = U^e_A + U^e_B \). This implies that the optimal levels of innovation intensity chosen by the merged firm are again given by (11), that is, the values the entrepreneurs would choose if the two firms were independent. Thus, if investors are uncertainty neutral, the merger does not add value.

In contrast, if investors are uncertainty averse, we have that \( \theta^m_\tau = \overrightarrow{\theta}^a \) which, from (8), depends on the choice of both \( y_A \) and \( y_B \). As shown in Lemma 4, for interior assessments (which we will

\(^{24}\)The two innovation processes may be sold to the public equity market either as a single multi-divisional firm, or as two independent firms. If the two innovations are sold in two separate firms, from Lemma 3, investors will optimally invest in both firms. Alternatively, a third firm which may acquire both entrepreneurs’ innovations.
show is the case in equilibrium), we now have that

\[ V_A = V_B = p(\theta_T) y_A^{\frac{1}{2}} y_B^{\frac{1}{2}}. \]

The maximization problem of the merged firm becomes

\[
\max_{y_A y_B} U^{m,a} = 2p(\theta_T) y_A^{\frac{1}{2}} y_B^{\frac{1}{2}} - \frac{1}{Z_A (1 + \gamma)} y_A^{1+\gamma} - \frac{1}{Z_B (1 + \gamma)} y_B^{1+\gamma}.
\]

**Theorem 4** When both entrepreneurs are successful, state SS, the entrepreneurs merge the two divisions, the merged firm will implement greater innovation intensity in both projects

\[
y_T^{m,a} = \left[ p(\theta_T) Z_A^{1-\gamma} Z_B \right]^{\frac{1}{\gamma}} > y_T^{a,SS},
\]

where \( \chi = \frac{2\gamma+1}{2(\gamma+1)} \), and will have a greater value than these firms would have as a stand-alone:

\[
V^{m,a} = 2 \left[ p(\theta_T) \right]^{\frac{2+1}{\gamma}} \left[ Z_A Z_B \right]^{\frac{1}{2\gamma}} > V_A^{a,SS} + V_B^{a,SS}.
\]

Theorem 4 shows mergers add value to the innovative process. By merging, the joint firm chooses innovation intensities greater than those that the two entrepreneurs would choose individually. Because of the positive externality between investment levels \( y_T \), inefficiently low levels of investment occur when each entrepreneur maximizes her own payoff. By merging, the post-acquisition firm internalizes the spillover effects of investment, leading to greater innovation intensities and firm valuation. An interesting implication of Theorem 4 is that, in our model, synergies are not exogenous, but are generated endogenously by the valuation externality due to uncertainty aversion.

We now examine the impact of mergers at the interim date, \( t = 2 \), on the entrepreneurs’ ex-ante incentives to innovate, that is, to sustain at \( t = 1 \) the discovery cost \( k_T \). The initial decision to innovate will depend on the conditions at which the entrepreneur anticipates the merger will take place. The acquisition price will depend on the allocation of the surplus generated by the acquisition. Allocation of the synergies created in the merger occurs through bargaining, and we will assume that the two entrepreneurs will split the surplus equally.\(^{25}\) Thus, if both innovations

\(^{25}\)It is easy to see that this surplus allocation will be the same as the one obtained through Shapley Values.
are successful in the first stage, entrepreneur $\tau$ earns

$$\gamma^{m,a}_\tau = \mathcal{U}^{d,SS}_\tau + \frac{1}{2} \left( \mathcal{U}^m - \mathcal{U}^{a,SS}_A - \mathcal{U}^{a,SS}_B \right).$$

**Theorem 5** There are critical thresholds $\{k^{m^\tau}_r, \hat{k}^{m^\tau}_r\}$, with $k^{m^\tau}_r < \hat{k}^{m^\tau}_r$, such that: (i) for low levels of discover cost, $k^m_r \leq k^{m^\tau}_r$, an entrepreneur always innovates, $d^a_r = 1$; (ii) for high levels of discovery cost, $k^m_r \geq \hat{k}^{m^\tau}_r$, an entrepreneur never innovates, $d^a_r = 0$; (iii) for intermediate levels of the discovery cost, $k^m_r \in (k^{m^\tau}_r, \hat{k}^{m^\tau}_r)$, an entrepreneur innovates only if the other entrepreneur innovates as well, $d^a_r = d^a_{r'}$. If both entrepreneurs have intermediate levels of discovery costs, there are two subgame perfect equilibria, namely one where both entrepreneurs innovate, $d^a_A = d^a_B = 1$, and one where neither innovate, $d^a_A = d^a_B = 0$. The equilibrium where both entrepreneurs innovate Pareto-dominates the no-innovation equilibrium. Finally $k^{m^\tau}_r = k^m_r < \hat{k}^m_r$: the possibility of a merger induces entrepreneurs to innovate more ex ante.

Theorems 4 and 5 have the interesting implication that an active M&A market promotes innovative activity and leads to greater innovation rates, stronger investor sentiment, and higher firm valuations. Synergies created in the merger are a direct consequence of endogenous investor sentiment due to uncertainty aversion. A merger allows entrepreneurs to internalize the positive impact that the choice of the innovation intensity in one innovation has on other innovations, and leads to greater innovation rates. Thus, the merger of innovations endogenously promotes stronger investor sentiment and leads to greater valuations.

### 4 Investor Sentiment and Innovation Waves

We now examine the effect of uncertainty on innovation waves in the context of a simple dynamic model. We show that entrepreneurs initiate innovation only when there is a sufficiently large number of potentially active innovators. In particular, when the number of potential innovators is low, entrepreneurs do not engage in innovation because they expect weak investor sentiment and, thus, the market for innovation to be “cold;” potential innovations remain latent in the economy. In contrast, when the number of innovators reaches critical mass, a wave of innovations is triggered amid improved investor sentiment.
We modify the basic model as follows. We consider a simple discrete-time, infinite-horizon, dynamic stochastic game, where \( t \in \mathbb{T} \equiv \{1, 2, 3, \ldots\} \) denotes time. At every date, a new entrepreneur endowed with a project-idea arrives in the economy at a constant rate \( \pi \). At each date \( t \in \mathbb{T} \), a new project-idea arrives with a constant, exogenous probability \( \pi \), where each project-idea is owned by a unique entrepreneur. Let \( \mathcal{E}_t \) be the set of entrepreneurs endowed with a project-idea at any given time \( t \in \mathbb{T} \), and let \( \nu_t \equiv |\mathcal{E}_t| \) the number of entrepreneurs endowed with a project-idea.

Different from the basic model, we now assume that an entrepreneur with a project-idea can delay its implementation to a future date. Waiting to implement the project-idea, however, is costly: entrepreneurs and investors are impatient and have discount factor \( \delta \), which is the same for both groups. For analytical tractability (and simplicity of exposition), we now assume that (i) if implemented, the first stage of the innovation process is always successful, setting \( q = 1 \); and (ii) that project ideas have the same productivity, \( Z \), and innovation intensity is fixed at \( y \).\(^{26}\)

An entrepreneur endowed with a project-idea at time \( t \) must decide whether to implement the innovation immediately or to delay its implementation to the next period. We assume that the decision of whether or not to initiate the innovation is made simultaneously by the entrepreneurs endowed with a project-idea only after observing a public signal \( \sigma_t \) which is informative on \( \nu_t \). For simplicity, we assume that the public signal is perfectly informative on \( \nu_t \) and that \( \sigma_t = \nu_t \).\(^{27}\)

If an entrepreneur decides to innovate at time \( t \), she must pay at that time discovery cost \( k \) to implement the first stage of the innovation process. At \( t + 1 \), successful entrepreneurs proceed with the second stage of the innovation process by implementing innovation intensity \( y \) at cost \( c(y) \), which is paid for by selling equity to investors, as described below. Finally, project-ideas implemented at time \( t \) have at time \( t + 2 \) a payoff \( y \) with probability \( p \), and 0 otherwise, after which the entrepreneur exits from the economy. If a project-idea available at \( t \) is not implemented, it is carried over at the following period, \( t + 1 \), where the entrepreneur faces again the choice of whether to implement the project-idea at that time, or to delay its implementation.

We model uncertainty is a way similar to the basic model. The success probability of the second stage of a project implemented by entrepreneur \( n \) at date \( t \) is uncertain, depending again on the value of a parameter \( \theta_{nt} \), and is equal to \( p(\theta_{nt}) = e^{\theta_{nt} - \theta_M} \). For simplicity, we assume that

\(^{26}\)Our results can be extended to the case where \( q_n < 1 \) and innovation intensity \( y_n \) is not fixed.

\(^{27}\)It is possible, although messy, to extend the model to the case in which the public signal is noisy.
uncertainty on \( p \) is stationary and independent across time, setting \( \theta_{nt} = \theta_n \) for all \((t, n)\).\(^{28}\) Thus, at any time \( t \), investors are uncertain over \( \theta \), and believe that

\[
\sum |\theta_n - \theta^*| \leq \eta \tag{23}
\]

for some \( \theta^* \) and \( \eta \geq 0 \). In this representation, the vector \( \vec{\theta}^* \) can be interpreted as the vector \( \vec{\theta} \) of parameters that support the reference probability \( \hat{p} \) in (4) and \( \eta \) as the degree of uncertainty on \( \vec{\theta} \) perceived by the decision maker.\(^{29}\)

We solve the model by examining first the sub-games in which (some) entrepreneurs are seeking financing at time \( t \), which means that they have initiated the innovation process at time \( t - 1 \). In this case, uncertainty-averse investors form at time \( t \) their portfolios of uncertain assets by buying equity from those entrepreneurs (if any) available at that time. We denote by \( S_t \) the set of entrepreneurs seeking financing at \( t \), and let \( s_t \equiv |S_t| \). Given our assumption that all innovations that are undertaken by an entrepreneur have a successful first-stage, \( S_t \) is also equal to the set of entrepreneurs that initiated their project-ideas at time \( t - 1 \).

Similar to the basic model, each investor chooses a portfolio of the uncertain assets, \( \{\omega_{nt}\}_{n \in S_t} \), given their market valuations \( \{V_{nt}\}_{n \in S_t} \). By identical reasoning as Lemma 4, investors optimally invest equally in all available innovations: \( \omega_{nt} = \omega_{n't} \) for all \( n, n' \in S_t \) and \( t \in T \). Further, given investor assessments \( \vec{\theta}_t^n \) at \( t \in T \), equity is priced at its expected value, that is \( V_{nt}^a = \hat{p}(\theta_{nt}^a) y \) for all \( n \in S_t \). We will interpret again \( \vec{\theta}_t^n \) as characterizing investor sentiment in the equity market at time \( t \). Importantly, similar to the basic model, investor assessment depends on the number \( s_t \) of entrepreneurs that are seeking financing in the equity market at \( t \), as established in the following.

**Lemma 6** If an entrepreneur develops innovation alone, \( s_t = 1 \), investors will be pessimistic and the value of equity will be

\[
V_{nt}^a (1) = \hat{p}(\theta^* - \eta) y; \tag{24}
\]

If an entrepreneur develops innovation with others, \( s_t > 1 \), investors will be more optimistic about

---

\(^{28}\)This assumption rules out, for example, interesting issues related to learning, which could be included in the analysis but we leave it open for future research.

\(^{29}\)If investors are do not have any short positions, the relevant portion of (23) is where \( \theta_{nt} \leq \theta^* \), so this specification, (23), nests the specification from Section 1. Formally, set \( N = 2 \), \( \theta_H = \theta^* \), \( \theta_T = \theta^* - \frac{\eta}{2} \), and \( \theta_L = \theta^* - \eta \).
innovation,

$$\theta_{nt}(s_t) = \theta^* - \frac{\eta}{s_t}. \quad (25)$$

Correspondingly, the market value of equity will be

$$V_{nt}^a(s_t) = \delta p[\theta_{nt}(s_t)] y, \quad (26)$$

where $V_{nt}^a(s_t)$ is increasing in $s_t$. If $s_t = 0$ the equity market is closed, and $V_{nt}^a = 0$.

Lemma 6 shows that investor sentiment at any given date $t$ depends on the number $s_t$ of entrepreneurs that have initiated the innovation process the previous period, $t - 1$, and that are actively seeking for financing at $t$. When an entrepreneur innovates unilaterally, $s_t = 1$, investor sentiment is weak, $\theta_{nt} = \theta^* - \eta$, and the capital market values innovations conservatively, $V_{nt}^a = \delta p(\theta^* - \eta)y$, generating a “cold market”.

In contrast, when entrepreneurs innovate together, for $s_t > 1$, from (25) investor sentiment is increasing in the number of projects available at that time, $s_t$, leading to a “hot market”. This happens because, with a larger number of projects available for investment, $s_t$, uncertainty-averse investors will be relatively less concerned on each individual project that is available at that time. When facing a larger number of projects, due to uncertainty-hedging, uncertainty-averse investors are able to (weakly) reduce their exposure to uncertainty related to each individual project. Reduced exposure to project uncertainty, in turn, leads to more optimistic beliefs for each project, to stronger investor sentiment and, thus, hot equity market. Similarly, investor sentiment is negatively affected by the extent of uncertainty in the economy (which we measure by $\hat{\eta}$). This happens because, a greater value of $\hat{\eta}$ will increase the total uncertainty faced by investors, which makes uncertainty-averse relatively more pessimistic and leads to lower equity valuations.

We can now focus on the equilibrium of the full game. Entrepreneurs endowed with a project-idea must decide at the beginning of each period $t$ whether to pay the discovery cost $k$ and innovate, $d_{tn}^a = 1$, or to delay the initiation of the innovation process to a later period, $d_{tn}^a = 0$. This decision is made by the entrepreneur after observing the public signal $\sigma_t$ which is informative on the number of entrepreneurs with a project idea at that time, $\nu_t$. Furthermore, because the signal $\sigma_t$ is perfectly informative on $\nu_t$, the relevant state variable in our economy can be represented at any time $t \in \mathbb{T}$.
by the number of entrepreneurs \( \nu_t \) that are endowed with a project-idea.

We will use the notion of Markov Perfect Equilibrium, (see, for example, Maskin and Tirole, 2001). A strategy is a Markov Strategy if and only if it depends only on the current state of the game, which in our setting is given by number of entrepreneurs present in the economy, \( \nu_t \). Thus, we let the development decision of entrepreneur \( n \) be denoted by \( d^a_n(\nu_t) \in \{0, 1\} \), and we focus on equilibria with symmetric pure Markov Strategies.

Payoffs for a given entrepreneur are determined as follows. Let the development decisions of entrepreneurs other than \( n \) be denoted by \( d^a_n,-(\nu_t) \). If an entrepreneur develops her project at time \( t \), \( d^a_n(\nu_t) = 1 \), she expects the total number of projects available to investors to be \( s_t = 1 + (\nu_t - 1) d^a_{n,-}(\nu_t) \). From Lemma 6, it implies that she will sell equity in her project next period for \( V^a_{nt}(s_t) \), yielding utility at \( t \) equal to \( u^a_n(v_t, 1, d^a_{n,-}) = \hat{u}(s_t) \) where

\[
\hat{u}(s_t) = \delta [V^a_{nt}(s_t) - c(y)] - k.
\] (27)

Alternatively, if the entrepreneur chooses not to develop her project at time \( t \), \( d^a_n(\nu_t) = 0 \), she will earn 0 at \( t \) but will still have the project at time \( t+1 \). Delaying the innovation has both a cost and a benefit. The cost of delaying the innovation is given by the discount factor \( \delta \), while the benefit is that a new entrepreneur may arrive in the economy, which will increase the expected market value of the equity of her company, (26). The overall utility can be represented recursively as

\[
U^a_n(v_t, d^a_{n,-}) = \max_{d^a_n \in \{0, 1\}} \left\{ d^a_n u^a_n(v_t, 1, d^a_{n,-}) + (1 - d^a_n) \delta EU^a_n(v_{t+1}, d^a_{n,-}) \right\}.
\] (28)

We now introduce the notion of equilibrium of our model.

**Definition 2** A Markov Perfect Equilibrium is a strategy combination \( \{d^a_n(\nu_t)\}_{n \in E_t} \) such that each entrepreneur maximizes (28) and investors value equity according to (24) and (26).

Given the strategies of the other entrepreneurs, \( d^a_{n,-} \), at each point in time entrepreneurs optimally decide whether or not to develop their innovation. Markov Perfect Equilibria for our game are characterized in the following theorem.

**Theorem 6** Let \( k \in (k_d, \bar{F}_d) \). In any Markov Perfect Equilibrium, there is a threshold \( \nu^* > 1 \) such
that $d_n^a(v_t) = 0$ for $v_t < v^*$ and $d_n^a(v^*) = 1$; when there are $v_t$ entrepreneurs with project-ideas, the corresponding equilibrium payoff is

$$U_n^a(v_t, d_n^a) = \left[ \frac{\delta \pi}{1 - \delta (1 - \pi)} \right]^{v^*-v_t} \hat{u}(v^*).$$

(29)

Let $\bar{v}$ be the smallest threshold such that there exists a Markov Perfect Equilibrium. The threshold values $(k_d, \bar{k}_d, \bar{v})$ are defined in the Appendix.

Lemma 6 and Theorem 6 imply that investor sentiment, market valuations of firm equity, and innovation decisions are endogenous, and depend on the number of innovative firms available on the market. If few entrepreneurs are endowed with a project-idea, from (25), they rationally anticipate that in the following period investor sentiment will be pessimistic, and correspondingly, market valuations will be low. This expectation of “cold equity markets” induces entrepreneurs to delay innovation to a later date. In contrast, when the number of entrepreneurs with an innovation is greater than a certain critical mass, $\bar{v}$, entrepreneurs anticipate that, if they all innovate, investors will have stronger sentiment in the following period, and, correspondingly, market valuations will be higher. The expectation of a “hot equity market” will thus induce entrepreneurs to innovate.

From Theorem 6, innovation occurs when the number of entrepreneurs with a project-ideas reaches critical mass $v^*$, for any $v^* \geq \bar{v}$. The following theorem shows that, if the probability that a new project-idea arrives, $\pi$, is sufficiently low it is best for entrepreneurs with project-ideas to pay the discovery cost $k$ and innovate as soon as their number $v_t$ exceeds the critical mass $\bar{v}$.

**Theorem 7** There a threshold $\bar{\pi}$ such that if $\pi \leq \bar{\pi}$ the efficient Markov Perfect Equilibrium is one in which entrepreneurs innovate as soon as their number exceeds critical mass, $v_t \geq \bar{v}$.

The factors affecting the value of the critical mass $\bar{v}$ are characterized in the following corollary.

**Corollary 3** The critical mass $\bar{v}$ is increasing in $\{\eta, k\}$, and decreasing in $\delta$.

The critical mass $\bar{v}$ depends positively on the level of uncertainty $\eta$. This happens because, from (25), a greater value of $\eta$ will make uncertainty-averse investors more pessimistic (all else equal). Thus, a greater number of project-ideas is needed to generate a level of investor sentiment
sufficiently strong to ignite innovation. Similarly, a greater discovery cost $k$ will require stronger investor sentiment, and thus greater equity valuations, to induce entrepreneurs to initiate the innovation process. Finally, a smaller discount factor $\delta$ will make entrepreneurs less patient, so they will require more positive sentiment to be willing to invest, requiring a larger critical mass.

Our model has the following implications for the innovation process in an economy. Theorem 7 implies that innovation activity remains latent in the economy when the number of entrepreneurs with project-ideas is below critical mass. During this time, entrepreneurs delay innovation, the market for entrepreneurial equity is “cold,” and dominated by low investor sentiment with a negative outlook. When the number of entrepreneurs with project-ideas reaches critical mass, entrepreneurs expect a substantial improvement in investor sentiment and a “hot” equity market for innovations. The improved expectations on the future market conditions spark an innovation wave that ripples through the economy. In addition, Corollary 3 implies that greater uncertainty, or a greater discovery cost, will lead to less frequent innovation waves, but when the wave takes place it will involve a larger number of innovations and will be characterized by stronger investor sentiment and equity valuations. Alternatively, if we interpret $\eta$ as characterizing the complexity of an industry, Corollary 3 implies that less complex industries are characterized by more frequent innovation waves, of smaller intensity, and with less ebullient equity markets. In contrast, more complex industries are characterized by relatively less frequent innovation waves but that, when they occur, are of greater intensity, and with more ebullient equity markets.

5 Discussion and Empirical Implications

5.1 Discussion and Extensions

In our model we make certain simplifying assumptions that facilitate analysis. We now discuss more closely these assumption and we assess their impact on our results. Explicit examination of models relaxing such assumptions is, however, beyond the scope of this paper.

First, and foremost, we develop our model under the simplifying assumptions that investors are uncertainty averse but risk neutral, and that entrepreneurs are both risk and uncertainty neutral. Our model can be extended to the case of investor risk aversion. In this case, risk-averse investors
would require a risk premium which will determine the appropriate risk-adjusted discount rates used to discount future expected cash flows. Such discount rates would reflect the appropriate market risk premium and the appropriate load factors for each firm. The main results of our paper will hold in this economy as well. The presence of risk-averse investors, however, may generate innovation waves in the same spirit as Acemoglu and Zilibotti (1997). In this case, the channel creating the wave relies crucially on a reduction of the market risk premium that is due to the large number of firms seeking financing in a wave. The feasibility of such waves, therefore, requires a volume of new firms that is sufficiently large to reduce the aggregate market risk premium in the economy. Thus, such a channel would not be able to explain sector-specific hot-market waves, where the overall equity market remains stable. In addition, risk premia will in general play little or no role in innovation process, where the risk faced by investors is essentially of technological nature, with little or no correlation with aggregate risk (as in Pastor and Veronesi, 2009).

Entrepreneurial uncertainty and risk aversion can also be introduced in the analysis at the cost of adding additional complexities which would not change the nature of our results. First, entrepreneurial uncertainty aversion will have only a negligible impact on our model because entrepreneurs are exposed uniquely to the uncertainty of their own firms. This means that entrepreneurs would use their “worst-case” scenario to assess the success probability of the innovation’s first stage, leaving our results unchanged. In contrast, risk-averse entrepreneurs will be concerned on the risk they face in the success of their own innovations, and they will require again a proper risk premium to initiate the innovation process. Interestingly, risk aversion of entrepreneurs amplifies the effect of investor uncertainty aversion. Risk-averse entrepreneurs will also be concerned about the likelihood that other entrepreneurs are successful as well, since the presence of other entrepreneurs will affect the market value of their firms at the interim date (given investors’ uncertainty aversion). This channel would complement the one we analyze in this paper, which is centered on investors.

In our paper it is also quite useful that the core belief set \( \mathcal{M} \) is a strictly convex set with a smooth boundary. Although this assumption greatly simplifies the analysis, it is not critical for our results. The main results of our paper depend only on the benefits of uncertainty hedging, a feature that is at the very heart of uncertainty aversion. This is because, by holding ambiguous assets in a portfolio, uncertainty hedging offers an uncertainty-averse investor an advantage that is analogous
to the benefits of diversification for traditional risk-averse investors in standard portfolio theory. In our context, loosely speaking, the benefits of uncertainty hedging are lost either when the worst-case probability for each investment can be taken on case-by-case basis, and independently from investor’s overall portfolio composition, or when there is a single source of uncertainty affecting all assets. Note that this situation is analogous to the loss of the benefits of diversification in traditional portfolio theory when assets are perfectly positively correlated.

Note that in our paper, for generality, we take the core-beliefs set of investors as a representation of their primitive preferences. The core-beliefs set, $C$, however could be obtained as the outcome of a “micro-foundation” that builds directly on investors’ uncertainty on economic fundamentals. In this spirit, in Appendix B, we present a model specification of that generates qualitatively identical results, but the source of uncertainty is consumer demand (formally, the proportion of consumers that exhibit a relatively stronger preference for each good). This specification of our model generates sector-specific innovation waves as described in our paper.

Finally, two additional important features that we deliberately ignore are the effects of learning and competition. First, learning about either technologies or the economic environment is clearly a key component of the innovation process. Our model suggests that, due to the complementarities we identify in our paper, learning in one project (or sector) may have important spillover effects in other projects (or sectors). In addition, learning may impact the extent of uncertainty present in the economy, affecting valuations, project investments, and investor sentiment. Second, the presence of product market competition may mitigate our results, because an early innovator may receive a monopoly rent, at least in the short-run (see, for example, Maksimovic and Pichler, 2001). A richer setting could be explored with the following trade-off. If an entrepreneurs innovates now, they must raise funds under negative market sentiment, but receive monopoly profits. If the entrepreneur waits, they raise funds at better terms under positive market sentiment, but receive lower profits ex post. Analysis of such a setting could provide for sector-specific implications on innovation waves. We leave consideration of these important cases to future research.
5.2 Empirical Implications

Our paper has several novel empirical implications on the relationship between innovation waves, equity valuations in the technology sectors, “hot” IPO markets and M&A activity.

1. Innovation waves. Strategic complementarity between entrepreneurs’ innovation decisions in our model creates the possibility of innovation waves. An innovation wave occurs if the number of potential entrepreneurs reaches critical mass. Arrival of innovation opportunities (i.e. project-ideas) in the economy may be random, and it may depend on classic “fundamentals” such as technological advances in certain sectors, say in Information Technologies or Life Sciences. Our paper suggests that such technological advances, while necessary, may not be sufficient to start a wave. Rather, an innovation wave occurs when a critical mass of potential innovators is attained which will spur a “hot” market for innovative companies.

Note that an innovation wave may start in one “sector” and then spill over to other “sectors,” even if they are unrelated. This can happen, for example, when a positive shock in the project idea of entrepreneurs in one sector lowers their discovery cost from a high level, $k_\tau > k_\tau$, to a low level, $k_\tau < k_\tau$, while the other entrepreneur faces a moderate discovery cost, $k_\tau' \in (k_\tau, k_\tau')$, $\tau \neq \tau'$. If the discovery costs of the first set of entrepreneurs are subject to a shock and decrease to a low level, $k_\tau < k_\tau$, it now becomes optimal for them to initiate the innovation process. This decision makes it profitable for other entrepreneurs to innovate as well, in anticipation of higher equity prices. Thus, a positive idiosyncratic shock to the technology in one sector spills over to other entrepreneurs, triggering an innovation wave in another sector. Note that the “contagion” across sectors works through an “equity valuation” channel which is driven by strong investor sentiment, rather than a pure technological channel. Similar results hold for the productivity of innovation, $Z_\tau$, and the probability of success, $q_\tau$. The beneficial spillover effect is more likely to occur the greater the degree of relatedness of the two technologies (the greater the value of $r$).

2. Innovation waves, investor sentiment, and hot IPO markets. Cyclical “hot and cold” markets for IPOs have been documented in the literature, and they largely remain a puzzle (see, for example, Ritter and Welch, 2002). Lowry (2003) finds that “hot” IPO markets are associated with strong

\[^{30}\text{For example, a positive technological shock to, say, LinkedIn may be a boost to Uber, even if no direct technological link is present.}\]
investor sentiment and high demand for capital by firms. In our model, the market value of an entrepreneur’s firm is (weakly) increasing in the number of successful firms in the market, and therefore on their demand for external capital. This is because uncertainty-averse investors are more optimistic when they can invest in the equity of a larger set of new firms, leading to higher equity valuations. Innovation waves will be associated with strong investor sentiment toward innovations and, thus, with booms in the equity of technology firms. This also means that innovation waves can be associated with hot IPO markets which are then followed by lower stock returns. Thus, our model can explain the relationship between IPO volume and stock market valuations and the subsequent lower returns documented in the literature.

3. Innovation waves and venture capitalists. An additional implication of our model is a new role for venture capitalists. If discovery costs fall in the intermediate range, \( k_r \in (\bar{k}_r, \overline{k}_r) \), entrepreneurs face an “assurance game” in that each entrepreneur will be willing to incur the discovery cost and innovate only if she is assured that other entrepreneurs will also do the same. Lacking such assurance, entrepreneurs may be confined to the inefficient equilibrium with no innovation. In this setting, a venture capitalist may play a positive role by addressing the coordination failure among entrepreneurs. By investing in several technology firms, the venture capitalist can help coordination among entrepreneurs and lead to greater innovation. Note that companies in the VC portfolio do not need to have directly related technologies for the VC to have a beneficial role. In addition, as discussed above, coordination among entrepreneurs’ innovative activities will be associated with greater equity market valuations. These observations imply that venture capital activity will be associated with innovation waves and greater equity valuations.

4. Venture capital cycles. The innovation cycle discussed in this paper also generates a VC cycle. Periods characterized by strong investor sentiment and a hot IPO market will be also associated with strong VC fundraising and commitment activity, as documented in Gompers et al (2008). VC investment will be active and at favorable terms for portfolio companies, driven by high expected exit multiples. Conversely, in periods where the market is cold, VCs will expect low exit multiples, generating limited fundraising and more unfavorable pricing of equity for portfolio companies.

5. Innovation, investor sentiment and merger activity. Our paper presents a new channel in which merger activity can generate synergies and spur innovation. Synergistic gains are the direct
outcome of the beneficial spillover effect of the merger on the expected value of the innovation. In the post-merger firm, innovators choose greater levels of innovation intensity, leading to greater innovation rates for the merged firms. Our model also predicts that merger activities involving innovative firms will be associated with strong investor sentiment and greater firm valuations.

6. Related Technologies. Our paper shows that entrepreneurs have an incentive to invest in related technologies. Formally, entrepreneurs’ utility is increasing in project relatedness, \( r \). Investors value innovation more when they can simultaneously invest in other technologies, so entrepreneurs have an incentive to develop innovation which is more likely to succeed at the same time that other innovations succeed. Further, this mitigates the coordination failure, because it makes innovation more attractive to the other entrepreneur as well.

7. Incubators. Our model also provides a new motivation for technological incubators. Incubators allow entrepreneurs to meet each other, and coordinate innovation decisions. Our model made the standard assumption of full information, but this may not hold in practice. By creating an incubator, entrepreneurs can meet each other, perhaps overcoming the coordination failure.

6 Conclusion

In this paper, we show that uncertainty aversion generates endogenous investor sentiment and innovation waves. Uncertainty aversion causes investors to treat different uncertain investments as complements, a property that depends on uncertainty hedging. Uncertainty hedging by investors generates strategic complementarity in entrepreneurial behavior, producing innovation waves. Specifically, when one entrepreneur has a successful first-stage project, equity valuation, entrepreneur utility, and the intensity of innovation increase for other entrepreneurs. Thus, entrepreneurs are more willing to innovate if they expect other entrepreneurs are going to innovate as well, resulting in multiple equilibria. Our model can thus explain why there are some periods when investment in innovation is “hot,” and investors are more willing to invest in risky investment projects tainted by significant uncertainty. Our model also has implications for the composition of venture capital portfolios, and the structure of the venture capital industry. This happens because of the possible beneficial role that venture capitalists can play to remedy a coordination failure that causes the inef-
cient no-innovation equilibrium. Finally, we argue that mergers can add value because the positive spillover effects of innovation due to uncertainty hedging. Thus, our model predicts simultaneous innovation waves, merger waves, and positive investor sentiment in “hot” equity markets.

References


A Appendix: Proofs

Proof of Lemma 1. Let $x = \{x_A, x_B\}$ be a vector of indicator variables for success of type $A$ and $B$ assets: $x \in \{0,1\}^2$. If the probability of success is $p = \{p_A, p_B\}$ the probability of $x$ is $p_A^x p_B^{x^c} (1 - p_A)^{1-x_A} (1 - p_B)^{1-x_B}$.

Thus, the relative entropy of $p$ w.r.t. $\hat{p}$ is

$$R(p|\hat{p}) = \sum_{x \in \{0,1\}^2} p_A^x p_B^{x^c} (1 - p_A)^{1-x_A} (1 - p_B)^{1-x_B} \ln \frac{p_A^x p_B^{x^c} (1 - p_A)^{1-x_A} (1 - p_B)^{1-x_B}}{p_A^{\hat{x}} p_B^{\hat{x}^c} (1 - p_A)^{1-\hat{x}_A} (1 - p_B)^{1-\hat{x}_B}}.$$ 

Because the log of a product is the sum of the logs, and probabilities sum to one, we can express this as

$$R(p|\hat{p}) = R(p_A|\hat{p}_A) + R(p_B|\hat{p}_B)$$

where $R(p_r|\hat{p}_r) = p_r \ln \frac{p_r}{\hat{p}_r} + (1 - p_r) \ln \frac{1 - p_r}{1 - \hat{p}_r}$. Because $rac{\partial^2 R}{\partial p_r^2} = \frac{\hat{p}_r}{p_r} + \frac{1 - \hat{p}_r}{1 - p_r}$, $R(p_r|\hat{p}_r)$ is strictly convex in $p_r$. Thus, $R(p|\hat{p})$ is strictly convex in $p = \{p_A, p_B\}$. Also, $\lim_{p_r \to 0} R(p_r|\hat{p}_r) = \ln \frac{1}{1 - \hat{p}_r}$ and $\lim_{p_r \to 1} R(p_r|\hat{p}_r) = \ln \frac{1}{\hat{p}_r}$. Define $\eta(\hat{p}) = \min_{\hat{p}_r \in Q} \ln \frac{1}{\hat{p}_r}$, where $Q = \{\hat{p}_A, 1 - \hat{p}_A, \hat{p}_B, 1 - \hat{p}_B\}$. Therefore, if $\eta < \eta(\hat{p})$, $M$, as the lower level set of a strictly convex function, is strictly convex. Note this generalizes: Theorem 2.5.3 of Cover and Thomas (2006) shows relative entropy is additively separable in independent variables, and Theorem 2.7.2 shows it is strictly convex.

Suppose an investor receives $y_A$ if innovation $A$ is successful and $y_B$ if innovation $B$ is successful, both of which are strictly positive. It can be quickly verified that $R$ achieves a minimum of zero at $p = \hat{p}$ and that $R$ is strictly convex in both arguments (most importantly $p$ here). This implies that $\frac{\partial R}{\partial p_r} < 0$ for $p_r < \hat{p}_r$ and $\frac{\partial R}{\partial p_r} > 0$ for $p_r > \hat{p}_r$. The worst-case scenario solves

$$\min \{pAy_A + pBy_B\}$$

$$R(p|\hat{p}) \leq \eta$$

Let $\lambda$ be the multiplier for the constraint, and $L$ be the Lagrangian function. Thus, $L = -(pAy_A + pBy_B) - \lambda (R(p|\hat{p}) - \eta)$, so $\frac{\partial L}{\partial p_r} = -y_r - \lambda \frac{\partial R}{\partial p_r}$. At the worst-case scenario, $\frac{\partial L}{\partial p_r} = 0$. Because $y_r > 0$, it must be that $\lambda \frac{\partial R}{\partial p_r} < 0$. This requires not only that the constraint binds, $\lambda > 0$, but also that $p_r$ is on the decreasing portion of $R$, or equivalently, that $p_r < \hat{p}_r$. If the investor has strictly positive exposure to only one uncertain act, but not the other, say $y_r > 0$ but $y_r^c = 0$, the worst-case scenario involves choosing the worst possible value of $p_r$, $R(p_r|\hat{p}_r) = \eta$ for $p_r < \hat{p}_r$, and setting $p_r^c = \hat{p}_r^c$. Finally, if $y_A = y_B = 0$, the claim holds WLOG.

Proof of Lemma 2. Investor’s worst-case scenario solves $U(\Pi) = \min u(\theta; \Pi)$ s.t. $\frac{1}{2} (\theta_A + \theta_B) = \theta r$. Let $L$ be the Lagrangian for the minimization problem, and $\lambda$ be the multiplier on the constraint. Thus, $\frac{\partial L}{\partial \theta_r} = -e^{\theta_r - \theta} \psi_r y_r + \frac{\lambda}{2}$. 

Because \( u \) is strictly convex in \( \theta \), FOCs are sufficient for a minimum. Setting \( \frac{\partial u}{\partial \omega_T} |_{\theta_T=\theta_T} = 0 \), and substituting into \( \frac{1}{2} (\theta_A + \theta_B) = \theta_T \), this implies
\[
\hat{y}_{\tau}^\theta = \theta_T + \frac{1}{2} \ln \frac{\omega_T e^{y_T}}{\omega_T y_T}.
\]
Thus, if \( \hat{y}_{\tau}^\theta (\Pi) \in [\theta_L, \theta_H] \), \( \hat{y}_{\tau}^\theta = \hat{y}_{\tau}^\theta \). If \( \hat{y}_{\tau}^\theta < \theta_L \), \( \frac{\partial}{\partial \omega_T} \omega_T < 0 \) for all \( \theta_T \in [\theta_L, \theta_H] \), so \( \hat{y}_{\tau}^\theta = \theta_T \). If \( \hat{y}_{\tau}^\theta > \theta_H \), \( \frac{\partial}{\partial \omega_T} \omega_T > 0 \) for all \( \theta_T \in [\theta_L, \theta_H] \), so \( \hat{y}_{\tau}^\theta = \theta_H \). Therefore, (8) corresponds to the worst-case scenario for an investor with portfolio II.

**Proof of Lemma 3.** Each investor’s objective function is
\[
U(\theta) = \min_{\theta \in C} u(\theta; \Pi) \text{ where } u(\theta; \Pi) = e^{\theta_A - \theta_B} \omega_A y_A + e^{\theta_B - \theta_A} \omega_B y_B + \omega_0 - \omega_A V_A - \omega_B V_B.
\]
Thus, for \( \tau \in \{A, B\} \),
\[
\frac{dU}{d\omega_T} = \frac{\partial u}{\partial \omega_T} + \frac{\partial u}{\partial \theta_A} \frac{d\theta_A}{d\omega_T} + \frac{\partial u}{\partial \theta_B} \frac{d\theta_B}{d\omega_T}.
\]
If investors are uncertainty-neutral, they believe \( \theta_T = \theta_T \), so the second & third terms disappear (\( \theta_T \) is constant). If investors are uncertainty averse, \( \theta_T \) solves the minimization problem. For interior solutions, by Lemma 2, \( \frac{\partial}{\partial \omega_T} \omega_T = \frac{\partial}{\partial \theta_T} \theta_T = 0 \). Therefore, \( \frac{dU}{d\omega_T} = \frac{\partial u}{\partial \omega_T} \) for \( \tau \in \{A, B\} \). Note \( \frac{d\theta_A}{d\omega_T} = \frac{d\theta_B}{d\omega_T} \) is identical for all investors. Note that WLOG optimal for all investors to set \( \omega_B = 1 \).

**Proof of Lemma 4.** From Lemma 2, \( \theta_1^\tau (\Pi) = \theta_L \) iff \( \hat{y}_{\tau}^\theta (\Pi) \leq \theta_L \) iff \( y_T \geq e^{2\theta_T} \). Thus, if \( y_T \geq e^{2\theta_T} \), \( V_T = p(\theta_L) y_T \) and \( V_T = p(\theta_T) y_T \). The \( \theta_2^\tau (\Pi) = \theta_H \) case is symmetric. Finally, from Lemma 2, \( \theta_0^\tau (\Pi) \in (\theta_L, \theta_H) \) iff \( y_T \in (e^{-2\theta_T}, e^{2\theta_T}) \). Because \( \theta_0^\tau (\Pi) = \hat{y}_{\tau}^\theta (\Pi) \), \( p(\theta_0^\tau) = e^{(\theta_0^\tau - \theta_L)} y_T \), which implies the market value entrepreneur \( \tau \)'s firm at \( V_T = e^{3 \theta_T - \theta_L} \gamma T \). There is strategic complementarity in production because \( \frac{\partial V_T}{\partial y_T} \geq 0 \) for \( \tau' = \tau \), with strict inequality for \( y_T \in (e^{-2\theta_T}, e^{2\theta_T}) \).

**Proof of Lemma 5.** Because only entrepreneur \( \tau \) has a successful first-stage project idea, \( y_T = 0 \). By Lemma 3, \( V_A = p(\theta_L) y_A \), so the entrepreneur’s payoff is
\[
U_T^{a, SF} = p(\theta_L) y_T - \frac{1}{Z_T (1 + \gamma)} y_T^{1+\gamma}.
\]
\[
\frac{\partial U_T^{a, SF}}{\partial y_T} = p(\theta_T) - \frac{1}{Z_T} y_T^{1+\gamma}, \text{ and } \frac{\partial^2 U_T^{a, SF}}{\partial y_T^2} = -\frac{1}{Z_T} y_T^{-1} < 0, \text{ so FOCs are sufficient. Entrepreneur } \tau \text{ selects } y_T^{a, SF} = [p(\theta_L) Z_T]^{\frac{1}{1+\gamma}}, \text{ sells for } U_T^{a, SF} = [p(\theta_L) Z_T]^{\frac{1}{1+\gamma}}, \text{ and earns continuation payoff } U_T^{a, SF} - [p(\theta_L) Z_T]^{\frac{1}{1+\gamma}}.
\]

**Proof of Theorem 2.** When both entrepreneurs have successful first-stage projects, they select innovation intensity \( y_T \) to maximize \( U_T^{a, SS} = U_T^{a, SS} (\Pi) - \frac{1}{Z_T (1 + \gamma)} y_T^{1+\gamma} \), where \( V_A (\Pi) \) is given in Lemma 4. For \( y_T < e^{-2\theta_T} \), \( \frac{\partial U_T^{a, SS}}{\partial y_T} = p(\theta_H) - \frac{1}{Z_T} y_T^{1+\gamma} \), and for \( y_T \in (e^{-2\theta_T}, e^{2\theta_T}) \), \( \frac{\partial U_T^{a, SS}}{\partial y_T} = \left( \frac{p(\theta_T)}{Z_T} \right) y_T^{1+\gamma} - \frac{1}{Z_T} y_T^{1+\gamma} \). Thus, \( \lim_{y_T \to e^{-2\theta_T}} \frac{\partial U_T^{a, SS}}{\partial y_T} = \frac{1}{Z_T} y_T^{1+\gamma} \), \( \lim_{y_T \to e^{2\theta_T}} \frac{\partial U_T^{a, SS}}{\partial y_T} = \frac{1}{Z_T} y_T^{1+\gamma} \), and \( \lim_{y_T \to e^{-2\theta_T}} \frac{\partial U_T^{a, SS}}{\partial y_T} = \frac{1}{Z_T} y_T^{1+\gamma} \). Therefore, any critical point \( y_T \leq e^{-2\theta_T} \) is a global maximum, but a critical point in \( (e^{-2\theta_T}, e^{2\theta_T}) \) could be just a local maximum, and must be compared to the critical point \( y_T \geq e^{2\theta_T} \).

We will now solve for the best-response function. It is optimal to select \( y_T < e^{-2\theta_T} \) only if \( y_T = \frac{1}{Z_T} y_T^{1+\gamma} < e^{-2\theta_T} \). Therefore, for \( y_T > \hat{y}_{\tau}^\theta \equiv e^{(2+\frac{1}{2}) \frac{1}{Z_T}} \), \( y_T \geq (p(\theta_T) Z_T)^{\frac{1}{1+\gamma}} \). It is optimal to select \( y_T < e^{-2\theta_T} \) only if \( \lim_{y_T \to e^{-2\theta_T}} \frac{\partial U_T^{a, SS}}{\partial y_T} \geq 0 \), which holds if \( y_T \in [\hat{y}_{\tau}^\theta, \hat{y}_{\tau}^\theta] \), where \( \hat{y}_{\tau}^\theta \equiv \frac{1}{Z_T} \left( \frac{p(\theta_T) Z_T}{1+\gamma} \right)^{\frac{1}{1+\gamma}} \) (\( \hat{y}_{\tau}^\theta \) defined above). The optimal \( y_T > e^{2\theta_T} \) is \( y_T^{SF} \) from Lemma 5, which provides \( U_T^{a, SF} = [p(\theta_L) Z_T]^{\frac{1}{1+\gamma}} y_T^{1+\gamma} \). For \( y_T \in (e^{-2\theta_T}, e^{2\theta_T}) \) to be optimal, it must not only be locally optimal, but also must provide greater utility.

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\[31] \text{For } y_T \in (e^{-2\theta_T}, e^{2\theta_T}), \frac{\partial^2 U_T^{a, SS}}{\partial y_T^2} = -\frac{1}{Z_T} y_T^{1+\gamma}, \text{ and for } y_T \notin (e^{-2\theta_T}, e^{2\theta_T}), \frac{\partial^2 U_T^{a, SS}}{\partial y_T^2} = -\frac{1}{Z_T} y_T^{1+\gamma} \]
Comparative Statics follow from inspection of Proof of Theorem 3. Theorem 2 showed $Z_{\alpha} < U_{\alpha}^* < U_{\alpha}^{SF}$ if $\gamma < \alpha$ and $Z_{\alpha} > U_{\alpha}^* > U_{\alpha}^{SF}$ if $\gamma > \alpha$. Therefore, if $\gamma < \alpha$, $y_r = \left[\frac{Z_r}{\theta_r} p(\theta_r) y_r \right]^{\frac{1}{4+\gamma}}$, which provides $U_{\alpha}^* = (p(\theta_r) \frac{Z_r}{\theta_r})^{\frac{2+\gamma}{4+\gamma}} \left[ 2^{\frac{2+\gamma}{4+\gamma}} \right] y_r^{\frac{1}{4+\gamma}}$. $U_{\alpha}^* > U_{\alpha}^{SF}$ if $y_r > y_r^* \equiv e^{-\alpha} [p(\theta_r)]^\frac{1}{2} Z_r^\frac{1}{2+\gamma} \left[ 2^{2+\gamma} \right]^{\frac{1}{4+\gamma}}.$

Restricting attention to pure strategy equilibria, it must be either the investors have interior beliefs, $y_r^* / y_r \in (e^{-2\alpha}, e^{2\alpha})$, or corner beliefs, $y_r^* / y_r \notin (e^{-2\alpha}, e^{2\alpha})$. Both entrepreneurs select innovation optimally. Because there is a kink at $y_r^* = e^{2\alpha} y_r$, it can never be that $\frac{y_r^*}{y_r} = e^{2\alpha}$ in equilibrium. Suppose to the contrary that investors have corner beliefs in equilibrium, so one entrepreneur selects $y_r > e^{2\alpha} y_r$. In that case, $y_r = e^\frac{\gamma}{2} [p(\theta_r)]^\frac{1}{2} Z_r^\frac{1}{2+\gamma}$, so $y_r > e^{2\alpha} y_r$ only if $\frac{Z_r}{y_r} > e^{-2\alpha}$, which holds by assumption. Therefore, when $\frac{Z_r}{y_r} \in \left( \frac{1}{\psi}, \psi \right)$, both entrepreneurs select innovation intensity so that investors have interior beliefs in equilibrium, selecting according to best-response function $y_r = \left[ \frac{Z_r}{\theta_r} p(\theta_r) y_r \right]^{\frac{1}{4+\gamma}}$, leading to equilibrium innovation $y_r^{\alpha} = \left[ \frac{1}{2} \right] p(\theta_r) Z_r^\frac{1}{2+\gamma} \left[ Z_r^\frac{1}{2+\gamma} \right]^{\frac{1}{4+\gamma}}$. Because the market price is $V_{\alpha}^* = p(\theta_r) \frac{1}{2} y_r^* V_{\alpha}^* = 2^{\frac{1}{4+\gamma}} [p(\theta_r)]^{\frac{2+\gamma}{4+\gamma}} Z_r \left[ Z_r^\frac{1}{2+\gamma} \right]^{\frac{1}{4+\gamma}}$, which can be expressed as $U_{\alpha}^{SF} = 2^{\frac{1}{4+\gamma}} [p(\theta_r)]^{\frac{2+\gamma}{4+\gamma}} Z_r \left[ Z_r^\frac{1}{2+\gamma} \right]^{\frac{1}{4+\gamma}}$, for $\gamma \in \left( A, B \right)$ and $\gamma \neq \tau$. Thus, there are strategic complementarities in production and profit.

**Proof of Corollary 1.** Theorem 2 showed $U_{\alpha}^{SF} > U_{\alpha}^*$ because $\frac{Z_r}{y_r} \in \left( \frac{1}{\psi}, \psi \right)$, where $\psi = \left[ \frac{1}{2} e^{2\alpha} \right]^{\frac{1}{4+\gamma}}$. $V_{\alpha}^* = 2^{\frac{1}{4+\gamma}} [p(\theta_r)]^{\frac{2+\gamma}{4+\gamma}} Z_r \left[ Z_r^\frac{1}{2+\gamma} \right]^{\frac{1}{4+\gamma}}$ and $V_{\alpha}^{SF} = [p(\theta_r)]^{\frac{2+\gamma}{4+\gamma}} Z_r \left[ Z_r^\frac{1}{2+\gamma} \right]^{\frac{1}{4+\gamma}}$, so $V_{\alpha}^* > V_{\alpha}^{SF}$ iff $\frac{Z_r}{y_r} > 4^{\frac{1}{4+\gamma}} e^{-2\alpha}$. Similarly, $y_r^{\alpha SF} = \left[ \frac{1}{2} \right] p(\theta_r) Z_r^\frac{1}{2+\gamma} \left[ Z_r^\frac{1}{2+\gamma} \right]^{\frac{1}{4+\gamma}}$ and $y_r^{\alpha SF} = [p(\theta_r)]^\frac{1}{2} Z_r \left[ Z_r^\frac{1}{2+\gamma} \right]^{\frac{1}{4+\gamma}}$, so $y_r^{\alpha SF} > y_r^{\alpha SF}$ iff $\frac{Z_r}{y_r} > 4^{\frac{1}{4+\gamma}} e^{-2\alpha(1+\gamma)}$. Therefore, $U_{\alpha}^{SF} > U_{\alpha}^*$, $V_{\alpha}^{SF} > V_{\alpha}^*$, and $y_r^{\alpha SF} > y_r^{\alpha SF}$.

**Proof of Theorem 3.** If only one entrepreneur innovates, she earns payoff $E_{\alpha}^{SF} = q_i U_{\alpha}^{SF} - k_\alpha$ (Lemma 5). Thus, if an entrepreneur does not expect the other entrepreneur to innovate, she will innovate iff $k_\alpha \leq \frac{k_\alpha}{\psi} \equiv q_i U_{\alpha}^{SF}$. Conversely, if the other entrepreneur innovates, the entrepreneur earns payoff $E_{\alpha}^{LI} = (q_i q_r + r) U_{\alpha}^{SF} + [q_r (1 - q_r) - r] U_{\alpha}^{SF} - k_\alpha$ if she innovates as well. Thus, if the other entrepreneur innovates, she innovates iff $k_\alpha \leq \frac{k_\alpha}{\psi (q_i q_r + r) U_{\alpha}^{SF} + [q_r (1 - q_r) - r] U_{\alpha}^{SF}}$. By Corollary 1, $U_{\alpha}^{SF} < U_{\alpha}^*$, so $k_\alpha < \frac{k_\alpha}{\psi}$. If both equilibria exist, $k_\alpha \in \left[ \frac{k_\alpha}{\psi}, \frac{k_\alpha}{\psi} \right]$ for both entrepreneurs. In the no-innovation equilibrium, entrepreneurs earn zero. In the innovation equilibrium, entrepreneurs earn $E_{\alpha}^{LI} > 0$ (strict inequality if $k_\alpha < k_\alpha$). Firms are priced so investors are indifferent. Everyone is better off, so the innovation equilibrium dominates the other.

**Proof of Corollary 2.** Comparative Statics follow from inspection of $k_\alpha$ and $\frac{k_\alpha}{\psi}$ from Theorem 3, and because $U_{\alpha}^{SF}$ is increasing in $Z_r$ and $Z_{\alpha}$, $U_{\alpha}^{SF}$ is increasing in $Z_r$, and $U_{\alpha}^{SF} > U_{\alpha}^{SF}$. **Proof of Theorem 4.** The merged firm seeks to maximize the combined value of the two projects. By identical

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32 This assumes that $y_r^{\alpha SF} > e^{2\alpha} y_r$. If not, selecting $y_r^{\alpha SF}$ would give entrepreneur $\alpha$ a strictly larger payoff than $y_r$, but a smaller payoff than $y_r$, which provides $U_{\alpha}^{SF}$ is strictly concave on $(e^{-2\alpha} y_r, e^{2\alpha} y_r)$. If $\frac{Z_r}{y_r} \in \left( \frac{1}{\psi}, e^{2\alpha(1+\gamma)} \right)$, the equilibrium is a mixed strategy equilibrium, where one firm selects $y_r = e^{-\gamma} [p(\theta_r)]^\frac{1}{2} Z_r^\frac{1}{2+\gamma} \left[ 2^{\frac{2+\gamma}{4+\gamma}} \right]$ and the other randomizes between $[p(\theta_r)]^\frac{1}{2} Z_r^\frac{1}{2+\gamma} \left[ 2^{\frac{2+\gamma}{4+\gamma}} \right]$ and $\frac{\theta_r}{2} p(\theta_r) y_r^{\frac{1}{4+\gamma}}$. 

33 If $\frac{Z_r}{y_r} \in \left( \frac{1}{\psi}, e^{2\alpha(1+\gamma)} \right)$, the equilibrium is a mixed strategy equilibrium, where one firm selects $y_r = e^{-\gamma} [p(\theta_r)]^\frac{1}{2} Z_r^\frac{1}{2+\gamma} \left[ 2^{\frac{2+\gamma}{4+\gamma}} \right]$ and the other randomizes between $[p(\theta_r)]^\frac{1}{2} Z_r^\frac{1}{2+\gamma} \left[ 2^{\frac{2+\gamma}{4+\gamma}} \right]$ and $\frac{\theta_r}{2} p(\theta_r) y_r^{\frac{1}{4+\gamma}}$. 

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reasoning to Lemma 3, \( V_r = p(\theta_t^n) y_r \), where \( \bar{\theta}^m \) is the market assessment at \( t = 2 \) on \( \bar{\theta} \). Thus, the merged firm’s objective is \( \mathcal{U}^{m,a} = p(\theta_t^n) y_A + (\theta_t^n) y_B - c_A(y_A) - c_B(y_B) \). Because investors are uncertainty averse, \( \bar{\theta}^m = \bar{\theta}^a \), which depends on the choice of \( y_A \) and \( y_B \). As shown in Lemma 4, \( V_A = V_B = p(\theta_t) y_A^\frac{1}{2} y_B^\frac{1}{2} \), so the objective becomes
\[
\mathcal{U}^{m,a} = 2p(\theta_t) y_A^\frac{1}{2} y_B^\frac{1}{2} \frac{1}{Z_A (1+\gamma)} y_A^{1+\gamma} - \frac{1}{Z_B (1+\gamma)} y_B^{1+\gamma}.
\]

Because \( \frac{\partial \mathcal{U}^{m,a}}{\partial y_r} = p(\theta_t) y_r^\frac{1}{2} y_B^\frac{1}{2} - \frac{1}{Z_r} y_r^\gamma \), for \( r \in \{A, B\} \) and \( r \neq \tau \), this implies \( Y^m_r(y_r) = \left[ p(\theta_t) Z_r y_r^\frac{1}{2} \right] ^\frac{1}{1+\gamma} \), so \( y^m_r = \left[ p(\theta_t) Z_r y_r^\frac{1}{2} \right] ^\frac{1}{1+\gamma} \). Thus, \( V^m_A = V^m_B = [p(\theta_t)]^{\frac{1}{1+\gamma}} \left[ Z_r Z_r^\gamma \right] ^\frac{1}{1+\gamma} \).

**Proof of Theorem 5.** If entrepreneur \( \tau \) does not expect entrepreneur \( \tau' \) to innovate, she innovates iff \( k_\tau \leq k_\tau^n = q_\tau \mathcal{U}_{\tau'}^{SS} \), the same cutoff as without the possibility of a merger. However, if entrepreneur \( \tau \) expects entrepreneur \( \tau' \) to innovate, she innovates iff \( k_\tau \leq k_\tau^n = (q_\tau q_{\tau'} + r) \mathcal{T}_{\tau'}^{m,a} + [q_\tau (1-q_{\tau'}) - r] \mathcal{U}_{\tau'}^{SS} \). Because \( \mathcal{T}_{\tau'}^{m,a} = \mathcal{U}_{\tau'}^{SS} + \frac{1}{2} \left( \mathcal{U}^{m,a} - \mathcal{U}_{\tau'}^{SS} - \mathcal{U}_{\tau'}^{SS} \right) \), if \( \mathcal{T}_{\tau'}^{m,a} > \mathcal{U}_{\tau'}^{SS} \), \( k_\tau^n > k_\tau \), so the cutoff will be larger when mergers are possible, resulting in more innovation. Thus, it is sufficient to show that \( \mathcal{U}^{m,a} > \mathcal{U}_{\tau'}^{SS} + \mathcal{U}_{\tau'}^{SS} \).

Because \( V^{m,a} = V^{m,a}_A + V^{m,a}_B \), the merged firm earns utility
\[
\mathcal{U}^{m,a} = 2 - \frac{\gamma}{1+\gamma} [p(\theta_t)]^{\frac{1}{1+\gamma}} \left[ Z_r Z_r^\gamma \right] ^\frac{1}{1+\gamma}.
\]

Each entrepreneur could earn utility \( \mathcal{U}_{\tau'}^{SS} = \frac{1}{2} \left[ p(\theta_t) \right]^{\frac{1}{1+\gamma}} \left[ Z_r Z_r^\gamma \right] ^\frac{1}{1+\gamma} \left[ 2^\gamma + 1 \right] ^\frac{1}{2\gamma} \) if they did not merge, so
\[
\mathcal{U}^{SS}_A + \mathcal{U}^{SS}_B = \mathcal{U}^{m,a} \frac{1}{2^\gamma} 2^\gamma + 1 \frac{1}{2\gamma}
\]

Because \( \frac{1}{2^\gamma} 2^\gamma + 1 \in (0,1) \) for all \( \gamma \in (0,\infty) \), the merger adds value: \( \mathcal{U}^{m,a} > \mathcal{U}_{\tau'}^{SS} + \mathcal{U}_{\tau'}^{SS} \), so \( T_{\tau} > U_{\tau}^{SS} \).

**Proof of Lemma 6.** At \( t+1 \), entrepreneurs in \( S_t \) chose to implement their project-ideas and thus have a successful first-stage innovation. Only implemented projects can be traded, so investors choose portfolio weights \( \{\omega_n\}_{n \in S_t} \) to maximize their minimum expected payoff, \( \min_{\bar{\theta} \in C} u \left( \bar{\theta} \right) \), where \( u \left( \bar{\theta} \right) = \sum_{n \in S_t} \omega_n \left[ \delta \left( \theta_{nt} \right) y_n - V_n \right] + \omega_0 \). By identical proof to Lemma 3, in equilibrium, \( \omega_n = 1 \) for all \( n \in S_t \) and \( V_n = \delta \left( \theta_{nt} \right) y_n \). Recall that \( \bar{\theta} \in C \) iff \( \sum_{n \in S_t} \left| \theta_n - \theta^n \right| \leq \bar{\theta} \). Let \( L \) be the Lagrangian function for the minimization problem and let \( \lambda \) be the multiplier for the sum. Thus, \( \frac{\partial L}{\partial \theta_{nt}} = -e^{\theta_{nt} - \theta M} y_n - \lambda I_n \). For all \( n \in S_t, y_n = y > 0 \), so \( -e^{\theta_{nt} - \theta M} y_n < 0 \). By complementary slackness, \( \lambda \geq 0 \), so it must be that \( \theta_n < \theta^* \). Therefore, \( \frac{\partial L}{\partial \theta_{nt}} = 0 \) if \( e^{\theta_{nt} - \theta M} y_n = \lambda \), so \( \theta_{nt} \) is constant for all \( n \in S_t \): \( \theta_{nt} = \theta^* - \frac{\theta}{y} \), because \( s_t = |S_t| \). Market valuation follows by substitution, and is increasing in \( s \) because \( \theta_{nt} \) is increasing in \( s \).

**Proof of Theorem 6.** We consider Markov Perfect Equilibrium with symmetric pure strategies. The state space is the number of entrepreneurs with projects, the action space is to develop or not, \( d_a \left( \nu_t \right) \in \{0,1\} \), the payoff solves (28). The transition probability is as follows: let \( D_t = \sum_{n \in \mathcal{E}_t} d_a \left( \nu_t \right) \). Because \( D_t \) projects are developed at time \( t \), and a new project arrives with probability \( \pi \), \( P \left( \nu_{t+1} = \nu_t + 1 \right) = \pi \) and \( P \left( \nu_{t+1} = \nu_t - D_t \right) = 1 - \pi \). The discount factor is assumed to be \( \delta \in (0,1) \) and, for simplicity, we assume that the initial state is \( \nu_0 = 0 \) (projects arrive during the game).

If an entrepreneur develops in a group of \( s (s-1 \text{ other entrepreneurs develop}) \), she earns at that time \( \hat{u} (s) = \delta^2 e^\delta s - \frac{\delta^2}{2} s \theta M y - \delta c (y) - k \), which is increasing in \( s \). To have innovation in equilibrium, \( u^n_a (s) > 0 \) for all equilibrium \( s \).

\[\text{Define } x = \frac{1}{2}, \text{ and } f (x) = 2^{-2x-1} (2 + x): f' (x) = 2^{-2x-1} [1 - (2 + x) \ln 2], \text{ which is strictly negative because } 2 \ln 2 > 1, \text{ and } \lim_{x \to \infty} f (x) = 0. \text{ Therefore, } \frac{1}{2} e^n (0,1) \text{ for all } \gamma \in (0, \infty).\]

By identical argument, undeveloped projects are not traded this period, so investors treat those projects as if \( y_n = 0 \). Thus, investors will assess any undeveloped or untraded project with \( \theta_{nt} = \theta^* \).
Thus, it must be that \( \lim_{s \to -\infty} \hat{u}(s) > 0 \) or equivalently, \( k < \bar{k}_d \equiv \delta^2 \epsilon^{\theta_{M}} y - \delta c(y) \). If each entrepreneur develops her innovation immediately upon discovering it, she earns utility \( \hat{u}(1) \). Thus, if \( \hat{u}(1) < 0 \), entrepreneurs optimally wait to innovate in waves. To achieve this, assume the cost of development is not too small: \( k > \bar{k}_d \equiv \delta^2 \epsilon^{\theta_{M}} y - \delta c(y) \), which holds iff \( \eta \) is big enough. Finally, define \( \bar{v} \equiv \min \{ s | \hat{u}(s) > 0, s \in \mathbb{N} \} \); the assumption that \( k \in (\bar{k}_d, \bar{k}_d) \) implies \( 1 < \bar{v} < +\infty \).

Suppose an entrepreneur believes all other entrepreneurs will develop according to \( d^*_n \), for \( n \in \mathbb{N} \). Because \( d^*_n \in (0, 1) \), let \( \nu^* \equiv \inf \{ n \in \mathbb{N} | d^*_n(\nu) = 1 \} \). Entrepreneurs can forgo innovation, earning 0, so \( \hat{u}(\nu^*) > 0 \), so \( \nu^* \geq \bar{v} \). We will guess and verify – that is, we will solve the value function (28) assuming the entrepreneur also plays \( d^*_n \), then verify \( d^*_n \) is optimal. Because \( \nu > \nu^* \) is off-equilibrium, \( U_n^g(v, d^*_n) \) is undefined for \( \nu > \nu^* \). Everyone develops when \( \nu_t = \nu^* \), so \( U_n^g(v^*, d^*_n) = \hat{u}(\nu^*) \). For \( \nu_t < \nu^* \), \( d^*_n = 0 \), so \( U_n^g(v_t, d^*_n) = \delta \delta E [U_n^g(v_{t+1}, d^*_n)] \) and \( U_n^g(v_t, d^*_n) = \pi U_n^g(v_t + 1, d^*_n) + (1 - \pi) U_n^g(v_t, d^*_n) \), which implies \( U_n^g(v_t, d^*_n) = \frac{\nu^*}{1 - \delta(1 - \pi)} U_n^g(v_t + 1, d^*_n) \).

Because this holds for all \( \nu_t < \nu^* \),

\[
U_n^g(v_t, d^*_n) = \left[ \frac{\delta \pi}{1 - \delta(1 - \pi)} \right]^\nu^* - \nu_t \hat{u}(\nu^*). 
\]

To show that it is optimal for the entrepreneur to play \( d^*_n \) when everyone else play it, we must show two things: it is optimal to wait when \( v_t < \nu^* \), and it is optimal to develop when \( \nu_t = \nu^* \). Because \( \hat{u}(\nu^*) > 0 \), \( U_n^g(v_t, d^*_n) > 0 \) for all \( \nu_t \). However, \( \hat{u}(1) < 0 \) because \( k > \bar{k}_d \), so it is optimal to wait until the wave to develop. Consider the development decision when \( \nu = \nu^* \). If an entrepreneur sets \( d = 0 \), because everyone else will develop, their expected utility is \( U_n^g(1, d^*_n) \). Because \( \frac{\delta \pi}{1 - \delta(1 - \pi)} < 1 \), this is strictly smaller than the payoff to innovating, \( \hat{u}(\nu^*) \), so they will innovate. Therefore, when everyone else plays \( d_n^* \), it is optimal to respond with \( d_n^* \) if \( \hat{u}(\nu^*) > 0 \), or equivalently, if \( \nu^* \geq \bar{v} \). Any positive NP cutoff, \( \nu^* \geq \bar{v} \), is also an equilibrium.

**Proof of Theorem 7.** An equilibrium is efficient (in a class) if it has a higher payoff to entrepreneurs that any other equilibrium (in that class), because investors are indifferent. Theorem 6 showed that any Markov Perfect Equilibrium is a threshold equilibrium: entrepreneurs wait until there are waves if \( \nu^* \geq \bar{v} \) projects to innovate. Consider when the efficient equilibrium is the quickest to develop: \( \nu^* = \bar{v} \). To do so, Compare the efficiency of equilibrium \( \nu^* \) to \( \nu^* + 1 \) (implied by equilibrium strategies \( d^* \) and \( d^{*+} \), respectively).

From Theorem 6, when an entrepreneur discovers a project, if everyone else plays \( d^* \), her continuation utility is

\[
U_n^g(v_t, d^*_n) = \left[ \frac{\delta \pi}{1 - \delta(1 - \pi)} \right]^\nu^* - \nu_t \hat{u}(\nu^*). 
\]

In contrast, if everyone else play \( d^{*+} \), her continuation utility is

\[
U_n^g(v_t, d^{*+}_n) = \left[ \frac{\delta \pi}{1 - \delta(1 - \pi)} \right]^{\nu^* - 1 - \nu_t} \hat{u}(\nu^* + 1). 
\]

Therefore, \( U_n^g(v, d^*_n) \geq U_n^g(v, d^{*+}_n) \) iff \( \hat{u}(\nu^*) \geq \frac{\delta \pi}{1 - \delta(1 - \pi)} \hat{u}(\nu^* + 1) \), which holds iff \( \pi \leq \frac{\hat{u}(\nu^*)}{\nu^*} = \frac{\nu^*}{1 - \delta(1 - \pi)} \hat{u}(\nu^* + 1) \). This cutoff is well-defined for because \( \hat{u} \) is strictly increasing. Finally, if this is satisfied for all \( \nu^* \geq \bar{v} \), \( \pi \leq \min_{\nu^* \geq \bar{v}} \frac{\hat{u}(\nu^*)}{\nu^*} \), the efficient equilibrium sets \( \nu^* = \bar{v} \).

**Proof of Corollary 3.** From the proof of Theorem 6, \( \bar{v} \equiv \min \{ s | \hat{u}(s) > 0, s \in \mathbb{N} \} \), where \( \hat{u}(s) = \delta^2 \epsilon^{\theta_{M}} y - \delta c(y) \). Because \( \hat{u'} > 0 \), anything that increases \( \hat{u} \) decreases \( \bar{v} \), and vice versa. \( \frac{\partial \bar{v}}{\partial \eta} = -\frac{\delta^2 \epsilon^{\theta_{M}} y - \delta c(y)}{\delta} < 0 \), so \( \bar{v} \) is increasing in \( \eta \), \( \frac{\partial \bar{v}}{\partial \eta} = -\frac{\delta^2 \epsilon^{\theta_{M}} y}{\delta} < 0 \), \( \bar{v} \) is increasing in \( k \). \( \frac{\partial \bar{v}}{\partial \eta} = 2 \delta \epsilon^{\theta_{M}} y - \delta c(y) \), so \( \bar{v} \) is increasing in \( \eta \), \( \frac{\partial \bar{v}}{\partial \eta} = -\frac{\delta^2 \epsilon^{\theta_{M}} y}{\delta} + \hat{u}(s) + \frac{\hat{u}(s)}{\delta} \), which is strictly positive in the neighborhood of \( \bar{v} \) (by definition of \( \bar{v} \)). Thus, \( \bar{v} \) is decreasing in \( \delta \).

36 Note \( \hat{u}(s) \) is increasing in \( s \) and bounded, \( \lim_{s \to -\infty} \hat{u}(s) = 0 \). Thus, the entrepreneur will not hold a project indefinitely in hopes of earning an infinite payoff.

37 Because \( k > \bar{k}_d \), it is also an equilibrium to set \( \nu^* = +\infty \): if entrepreneurs believe innovation never occurs, they will refuse to ever innovate. We could still have waves if \( k < \bar{k}_d \): \( \nu^* \) must satisfy \( \left[ \frac{\delta s}{1 - \delta(1 - \pi)} \right]^{\nu^*-1} \hat{u}(\nu^*) > \hat{u}(1) \).
B Appendix: Demand Uncertainty

A key ingredient of our paper is that program (6) is a strictly convex programming problem which generates “interior beliefs” for well-diversified portfolios. In the main body of the paper, the possibility of such interior beliefs is a consequence of (strict) convexity of the relative entropy function $R(\cdot)$, which produces a strictly convex core beliefs set $\mathcal{M}$ (see Figure 1). Thus, no specific parametric restriction on the joint probability $p$ is needed to generate our results. In this appendix, we present an alternative “micro-foundation” where interior beliefs are the outcome of uncertainty about consumer demand. All results in our paper remain qualitatively the same in this specification.

Consider a simple extension of our three-dates model. There are three types of goods: type $\tau$ goods, $\tau \in \{A, B\}$, and the numeraire. There are two firms, each specializing in the production of goods of type $\tau$. At $t = 1$, entrepreneurs decide whether to pay the discovery cost to innovate. If successful, at $t = 2$, each entrepreneur will select the optimal investment into the project, $y_t$, financed by issuing equity to uncertainty-averse investors. The investment decision is made under demand uncertainty for each product (as described below). At $t = 3$, consumer demand is revealed and production decisions of firms are made. If successful, entrepreneurs will be monopolists in their innovative good market. For tractability, we assume that entrepreneur $\tau$ has production costs $c_\tau(Q_\tau) = K_\tau Q_\tau$, and that the intermediate investment $y_t$ lowers, at a cost $\xi(y_t) = \frac{\xi}{2} y_t^2$, the per-unit production cost: $K_\tau = K_0 - K_1 y_t$.

There are two types of consumers, type $A$ and type $B$, with a total mass of 1. Consumers value both goods, as well as the numeraire, but each consumer values one good more than the other, which determines their type. The price of the numeraire is fixed to 1, while the price of type $\tau$ good, $P_\tau$, is determined in equilibrium. For simplicity, we assume quadratic utility for each type of consumer. Thus

$$U^\tau(q^\tau_A, q^\tau_B) = (D + \Delta) q^\tau_A - \frac{\beta}{2} (q^\tau_A)^2 + D q^\tau_B - \frac{\beta}{2} (q^\tau_B)^2 + w - P_\tau q^\tau_A - P_\tau q^\tau_B,$$

where $D$, $\Delta$, and $\beta$ are strictly positive parameters. For simplicity, we assume that $w$ and $D$ large enough so that consumers (in equilibrium) always consume a positive amount of all goods available in the market. It is easy to verify that the consumer $\tau$’s demand function for good $\tau$ is $q^\tau_\tau = \frac{1}{\beta} (D + \Delta - P_\tau)$, and for good $\tau'$ is $q^\tau_{\tau'} = \frac{1}{\beta} (D - P_{\tau'})$. Let $m_\tau \in [m_A, m_B]$ be the proportion of consumers of type $\tau$, with $m_A + m_B = 1$. Market clearing condition for good $\tau$ requires that $m_\tau q^\tau_A + m_{\tau'} q^\tau_{\tau'} = Q_\tau$, where $Q_\tau$ is the output of a firm type $\tau$. Thus, market clearing requires that

$$P_\tau(Q_\tau) = D + m_\tau \Delta - \beta Q_\tau,$$

and the price of type-$\tau$ goods is increasing in $m_\tau$. Because producers know $m_\tau$ when making their production decisions $Q_\tau$, they maximize

$$\pi_\tau(Q_\tau) = P_\tau(Q_\tau) Q_\tau - K_\tau Q_\tau,$$

which gives

$$Q_\tau = \frac{D + m_\tau \Delta - K_\tau}{2\beta}.$$

Letting $\Pi_\tau = \max_{Q_\tau} \pi(Q_\tau)$, we have that entrepreneur $\tau$ profits are

$$\Pi_\tau = \frac{[D + m_\tau \Delta - K_\tau]^2}{4\beta}.$$

This implies that, when both entrepreneurs are successful, investors beliefs are determined by solving:

$$\min_{(m_A, m_B)} U = \omega_A \left[ \frac{D + m_A \Delta - K_A (y_A)}{4\beta} - V_A \right] + \omega_B \left[ \frac{D + m_B \Delta - K_B (y_B)}{4\beta} - V_B \right] + \omega_0$$

s.t. $m_A + m_B = 1$,

which is a (strictly) convex programming problem, with the same qualitative properties as (6).
Figure 1: Core Belief Set. This figure represents the core belief set implied by the relative entropy criterion.

This figure shows the core of beliefs when the maximum relative entropy criteria is applied. That is, it shows the set of probability distributions $p = (p_A, p_B)$ that satisfy $\{ p | R(p|\tilde{p}) \leq \eta \}$ when $\tilde{p}_A = \tilde{p}_B = \frac{1}{2}$ and $\eta = \frac{1}{30} \ln 2$. If $p_B = \tilde{p}_B = \frac{1}{2}$, the relative entropy criteria implies that $p_A \in [0.1893, 0.8107]$. In contrast, if $p_A = p_B$, then the relative entropy criterion implies that $p_A \in [0.2760, 0.7240]$.

This illustrates that an uncertainty-averse investor treats different uncertain lotteries as complements. The lower left boundary, which is darkened, represents the relevant portion of the core beliefs for investors with long positions in both risky assets. Specifically, if an uncertainty-averse investor invested in only one innovative project, he would value it as though it would succeed with probability 18.93%. In contrast, if he could invest equally in both, he would believe that each would succeed with probability 27.60%. Therefore, there will be strategic complementarities in innovation, because entrepreneurs rationally anticipate that they will face more positive sentiment if they both arrive at the same time.