Innovation Waves, Investor Sentiment, and Mergers*

David Dicks
Hankamer School of Business
Baylor University

Paolo Fulghieri
Kenan-Flagler Business School
University of North Carolina
CEPR and ECGI

December 18, 2017

Abstract

We develop a theory of innovation waves, investor sentiment, and merger activity based on uncertainty aversion. Investors must typically decide whether to fund an innovative project with very limited knowledge of the odds of success, a situation best described as "Knightian uncertainty." We show that uncertainty-averse investors are more optimistic on an innovation if they can also make contemporaneous investments in other innovative ventures. This means that uncertainty aversion makes investment in innovative projects strategic complements, which results in innovation waves. Innovation waves occur in our economy when there is a critical mass of innovative companies and are characterized by strong investor sentiment, high equity valuation in the technology sector, and "hot" IPO and M&A markets. We also argue that M&A promotes innovative activity and leads to greater innovation rates and firm valuations.

Keywords: Innovation, Ambiguity Aversion, Hot IPO Markets

*We would like to thank Hengjie Ai, Utpal Bhattacharya, Thomas Chemmanur, Chong Huang, Ramana Nanda, Scott Rockart, Jacob Sagi, Martin Schmalz, and seminar participants at Baylor, Bocconi, Calgary, Duke, Georgia State, HKUST, Imperial, Kellogg, Michigan State, Northeastern, Oklahoma, UNC Interdisciplinary Research Seminar, UNC-Charlotte, UT-Dallas, Wharton, York, the IAS Conference on Entrepreneurship and Finance, the CEPR Spring Symposium in Financial Economics, the FSU SunTrust Beach Conference, the Society for Financial Studies Cavalcade, the Financial Intermediation Research Society, the 5th Annual Corporate Finance Conference at Lancaster University (keynote), the American Finance Association, and the Midwest Finance Association. We can be reached at David_Dicks@baylor.edu and Paolo_Fulghieri@kenan-flagler.unc.edu

"Most, probably, of our decisions to do something positive, the full consequences of which will be drawn out over many days to come, can only be taken as the result of animal spirits—a spontaneous urge to action rather than inaction, and not as the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities."
Keynes, The General Theory of Employment, Interest and Money (1936).

Innovation is arguably one of the most important value drivers in modern corporations and a key source of economic growth (Solow, 1957). There are times when innovation is stagnant, but other times when technology leaps forward in innovation waves. These waves of innovation activity are often associated with high stock market valuations for technology firms and strong investor sentiment (see Shiller, 2000, Perez, 2002, and Baker and Wurgler, 2007). Interestingly, such flare-ups in stock market valuations may be limited to the technology sectors, with a relatively smaller impact on the general ("traditional") market (Pastor and Veronesi, 2009). Innovation can also be an important determinant of mergers and acquisitions (M&A) activity (Bena and Li, 2014).

In this paper we derive a joint theory of innovation waves, investor sentiment, and merger activity based on uncertainty aversion. Innovation, by its very nature, is characterized by a very limited knowledge of the probability distributions relevant for the innovation process, a situation best described as "Knightian uncertainty" (Knight, 1921). In this situation, investors must typically decide whether to fund an innovative project with very limited knowledge of the odds of success. We show that uncertainty aversion can generate innovation waves associated with strong investor sentiment, high stock market valuations, and an active M&A market.

There are many reasons why innovation develops in waves. These include fundamental reasons such as random scientific breakthroughs in the presence of externalities and technological spillovers. In this paper, we focus on the interaction between financial markets and the incentives to create innovation. We argue that innovation waves can be the product of investor uncertainty aversion: investor uncertainty aversion creates externalities in innovative activities which results in innovation waves characterized by strong investor sentiment and high stock market valuations. We also show that innovation waves lead to an active M&A market which further promotes innovation activities. Finally, our model suggests that innovation waves may lead to "hot" IPO markets.

We study an economy with multiple entrepreneurs endowed with project-ideas. Project-ideas are risky and, if successful, may lead to innovations. The innovation process consists of two stages. In the first stage, entrepreneurs must decide whether or not to invest personal resources, such as effort, to innovate. If the first stage is successful, further development of the innovation requires additional investment in the second stage. Entrepreneurs raise funds for the second-stage investment by selling shares of their firms to uncertainty-averse investors. The second stage of the innovation process is uncertain: outside investors are uncertain of the exact distribution of the residual success probability of the innovation process. We model uncertainty aversion by assuming outside investors maximize Minimum Expected Utility (MEU), as in Gilboa and Schmeilder (1989).

An important implication of uncertainty aversion, which plays a key role in our paper, is that probabilistic assessments (or "beliefs" in the sense of de Finetti, 1974) held by uncertainty-averse investors on future returns are not uniquely determined by a single prior but, rather, are determined endogenously as the solution of a minimization problem. A direct consequence is that investors prefer to hold an uncertain asset if they can also hold other uncertain assets in their portfolios, a feature that is known as "uncertainty hedging" (see Epstein and Schneider, 2007, and 2010). By holding uncertain assets in a portfolio, investors can lower their overall exposure to the sources of uncertainty in the economy. Because of uncertainty hedging, an investor will also hold more favorable probabilistic assessments toward an innovation – and thus be more optimistic – if he/she is able to invest in other innovations as well.[1] We will refer to the probabilistic assessments held by investors on the success of innovations as characterizing their "sentiment." Thus, our paper provides a decision-theoretic foundation of the notion of "sentiment" that has been suggested to play an important role in the economy (Baker and Wurgler, 2007).

In our economy, uncertainty-averse investors are more optimistic (i.e., have a stronger sentiment) and are willing to pay more for equity in a given entrepreneurial firm when other entrepreneurs innovate as well. By investing in a portfolio of (possibly independent) R&D processes, an uncertainty-averse investor will deem as very unlikely the state of the world in which all such R&D efforts will fail.[2] Thus, investments in different innovative companies are effectively complements,

[1] As discussed later, "uncertainty hedging" is the analog to traditional risk diversification in portfolio theory, but in the context of uncertainty on the true probability distributions that govern the random variables that are relevant to the decision maker. For further (closely related) discussion on the effect of uncertainy-hedging on investors' probabilistic beliefs, see Dicks and Fulghieri (2017).

[2] For example, there is currently considerable uncertainty on the technical difficulties related to the development of self-driving cars, an area in which several companies are engaged in substantial R&D effort. Clearly, there is very

associated with high valuations and more optimistic investor sentiment. This further implies that the market value of a new firm will be greater when multiple new firms are also on the market.

The key feature of our model is that strategic complementarity between innovative activities due to uncertainty aversion can generate innovation waves.[3] An innovation wave occurs when the number of innovators in a technological sector reaches critical mass. Arrival of innovation opportunities in the economy may be random and due to exogenous technological progress. We argue that such technological advances, while seeding the ground for an innovation wave, may not be sufficient to ignite one. Rather, an innovation wave will start when a critical mass of innovators is attained, which will spur a "hot" market for innovative companies. Thus, innovation waves are characterized by strong investor sentiment and a wave of "rational exuberance" with high equity market valuations. In our model, equity market "booms" in technology markets can materialize, and these booms are beneficial since they spur valuable innovation.

Our paper can be extended in several ways. An important feature that we deliberately ignore is the effect of learning. Learning about either technologies or the economic environment is clearly an key component of the innovation process. Our model suggests that, due to the complementarities we identify in our paper, learning in one project (or sector) may have important spill-over effects in other projects (or sectors). In addition, learning may affect the extent of uncertainty present in the economy, affecting valuations, project investments, and investor sentiment. We leave these important issues to future research.

The channel we propose in our model, based on uncertainty aversion, differs substantially from traditional "neoclassical" explanations.[4] Shleifer (1986) argues that innovations in one sector have a positive externality on innovators in other sectors, because of the positive effect that innovations have on aggregate demand. Similar to our paper, innovators prefer to postpone their innovation to periods of time when other innovators undertake theirs, generating self-fulfilling boom-and-bust

little information on the true odds of discovery that are relevant for each producer. Most likely, however, one of such innovators will generate a workable technology that will become the industry standard. By investing in a portfolio of companies, investors (such as VCs) limits the exposure to the event that all projects fail, and will increase the exposure to the possibility of having a very successful project (a "unicorn") in their portfolio.

[3] Note that, by design, we ignore the potential adverse effect on innovation incentives due to competitive pressure of the product market. The effect of competition (and patent races) on innovations incentives has been extensively examined in the literature (see, for example, Aghion et al, 2005, among many others).

[4] Note that the potential observational equivalence between models based on uncertainty aversion and those based on standard risk-averse models is an issue discussed in the literature (see, for example, the discussion in Maenhout, 2004, and Skiadas, 2003, among others).

cycles. Different from our paper, the boom-and-bust cycle occurs through the effect that a favorable aggregate macroeconomic environment has on the value of innovation, while in our model waves may be localized in a specific sector, even if the overall economy is not booming. Thus, our model can explain the boom in the biotech market in 1980-1992, which occurred around the economic recession of early 1990's and within a relatively calmer overall stock market (see Booth, 2016).

Acemoglu and Zilibotti (1997) argues that at the early stages of economic development, when capital is critically limited, the presence of project indivisibilities caps the range of risky investment projects that will be implemented in an economy, reducing the benefits of risk sharing, thus discouraging investment in risky assets. Our paper differs from Acemoglu and Zilibotti in many important dimensions. First, our results do not rely on the limited supply of capital but, rather, are driven by the random arrival of innovative ideas in the economy. In our model capital is abundant and, thus, it is better suited to explain innovation waves in more mature economies, while Acemoglu and Zilibotti is better suited to explain the random growth rates of economies at the earlier stages of their development. Second, in Acemoglu and Zilibotti a "wave" (or, perhaps a "crash") may occur as the outcome of negative production shocks that reduce capital available in the economy and, thus, restricts its diversification opportunities, setting back its growth path. In contrast, in our paper, a wave comes to end when the innovations that were initiated in that wave are completed, and a new wave starts when a new critical mass is achieved.

More generally, traditional portfolio-diversification arguments can only generate innovation waves and high stock market valuations as the outcome of a reduction of the economy-wide market price of risk. In this case, innovation waves will necessarily be associated with economy-wide equity market booms. Our approach, in contrast, can explain the apparent "boom and bust" behavior that are concentrated in technology sectors, such as the Life Sciences and the Information Technology, where hot periods alternate with cold periods in innovation rates, merger activity and asset valuations. For example, in the boom years of 1998-2000, the NASDAQ index, which is dominated by technology companies, more than doubled while the general market, as measured for example by the S&P500 index, remained substantially stable. The divergent behavior between a technology sector and the general market would be difficult to reconcile on the basis of risk aversion. Thus, Acemoglu and Zilibotti (1997), as Shleifer (1986), can only explain innovation waves that are (perfectly) correlated with aggregate variables such as the overall stock market or the general level economic activity. In contrast, our paper can explain sector-specific innovation waves that are not necessarily correlated with the aggregate market.

Our paper also has implications for the impact of M&A activity and, more generally, of the ownership structure on innovation rates. In the new channel we propose, mergers of innovative firms create synergies and spur innovation. In our paper, positive synergies in an acquisition are endogenous, and are the direct outcome of the beneficial spillover (i.e., externality) on the probabilistic assessments of future returns on innovation due to uncertainty aversion.[5] In addition, our model predicts that merger activities involving innovative firms will be associated with strong investor sentiment and, thus, greater valuations.

Finally, we argue that uncertainty aversion has implications for the composition of venture capital portfolios, and the structure of the venture capital industry. This happens because of the possible beneficial role that venture capitalists can play to remedy a coordination failure that causes the inefficient no-innovation equilibrium.

Literature review. Our paper contributes insights from uncertainty aversion to three strands of literature. First, and foremost, our paper belongs to the rapidly expanding literature on the determinants of innovation and innovation waves (see Fagerberg, Mowery and Nelson, 2005, for an extensive literature review).[6] The critical role of innovation and innovation waves in modern economies has been extensively studied at least since Schumpeter (1939) and (1942), Kuznets (1940), Schmookler (1966), and, more recently, Kleinknecht (1987), and Aghion and Howitt (1992). In this early research, which is focused mostly on the technological "fundamentals" behind innovation, innovation waves are driven by a technological breakthrough that affects an entire sector, such as the positive spillover effects across different technologies.

More recent research focuses on the link between innovation waves, the availability of financing, and stock market booms. Scharfstein and Stein (1990) suggest reputation considerations by investment managers may induce them to herd their stock market behavior facilitating the financing of technology firms. Gompers and Lerner (2000) find higher venture capital valuations are not

[5] Hart and Holmstrom (2010) develop a model where mergers create value by internalizing externalities, such as coordinating on a technological standard.

[6] Chemmanur and Fulghieri (2014) discuss current issues related to entrepreneurial finance and innovation.

necessarily linked to better success rates of portfolio companies. Perez (2002) shows technological revolutions are associated with "overheated" financial markets. Gompers et al. (2008) suggest that increased venture capital funding is the rational response to positive signals on technology firms' investment opportunities. Nanda and Rhodes-Kropf (2013) find that in "hot markets" VCs invest in riskier and more innovative firms. Nanda and Rhodes-Kropf (2016) argue favorable financial market conditions reduce refinancing risk for VCs, promoting investment in more innovative projects. A positive effect of investor sentiment on innovation is documented in Aramonte (2016).

To our knowledge, ours is the first paper that models explicitly the role of uncertainty aversion on the innovation process and its impact on innovation waves and stock market valuations. We show that investor uncertainty aversion can generate innovation waves that are driven by investors' optimism, that is, their positive sentiment. In our model, due to uncertainty aversion, investors' probabilistic assessments are endogenous, and they respond to the availability of investments in innovative projects. Innovation waves and stock market "exuberance" are jointly determined in equilibrium in a model where investors are sophisticated. In our model, greater investment in innovation activities occurs simultaneously with investor optimism and stock market booms.

Thus, our work also contributes to the emerging literature on uncertainty aversion in financial decision making and asset pricing.[7] Uncertainty aversion has been proposed as an alternative to Subjective Expected Utility (SEU) to describe decision making in cases where agents have limited information on probability distributions. This stream of research was motivated by a large body of work documenting important deviations from SEU and the classic Bayesian paradigm (see Etner, Jeleva, and Tallon, 2012). While the degree of ambiguity aversion may vary across treatments and subjects, the presence of ambiguity aversion appears to be a robust experimental regularity. Interestingly, Chew, Ratchford, and Sagi (2013) document that ambiguity-averse behavior is particularly relevant among more educated (and analytically sophisticated) subjects.

Uncertainty aversion has also been shown to be an important driver of asset pricing, providing an explanation for observed behavior that would otherwise be puzzling in the context of SEU. For

[7] This paper is part of the growing literature studying ambiguity aversion in finance, including Mukerji and Tallon (2001), Maenhout (2004), Epstein and Schneider (2008) and (2010), Easley and O'Hara (2009) and (2010), Caskey (2009), Bossaerts et al. (2010), Illeditsch (2011), Drechsler (2013), Jahan-Parker and Liu (2014), Byun (2014), Mele and Sangiorgi (2015), Gallant, Jahan-Parvar and Liu (2015), Dimmock et al. (2016), Garlappi, Giammarino, and Lazrak (2016), Miao and Rivera (2016), and Dicks and Fulghieri (2016) and (2017). For a thorough literature review, see Epstein and Schneider (2008) and (2010).

example, Anderson, Ghysels, and Juergens (2009) find stronger empirical evidence for uncertainty than for traditional risk aversion as a driver of cross-sectional expected returns. Jeong, Kim, and Park (2015) estimate that ambiguity aversion is economically significant and explains up to 45% of the observed equity premium. Boyarchenko (2012) shows that the sudden increase in credit spreads during the financial crisis can be explained by a surge in uncertainty faced by uncertainty-averse market participants. Dimmock et al. (2016) show that ambiguity aversion helps explain several household portfolio choice puzzles, such as low stock market participation, low foreign stock ownership, and high own-company stock ownership.

The second stream of literature is the recent debate on the links between technological innovation and stock market prices. Nicholas (2008) shows that an important driver of the stock market run-up experienced in the American economy in the late 1920's was the strong innovative activity by industrial companies which affected the market valuation of corporate "knowledge assets."

Two closely related papers are Pastor and Veronesi (2005) and Pastor and Veronesi (2009). The first paper argues that IPO waves can be the outcome of a change in the "fundamentals" characterizing a firm and its environment, such as an exogenous decrease in the market expected return. In our paper, in contrast, IPO waves can also occur in a stationary environment, and endogenously occur with high stock market valuations. The second paper argues that technological revolutions can generate dynamics in asset prices in innovative firms observationally similar to assets bubbles followed by a valuation crash. Their paper argues that this "bubble-like" behavior of stock prices is the rational outcome of learning about the productivity of new technologies, where the risk is essentially idiosyncratic, followed by the adoption of the new technologies on large scale, where the risk becomes systematic. Our paper proposes a new explanation for the link between innovative activity and stock market booms. In Pastor and Veronesi (2009) stock market booms (and subsequent crashes) are the outcome of the changing nature of risk that characterizes technological revolutions, from idiosyncratic to systematic, and its impact on discount rates. In our model, periods of strong innovative activity are accompanied by high valuations because innovation waves are, in equilibrium, associated with more optimistic expectations on future cash flows from innovations. Thus, our model, which focuses on expected cash flows, complements theirs, that focus on discount rates.

The third stream of literature focuses on the drivers of merger waves and the impact of M&A activity – and, more generally, of ownership structure – on incentives to innovate. High stock market valuations are also associated with strong M&A activity in merger waves (Maksimovic and Phillips, 2001, and Jovanovic and Rousseau, 2001). Rhodes-Kropf and Viswanathan (2004) argue that such correlation is the outcome of misvaluation of the true synergies created in a merger when the overall market is overvalued. The impact of M&A activity on corporate innovative activity has been documented by several empirical studies. Phillips and Zhdanov (2013) show that a firm's R&D expenditures increase in periods of strong M&A activity in the same industry. Bena and Li (2014) argue that the presence of technological overlap between two firms innovative activities is a predictor of the probability of a merger between firms. Bernstein (2015) documents that in the three years after their IPO, firms engage in strong M&A activity, acquiring a substantial number of patents. Sevilir and Tian (2012) show that acquiring innovative target firms is positively related to acquirer abnormal announcement returns and long-term stock return performance. The importance of the presence of technological overlaps between acquiring firms and targets is confirmed by Seru (2014), which finds that innovation rates are lower in diversifying mergers, where the technological benefits of a merger are likely to be absent. Entezarkheir and Moshiri (2016) show mergers are more likely among innovative firms.

In our model we are able to jointly generate the observed positive correlations between stock market valuations, the level of M&A activity, and innovation rates. Specifically, our paper creates a novel direct link between stock price valuations, M&A activity, and greater innovation rates that is based on investors' uncertainty aversion. Endogeneity of probabilistic assessments creates an externality between innovations that is at the heart of synergy creation in mergers of innovative companies. This externality results in greater innovation rates and innovation waves that are characterized by strong investor sentiment and greater stock market valuations.

Finally our paper is linked to the recent literature on investor sentiment and stock market valuations. Baker and Wurgler (2007) suggest that investor sentiment, in the form of "optimism or pessimism about stocks," is likely to affect more those stocks that are harder to evaluate or, in our context, stocks that are surrounded by more uncertainty. These include stocks of companies that are younger, smaller, or with extreme growth potential, such as highly innovative companies.

Thus, our paper provides a new decision-theoretic foundation of notion of "investor sentiment" and its effect on innovation activities and market valuations.

The paper is organized as follows. In Section 1, we introduce the basic model of our paper. In Section 2, we derive the paper's main results. Section 3 examines the impact of mergers on the incentives to innovate. Section 4 develops the dynamic version of our model, Section 5 shows our results hold for process innovation, and Section 6 presents the main empirical implications of our model. Section 7 concludes. All proofs are in the appendix.

1 The Basic Model

We study a two-period model, with three dates, $t \in \{0, 1, 2\}$. The economy has two classes of agents: investors and entrepreneurs. Entrepreneurs are endowed with unique project-ideas that may lead to an innovation. Project-ideas are risky and require investment both at the beginning, $t = 0$, and at the interim date, $t = 1$. If successful, project-ideas generate a valuable innovation at $t = 2$. If a project-idea is unsuccessful, it has zero payoff. For simplicity, we assume initially that there are only two types of project-ideas, denominated by τ, with $\tau \in \{A, B\}$.

Entrepreneurs are penniless and require financing from investors. There is a unit mass of investors, endowed at $t = 0$ with w_0 units of the riskless asset. The riskless asset can either be invested in one (or both) of the two types of project-ideas, or it can invested in the riskless technology. A unit investment in the riskless technology can be made either at $t = 0$ or $t = 1$, and yields a unit return in the second period, $t = 2$, so that the (net) riskless rate of return is zero.

We assume that project-ideas are specific to each entrepreneur: an entrepreneur can invest in only one type of project-idea. This assumption captures the notion that project-ideas are creative innovations that can be successfully pursued only by the entrepreneur who generated them.

The innovation process is structured in two stages. To implement a project-idea, and thus "innovate," an entrepreneur must first make at $t = 0$ a fixed, non-pecuniary investment k_τ. We interpret the initial investment k_τ as representing all the preliminary personal effort that the entrepreneur must exert in order to generate the idea and make it potentially viable. We will denote the initial personal investment made by the entrepreneur, k_τ, as a "discovery cost" necessary for innovation.

The innovation process is inherently risky, and we denote with q_τ the success probability of the first stage of the process. We allow the first-stage success probabilities of the two project-ideas to be correlated. Specifically, we assume that the probability that both entrepreneurs are successful in the first stage is $q_A q_B + r$, while the probability that only entrepreneur τ is successful is $q_\tau(1 - q_{\tau'}) - r$, with $\tau', \tau \in \{A, B\}$, $\tau' \neq \tau$ and $r \in \left(-\min\{q_A q_B, (1-q_A)(1-q_B)\}, \min_\tau q_\tau (1-q_\tau)\right)$.[8] The parameter r captures similarities between entrepreneurial project-ideas, and thus characterizes the degree of "relatedness" of the innovations.

If the first stage is successful, at $t = 1$ entrepreneurs enter the second stage of the process. In this second stage, the entrepreneur decides the level of intensity of the innovation process, for example, the level of R&D expenditures. Innovation intensity will affect the ultimate value of the innovation that can be realized at $t = 2$, and is denoted by y_τ. Innovation intensity is costly: entrepreneur τ choosing an innovation intensity y_τ will sustain a cost $c_\tau(y_\tau) = \frac{1}{Z_\tau(1+\gamma)} y_\tau^{1+\gamma}$, where Z_τ represents the productivity of entrepreneur τ's project-idea. To obtain interior solutions, we assume that the productivity parameters, Z_τ, for the two entrepreneurs are not too dissimilar.[9] Entrepreneurs will pay for the cost $c_\tau(y_\tau)$ by selling equity to a large number of well-diversified investors.[10] The second stage of the innovation process is also uncertain and, if successful, the innovation will generate at the end of the second period, $t = 2$, the payoff y_τ with probability p, and zero otherwise (if the project fails in the first stage, it is similarly worthless). We assume, for simplicity, that the success probabilities of the second stage are independent, and will show that innovation waves can occur.[11]

We assume entrepreneurs are impatient and that they will sell at the interim period, $t = 1$, their firms to outside investors at total price V_τ. An important feature of our model is that investors are uncertain about the success probability of the second stage of project-ideas, p.[12] We model uncertainty (or "ambiguity") aversion by adopting the minimum expected utility (MEU) approach developed in Gilboa and Schmeilder (1998).[13] In this framework, economic agents do not have a

[8] It can be quickly verified that the correlation of the first-stage projects is $r\left[q_A(1-q_A) q_B (1-q_B)\right]^{-\frac{1}{2}}$.

[9] Formally, we assume $\frac{Z_A}{Z_B} \in \left(\frac{1}{\psi}, \psi\right)$ where $\psi \equiv \frac{1}{4} e^{2\alpha(\gamma+1)} \left(1 + \frac{1}{2\gamma}\right)^{2\gamma}$, which implies that if both first-stage projects are successful, entrepreneurs execute innovation intensity levels so investors have interior beliefs in equilibrium.

[10] The sale of equity may, for example, take place in the form of an Initial Public Offering, IPO.

[11] Our model can easily be extended to the case where second-stage success probabilities are correlated.

[12] Note that our model can easily be extended to the case where entrepreneurs are uncertainty averse as well.

[13] An alternative approach is "smooth ambiguity" developed by Klibanoff, Marinacci, and Mukerji (2005). In their

single prior on future events but, rather, they believe that the probability distribution of future events belongs to a given set $\mathcal{M}$, denoted as the investor's "core beliefs set." Thus, uncertainty-averse agents maximize $\mathcal{U}$, where

$$\mathcal{U} = \min_{\mu \in \mathcal{M}} E_\mu \left[u\left(\cdot\right) \right], \tag{1}$$

where μ is a probability distribution over future events, and $u(\cdot)$ is a von-Neumann Morgenstern (vNM) utility function.[14] In addition, we assume that uncertainty-averse agents are sophisticated with consistent planning. In our setting, agents are sophisticated in that they correctly anticipate their future uncertainty aversion and, thus, correctly take into account how they will behave at future dates in different states of the world.[15]

We model investor uncertainty aversion by assuming that investors are uncertain on the success probability of the second stage of the innovation process, p. Following Hansen and Sargent (2001) and (2008), we characterize the core beliefs set $\mathcal{M}$ in (1) by using the notion of relative entropy.[16] For a given pair of (discrete) probability distributions $(p, \hat{p})$, the *relative entropy* of p with respect to $\hat{p}$ is the Kullback-Leibler divergence of p from $\hat{p}$:

$$R(p|\hat{p}) \equiv \sum_i p^i \log \frac{p^i}{\hat{p}^i}. \tag{2}$$

The core beliefs set for the uncertainty-averse investors in our economy is

$$\mathcal{M} \equiv \{p : R(p|\hat{p}) \leq \eta\}, \tag{3}$$

where p is the joint distribution of the success probability of the second stage of the two project-ideas, and $\hat{p}$ is an exogenously given "reference" probability distribution of such success probabilities. From (2), it is easy to see that the relative entropy of p with respect to $\hat{p}$ represents the

model, agents maximize expected felicity of expected utility, and agents are uncertainty averse if the felicity function is concave. The main results of our paper will hold in this approach (if the felicity function is sufficiently concave), but at the cost of requiring a substantially greater analytical complexity. Similarly, our results also hold under variational prefences of Maccheroni, Marinacci, and Rustichini (2006) if the ambiguity index $c(p)$ has a positive cross-partial.

[14] In the traditional framework, players have a single prior μ and maximize expected utility $E_\mu [u(\cdot)]$.

[15] Siniscalchi (2011) describes this framework as preferences over trees. See Epstein and Schneider (2010) and Barilla, Hansen and Sargent (2009).

[16] This specification of ambiguity aversion, which is often referred to as the "constrained preferences" approach, is a particular case of the larger class of "variational preferences." Strzalecki (2011) provides a general characterization of different approaches to modeling ambiguity aversion.

(expected) log-likelihood ratio of the pairs of distributions $(p, \hat{p})$, when the "true" probability distribution is p. Thus, the core beliefs set $\mathcal{M}$ includes the set of probability distributions, p, with the property that, if true, the investor would expect not to reject the ("null") hypothesis $\hat{p}$ in a likelihood-ratio test. Note that our results will go through, more generally, as long as the core belief set $\mathcal{M}$ is a strictly convex set with smooth boundaries.

Intuitively, the core belief set $\mathcal{M}$ can be interpreted as the set of probability distributions that are not "too unlikely" to be the true (joint) probability distribution that characterizes the two technologies, given the reference distribution $\hat{p}$. Note that a small value of η represents situations where agents have more confidence that the probability distribution $\hat{p}$ is a good representation of the success probability of the two technologies, while a large value of η corresponds to situations where there is great uncertainty on the true probabilities underlying the two technological processes.[17]

Lemma 1 *Let $\eta < \underline{\eta}(\hat{p})$, defined in the appendix. The core beliefs set $\mathcal{M}$ is a strictly convex set with smooth boundary. If investors have nonnegative investments in both innovations, the solution to (1) is on the lower left-hand boundary of $\mathcal{M}$.*

Lemma 1 is a direct implication of the fact that relative entropy $R(p|\hat{p})$ is a strictly convex function.[18] Lemma 1 also shows that uncertainty-averse investors with positive investment in both project ideas will select their probability assessments that lie in the "lower-left" boundary of the core beliefs set $\mathcal{M}$. Thus, the relevant part of the core beliefs set $\mathcal{M}$ is a smooth, decreasing, and convex function (see Figure 1).

It is easy to see that restricting investors' beliefs to belong to the core beliefs set (3) has the effect of ruling out probability distributions that are "too far" from the reference probability $\hat{p}$. In other words, the maximum entropy criterion implied by (3) excludes from the core-belief set probability distributions that give "too much" weight to extreme events. In addition, because from Lemma 1 uncertainty-averse investors are essentially concerned about "left-tail" events, we denote this property as "trimming pessimism."[19]

[17] As in Hansen and Sargent (2001), (2007), (2008), and Epstein and Schneider (2010), relative entropy can be interpreted as characterizing the extent of "misspecification error" that affects investors.

[18] For a general discussion, see Theorem 2.5.3 and 2.7.2 of Cover and Thomas (2006).

[19] Referring back to our example on self-driving cars, the relative entropy criterion eliminates from the core belief set $\mathcal{M}$ probability distributions that give "too much weight" to the extreme event that all technologies currently under development will fail.

Because there is no closed-form solution for the level set of relative entropy for binomial distributions in (3), for ease of exposition, we model the relevant portion of the core beliefs set (namely, the decreasing and convex "lower-left" boundary) by using a lower-dimensional parametrization, as follows. We assume that the success probability of project idea τ depends on the value of an underlying parameter θ_τ, and is denoted by $p(\theta_\tau)$, with $\theta_\tau \in [\theta_L, \theta_H] \subseteq [\theta_m, \theta_M]$. For analytical tractability, we assume that $p(\theta_\tau) = e^{\theta_\tau - \theta_M}$, with $\tau \in \{A, B\}$. Uncertainty-averse agents treat the vector $\vec{\theta} \equiv (\theta_A, \theta_B)$ as ambiguous and assess that $\vec{\theta} \in C \subset \{(\theta_A, \theta_B) : (\theta_A, \theta_B) \in [\theta_L, \theta_H]^2\}$. We interpret the parameter combination $\vec{\theta}$ as describing the state of the economy at $t = 2$ and we denote C as the set of "core beliefs" of our uncertainty-averse investors. In light of Lemma 1 and subsequent discussion, we assume that for $\vec{\theta} \in C$ we have that $(\theta_A + \theta_B)/2 = \theta_T$, where $\theta_T \equiv (\theta_H + \theta_L)/2$. Importantly, given θ, the success probabilities of the second-stage of project-ideas are independent. We will characterize the extent of technological uncertainty as $\alpha \equiv \theta_T - \theta_L$.[20]

Payoffs are determined as follows. If entrepreneur τ innovates, and the first stage of the innovation process is successful, he develops an innovation with a (potential) value y_τ. At the interim date, $t = 1$, each entrepreneur sells her entire firm to outside investors for a value V_τ, which thus represents her payoff from innovation. In turn, an uncertainty-averse investor can purchase a fraction ω_τ of firm τ, with $\tau \in \{A, B\}$, and thus holding the residual value $w_0 - \omega_A V_A - \omega_B V_B$ in the risk-free asset. To avoid (uninteresting) corner solutions, we assume that the endowment of the risk-free asset is sufficiently large that the budget constraint will not be nonbinding in equilibrium: $w_0 > \omega_A V_A + \omega_B V_B$. Investors' final payoff will then depend on their holdings of the risk-free asset and on the success/failure of each innovation at the second stage and on their holdings in the innovation, ω_τ. Finally, we assume that, while outside investors are uncertainty averse with respect to the parameter θ, there are no other sources of uncertainty (as opposed to "risk") in the economy,[21] and that all agents (investors and entrepreneurs) are otherwise risk-neutral.

We will at times benchmark the behavior of uncertainty-averse agents with the behavior of an uncertainty-neutral SEU agent, and we will assume that an uncertainty-neutral investor has

[20] Alternatively, the core-beliefs set C could be obtained from investors' uncertainty over consumer demand. In Appendix B, we present a model specification that generates qualitatively identical results, where the source of uncertainty is the proportion of consumers that exhibit a relatively stronger preference for each good in the economy.

[21] If there is uncertainty on q or r, entrepreneurs will assume the worst, selecting $q_{\min}$ and $r_{\min}$, because entrepreneurs' payoffs are increasing in q and r.

$\theta_L = \theta_H$, so that she assesses $\theta_\tau = \theta_T$. This assumption guarantees that the uncertainty-neutral investor has the same probability assessment on the success probability of each project-idea as a well-diversified uncertainty-averse investor (and thus there is no "hard-wired" difference between the two types of investors).

1.1 Endogenous Investor Sentiment

An important implication of uncertainty aversion is that the investor's probabilistic assessment at the interim date on the parameter θ depends on their overall exposure to the source of risk and, thus, on the structure of their portfolios. This means that the probability assessment (i.e., the "beliefs") held by an uncertainty-averse investor (that is, the parameter combination $\vec{\theta}$) are endogenous, and depend on the agent's overall exposure to the risk factors.

Endogeneity of beliefs is the outcome of the fact that the minimization operator in (1), which determines the probability assessment held by an investor on the success probability of the second stage of the project-ideas, in general depends on the composition of the investor's overall portfolio. Note that this property, which plays a critical role in our paper, implies that uncertainty-averse agents are more willing to hold uncertain assets if they can hold such assets in a portfolio rather than in isolation. By holding uncertain assets in a portfolio, investors can lower their overall exposure to the sources of uncertainty in the economy. Namely, by investing in both project-ideas, the investor will limit her exposure to the "tail event" that both project-ideas have a very low success probability in the second stage, a property that we refer to as uncertainty hedging.

The effect of uncertainty hedging in our model is that investors hold more favorable probability assessments on the success probability of project-ideas if they invest in both projects, rather than in just one project. Specifically, if an investor decides to purchase a proportion ω_τ of entrepreneur τ's firm, with innovation intensity y_τ, the investor will hold a risky portfolio $\Pi = \{\omega_A y_A, \omega_B y_B, w_0 - \omega_A V_A - \omega_B V_B\}$. Because investors are uncertainty averse but otherwise risk neutral, portfolio Π provides the investor with utility $U(\Pi) = \min_{\overrightarrow{\theta} \in C} u\left(\Pi, \overrightarrow{\theta}\right)$

$$u\left(\Pi, \overrightarrow{\theta}\right) = e^{\theta_A - \theta_M}\omega_A y_A + e^{\theta_B - \theta_M}\omega_B y_B + w_0 - \omega_A V_A - \omega_B V_B. \tag{4}$$

Because of uncertainty aversion, the investor's assessment at $t = 1$ on the state of the economy,

$\overrightarrow{\theta}^{a}$, is the solution to the minimization problem

$$\overrightarrow{\theta}^{a}(\Pi) = \arg\min_{\theta \in C} u\left(\Pi, \overrightarrow{\theta}\right). \tag{5}$$

and is characterized in the following lemma.

Lemma 2 *Increasing an investor's exposure to one innovation risk induces a more favorable assessment of the other innovation risk. Formally, given a portfolio* Π*, and letting*

$$\check{\theta}^{a}_{\tau}(\Pi) = \theta_T + \frac{1}{2}\ln\frac{\omega_{\tau'}y_{\tau'}}{\omega_{\tau}y_{\tau}}, \tag{6}$$

an uncertainty-averse investor holds an assessment θ^{a}_{τ} *on the uncertain parameter* θ_{τ} *equal to*

$$\theta^{a}_{\tau}(\Pi) = \begin{cases} \theta_L & \check{\theta}^{a}_{\tau}(\Pi) \leq \theta_L \\ \check{\theta}^{a}_{\tau}(\Pi) & \check{\theta}^{a}_{\tau}(\Pi) \in (\theta_L, \theta_H) \\ \theta_H & \check{\theta}^{a}_{\tau}(\Pi) \geq \theta_H \end{cases}. \tag{7}$$

Lemma 2 shows that an investor's assessment on $\overrightarrow{\theta}$ is endogenous, and it depends crucially on the composition of her portfolio, Π. Thus, we will at times refer to $\overrightarrow{\theta}^{a}(\Pi)$ as the "portfolio-distorted" assessments. We will say that the agent has "*interior assessments*" when $\theta^{a}_{\tau} \in (\theta_L, \theta_H)$, in which case, the agent's assessments are equal to $\check{\theta}_{\tau}(\Pi)$ as in (6). Otherwise, we will say that the investor holds "*corner assessments.*" Further, an uncertainty-averse investor's assessment of $\overrightarrow{\theta}$ determines the views held by the investor on the future state of the economy. Thus, we will also refer to the assessment θ^{a}_{τ} as *"investor sentiment."*

Lemma 2 shows that when an investor has a relatively smaller proportion of her portfolio invested in innovation τ, $\omega_{\tau}y_{\tau} < \omega_{\tau'}y_{\tau'}$, she will be relatively more optimistic about the return on that innovation. This happens because a smaller exposure to the risk generated by a given innovation, relative to another innovation, will make an uncertainty-averse investor relatively more concerned about priors that are less favorable to the other innovation. Correspondingly, the investor will give more weight to the states of nature that are more favorable to the first innovation. In other words, the investor will be more "optimistic" on the success probability of that innovation (i.e., will have a stronger sentiment), and more "pessimistic" with respect to the other innovation.

Suppose entrepreneur A decides to innovate, but entrepreneur B decides not to innovate. Because $y_B = 0$, by Lemma 2, $\theta_A^a(\Pi) = \theta_L$ for any $\omega_A y_A > 0$. Correspondingly, if entrepreneur B decides to innovate, but entrepreneur A does not, $\theta_B^a(\Pi) = \theta_L$. Similar situations emerge if only one entrepreneur has a successful first-stage project-idea, while the other entrepreneur fails. In this case, at the interim date, $t = 1$, investors hold more pessimistic assessments about the successful innovation than if both entrepreneurs have a successful first-stage project-idea. This means that investors, when facing only one innovation, will be more pessimistic on that innovation than when facing both innovations. This happens because, by investing in only one project-idea, investors forego the benefits of uncertainty-hedging and hold a portfolio with greater exposure to the possibility that the second-stage success probability is very low. In contrast, by investing in both technologies, the investor protects herself from the situation that both technologies have very low success probability, a hypothesis rejected by the relative entropy criterion (3).

In our model, portfolio-distorted assessments determine investors' expectations on the ultimate success probability of the innovation processes in the economy, and thus characterize investor "sentiment" toward innovations. An important implication of Lemma 2 that will play a key role in our analysis is that investor sentiment about one innovation will crucially depend on the availability of other innovations in the economy, and their innovation intensity. In particular, an investor will be more optimistic about an innovation success probability, and she values it more, if she will be able to also invest in the other innovation. Thus, investors' probabilistic assessments create an externality for entrepreneurs, in that an entrepreneur's successful innovation will be more valuable if other entrepreneurs have successful innovations as well. In other words, if both entrepreneurs innovate and are successful at the first stage, investor sentiment toward both innovations improves making both innovations more valuable. Thus, the spillover effect from one innovation to another is driven by endogenous investor sentiment.

2 The Innovation Decision

We will solve the model recursively. First, we find the choice by entrepreneurs that are successful at the first stage of the innovation process of the optimal innovation intensity, y_τ, and the value

V_τ that investors are willing to pay at the interim date for innovations. Next, we solve for the initial choice by entrepreneurs on whether or not to initiate the innovation process by incurring the initial discovery cost k_τ. As a benchmark, we start the analysis by characterizing the two entrepreneurs' innovation decisions when investors are uncertainty-neutral, then we consider the case where investors are uncertainty-averse.

The implementation of the second stage of the innovation process requires entrepreneurs to raise capital from investors by selling equity in the capital markets at $t = 1$. For simplicity, we assume that entrepreneur τ sells her entire firm to investors, uses the proceeds to pay cost $c_\tau(y_\tau)$, and pockets the difference. Further, y_τ is observable and contractible with outside investors, thus ruling out moral hazard. In this case, the choice of innovation intensity y_τ entrepreneur τ depends on the price that outside investors are willing to pay for her firm, that is, on the market value of the equity of the firm. This, in turn, depends on the assessments held by investors on the success probability of the innovation, $p(\theta_\tau)$.

Lemma 3 *Given investor assessments and risk-neutrality, entrepreneurs' firms are priced at their expected value, that is,* $V_\tau = p(\theta_\tau^a)y_\tau$ *for uncertainty-averse investors, and* $V_\tau = p\left(\theta_T\right)y_\tau$ *for uncertainty-neutral investors, with* $\tau \in \{A, B\}$*. In equilibrium, it is (weakly) optimal for investors to hold a balanced portfolio:* $\omega_A^* = \omega_B^*$ *for both type of investors.*

Lemma 3 shows that, given our assumption of risk-neutrality, investors price equity at its expected value, given their assessments. Investor assessments, however, depend on their attitude toward uncertainty. Endogeneity of assessments is critical because it will lead to different market valuation of equity, and thus, different behavior by entrepreneurs. Also, it is weakly optimal for investors to hold balanced portfolios. Uncertainty-neutral investors are indifferent on their portfolio composition, because of risk neutrality. In contrast, uncertainty-averse investors strictly prefer a balanced portfolio, due to uncertainty-hedging. For notational simplicity, we normalize investor portfolio holding and set $\omega_A^* = \omega_B^* = 1$.

2.1 The Uncertainty-Neutral Benchmark

As a benchmark, we start with the simpler case in which investors are uncertainty-neutral. When investors are uncertainty-neutral, equity prices depend only on their prior θ_T and on the level of innovation intensity, y_τ, chosen by the firm, giving

$$V_\tau^S = p(\theta_T)\, y_\tau, \quad \text{for} \quad \tau \in \{A, B\}. \tag{8}$$

Equation (8) shows that equity value for innovation τ depends only on the investor assessments of the success probability of the second stage of the innovation process, $p(\theta_T)$, and its level of innovation intensity, y_τ: it does not depend on the innovation intensity decision of the other firm, $y_{\tau'}$. Without uncertainty aversion, there are no interactions between the choice of the innovation intensities by the two entrepreneurs. In this case, if the first stage of the project-idea was successful, entrepreneur τ's chooses the level of innovation intensity for the second stage, y_τ, by solving

$$\max_{y_\tau} \quad \mathcal{U}_\tau^S \equiv V_\tau^S - c_\tau(y_\tau) = p(\theta_T)\, y_\tau - \frac{1}{Z_\tau(1+\gamma)} y_\tau^{1+\gamma}. \tag{9}$$

From (9) it follows that the optimal innovation intensity, y_τ, is

$$y_\tau^* \equiv [p(\theta_T)\, Z_\tau]^{\frac{1}{\gamma}}, \tag{10}$$

By direct substitution of y_τ^* into (9),[22] we obtain the ex-ante expected payoff for entrepreneur τ from initiating the innovation process, and thus incurring discovery cost k_τ:

$$E\mathcal{U}_\tau^S = q_\tau \frac{\gamma}{1+\gamma} [p(\theta_T)]^{\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}} - k_\tau.$$

Thus, entrepreneur τ innovates at $t = 0$ if $E\mathcal{U}_\tau^S \geq 0$, leading to the following theorem.

Theorem 1 *When investors are uncertainty-neutral, entrepreneurs of type τ innovate iff*

$$k_\tau \leq k_\tau^S \equiv q_\tau \frac{\gamma}{1+\gamma} [p(\theta_T)]^{\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}}, \quad \tau \in \{A, B\},$$

and the innovation processes of the two entrepreneurs are independent.

[22] Because $\frac{\partial^2 U_\tau^S}{\partial y_\tau^2} = -\frac{\gamma}{Z_\tau} y_\tau^{\gamma-1} < 0$, first-order conditions are sufficient for a maximum.

Theorem 1 shows that when investors are uncertainty neutral, the investment decisions by the two entrepreneurs are effectively independent from each other, with no spillover effects. When investors are uncertainty averse, however, the innovation processes of the two firms are interconnected.

2.2 Uncertainty Aversion and Innovation

We now derive optimal innovation decisions when investors are uncertainty averse. From Lemma 2, investor sentiment toward each innovation, $p(\theta_\tau^a)$, depends on the overall risk exposure of their portfolios. Sentiment is endogenous, and depends on the innovation intensities of both firms, y_τ.

Lemma 4 *If investors are uncertainty averse, the market value of entrepreneur τ's firm is*

$$V_\tau^U(\Pi) = \begin{cases} p(\theta_H)\, y_\tau & y_\tau \le e^{-2\alpha} y_{\tau'} \\ p(\theta_T)\, y_\tau^{\frac{1}{2}} y_{\tau'}^{\frac{1}{2}}, & y_\tau \in \left(e^{-2\alpha} y_{\tau'}, e^{2\alpha} y_{\tau'}\right) \\ p(\theta_L)\, y_\tau & y_\tau \ge e^{2\alpha} y_{\tau'} \end{cases}, \tag{11}$$

where y_τ is the innovation intensity selected by entrepreneur τ, with $\tau, \tau' \in \{A, B\}$, $\tau \neq \tau'$.

Lemma 4 shows that, when investors are uncertainty averse, the market value of one firm depends on the level of innovation intensity chosen by its entrepreneur as well as on the level chosen by the other firm. The interaction between equity market values of the two firms creates a strategic externality between the two entrepreneurs.

The linkage between the market value of the two firms occurs through endogenous investor sentiment. From Lemma 2 an increase of the innovation intensity of one firm will increase the relative exposure of investors to that firm's risk relative to the other firm's risk, making (all else equal) investors relatively more pessimistic about that firm's success probability and, correspondingly, relatively more optimistic about the other firm's success probability.

Lemma 4 also implies that an increase of the level of innovation intensity in one firm, y_τ, has two opposing effects on its value V_τ^U. First is the positive direct effect that greater innovation intensity has on the ultimate value of the innovation. This positive effect can however be mitigated by a second negative effect that an increase in innovation intensity has on investor sentiment. This implies that firm value is an increasing function of the innovation intensities of both firms.

Note that if one of the two firms does not innovate or the innovation is unsuccessful in the first stage, the level of innovation intensity for that firm is necessarily zero. Lemma 4 implies that the market value of the one firm is the worst-case scenario value: $V_\tau(\Pi) = p(\theta_L)\, y_\tau$.

We can now determine the optimal level of innovation intensity for each entrepreneur. If the first stage of the project-idea was successful, entrepreneur τ solves

$$\max_{y_\tau} \mathcal{U}_\tau^U \equiv V_\tau(\Pi) - \frac{1}{Z_\tau(1+\gamma)} y_\tau^{1+\gamma}, \tag{12}$$

where $\Pi = \{y_A, y_B, w_0 - V_A - V_B\}$ and $V_\tau(\Pi)$ is given in (11). To simplify exposition, we assume that the two projects are not too dissimilar. Specifically, we assume that the values Z_A and Z_B are not too far away from each other: $\frac{Z_A}{Z_B} \in \left(\frac{1}{\psi}, \psi\right)$ where $\psi \equiv \frac{1}{4} e^{2\alpha(\gamma+1)} \left(1 + \frac{1}{2\gamma}\right)^{2\gamma}$. This assumption ensures that if both firms have successful first-stage projects, they find it optimal to chose levels of innovation intensity $\{y_A, y_B\}$ that result in interior assessments for investors.

The solution to problem (12) depends on whether one or both firms decide to initiate the innovation process and pay the discovery costs k_τ and whether they are successful at the first stage of the innovation process. Thus, there are four possible states of the world that we need to analyze: (i) when both entrepreneurs had a successful first stage, state SS; (ii) when only one entrepreneur has a successful first-stage, state SF with the symmetric FS state, (iii) when both entrepreneur fail in the first stage and no innovation can take place, state FF. Since the last state FF is trivial, we focus on the first two.

2.2.1 Only One Firm Has Successful First-Stage Project, State SF

Consider first the case in which only one entrepreneur had a successful first-stage project-idea, state SF. This state may emerge either because the other entrepreneur has not initiated the innovation process (that is, she did not sustain the discovery cost), or because the first stage was unsuccessful.

Lemma 5 *If only one entrepreneur has a successful first stage project-idea (state SF), she selects innovation intensity equal to*

$$y_\tau^{U,SF} = \left[p(\theta_L)\, Z_\tau\right]^{\frac{1}{\gamma}}; \tag{13}$$

the market value of the entrepreneur's firm is equal to

$$V_\tau^{U,SF} = \left[p\left(\theta_L\right)\right]^{\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}}, \tag{14}$$

giving a continuation utility for the entrepreneur equal to

$$\mathcal{U}_\tau^{U,SF} \equiv \left[p\left(\theta_L\right)\right]^{\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma}. \tag{15}$$

If only one entrepreneur successfully develops a first-stage project, there will only be one innovation available to investors, so they will believe the worst-case scenario about that innovation resulting in negative investor sentiment and low equity valuations. Therefore, the lone entrepreneur will chose a low level of innovation intensity, consistent with negative sentiment.

2.2.2 Both Firms Have Successful First-Stage Projects, State SS

If both entrepreneurs have successful first-stage projects, market valuation is given in Lemma 4, which leads to the following theorem.

Theorem 2 *Let* $\frac{Z_A}{Z_B} \in \left(\frac{1}{\psi}, \psi\right)$*. If both entrepreneurs innovate and have a successful first-stage project-idea (state SS), they select innovation intensities with best-response function*

$$y_\tau^{U,SS}(y_{\tau'}) = \left[\frac{Z_\tau}{2} p\left(\theta_T\right)\left(y_{\tau'}\right)^{1/2}\right]^{\frac{1}{\gamma+\frac{1}{2}}}, \quad \textit{with} \quad \tau \neq \tau', \ \textit{and} \ \tau, \tau' \in \{A, B\}, \tag{16}$$

which implies that equilibrium innovation intensity for each entrepreneur is

$$y_\tau^{U,SS} = \left[\frac{1}{2} p\left(\theta_T\right) Z_\tau^{\frac{2\gamma+1}{2\gamma+2}} Z_{\tau'}^{\frac{1}{2\gamma+2}}\right]^{\frac{1}{\gamma}}. \tag{17}$$

Equilibrium firm value is

$$V_\tau^{U,SS} = 2^{-\frac{1}{\gamma}} \left[p\left(\theta_T\right)\right]^{\frac{1+\gamma}{\gamma}} \left(Z_\tau Z_{\tau'}\right)^{\frac{1}{2\gamma}}, \tag{18}$$

and continuation utility is

$$\mathcal{U}_\tau^{U,SS} = 2^{-\frac{1}{\gamma}} \left[p\left(\theta_T\right)\right]^{\frac{1+\gamma}{\gamma}} \left(Z_\tau Z_{\tau'}\right)^{\frac{1}{2\gamma}} \frac{2\gamma+1}{2\gamma+2}. \tag{19}$$

Theorem 2 establishes that there is strategic complementarity in entrepreneurs' production decisions. In particular, an entrepreneur's choice of innovation intensity, $y_\tau^{U,SS}(y_{\tau'})$, is an increasing function of the other entrepreneur's innovation intensity, $y_{\tau'}$. The strategic complementarity originates in investor uncertainty aversion and endogenous investor sentiment. From Lemma 2 and Lemma 4, the sentiment of uncertainty-averse investors on the success probability of the second stage of an innovation process and, thus, their market valuations at the interim date, depend on the innovation intensities chosen by both entrepreneurs. Thus, investors perceive innovations effectively as complements. This complementarity is then transferred from investors' sentiment to entrepreneurs' innovation decisions.

This allows us to determine the equilibrium levels of innovation intensities, market valuation, and entrepreneur utility, when both entrepreneurs have successful first-stage projects, as described in equations (17), (18), and (19), respectively. The following corollary examines how these values are affected by success of the other entrepreneur.

Corollary 1 *An entrepreneur is better off when the other also has a successful first-stage project:* $U_\tau^{U,SS} > U_\tau^{U,SF}$. *If* $\frac{Z_{\tau'}}{Z_\tau} \in \left(\frac{1}{\psi_1}, \psi_1\right)$, *equity values are higher when both entrepreneurs have successful first-stage projects:* $V_\tau^{U,SS} > V_\tau^{U,SF}$. *If* $\frac{Z_{\tau'}}{Z_\tau} \in \left(\frac{1}{\psi_2}, \psi_2\right)$, *entrepreneurs innovate with greater intensity when both have successful first-stage projects:* $y_\tau^{U,SS} > y_\tau^{U,SF}$. *Finally,* $\psi_2 < \psi_1 < \psi$.

An important implication of Corollary 1 is that, if entrepreneurs' productivities are not too dissimilar, because of the complementarity of innovations generated by uncertainty aversion, investors value an innovation more when they can also invest in the other innovation: $V_\tau^{U,SS} > V_\tau^{U,SF}$.

2.3 The Innovation Decision

We have shown that investor uncertainty aversion affects equity valuation and generates strategic complementarity in the interim choice of innovation intensity, y_τ. The interim strategic complementarity of the choice of innovation intensity generates a strategic complementarity also in the entrepreneurs' decisions to innovate at the beginning of the innovation process, $t = 0$.

If entrepreneur τ' chooses to innovate, the expected utility for entrepreneur τ from sustaining at $t = 0$ the initial discover cost k_τ and, thus, initiating the innovation process is

$$E\mathcal{U}_\tau^{U,I} = (q_\tau q_{\tau'} + r)\mathcal{U}_\tau^{U,SS} + (q_\tau(1 - q_{\tau'}) - r)\mathcal{U}_\tau^{U,SF} - k_\tau$$

for $\tau, \tau' \in \{A, B\}$ and $\tau \neq \tau'$. Conversely, if entrepreneur τ' does not innovate at $t = 0$, the expected utility for entrepreneur τ from choosing to innovate at $t = 0$ is

$$E\mathcal{U}_\tau^{U,N} = q_\tau \mathcal{U}_\tau^{U,SF} - k_\tau.$$

We can now characterize equilibrium innovation decisions at $t = 0$.

Theorem 3 *For low levels of discover cost, $k_\tau \leq \underline{k}_\tau$, the entrepreneur always innovates. For high levels of discovery cost, $k_\tau \geq \bar{k}_\tau$, the entrepreneur never innovates. For intermediate levels of the discovery cost, $k_\tau \in \left(\underline{k}_\tau, \bar{k}_\tau\right)$, the entrepreneur is willing to innovate only if the other entrepreneur innovates. If both entrepreneurs have intermediate levels of discovery cost, there are multiple equilibria, one where both entrepreneurs innovate and one where neither innovate. The innovation equilibrium dominates the no-innovation equilibrium.*

For very small levels of discovery costs, $k_\tau \leq \underline{k}_\tau$, it is a dominant strategy for the entrepreneur to innovate. For very large levels of discovery costs, $k_\tau \geq \bar{k}_\tau$, it is a dominant strategy for the entrepreneur to not innovate. For intermediate levels of discovery costs, $k_\tau \in \left(\underline{k}_\tau, \bar{k}_\tau\right)$, entrepreneur τ wishes to innovate only if the other entrepreneur innovates as well. Theorem 3 shows this strategic complementarity in entrepreneurs' innovation decisions.

The effect of the strategic complementarity created by uncertainty aversion is to create the possibility of multiple equilibria. When both entrepreneurs have intermediate levels of the discovery cost, there are equilibria with and without innovation. In this case, entrepreneurs face a classic "assurance game," in which there is a Pareto-dominant equilibrium, where both entrepreneurs innovate, yet there is also an inefficient, Pareto-inferior equilibrium, where neither entrepreneur innovates. Multiplicity of equilibria depends on the fact that it is profitable for one entrepreneur to innovate only if he expects the other entrepreneur to innovate as well.

Corollary 2 *The threshold levels* $\{\bar{k}_\tau\}_{\tau\in\{A,B\}}$ *are increasing functions of* $q_\tau, q_{\tau'}, Z_\tau, Z_{\tau'}$ *and* r*, and the threshold levels* $\{\underline{k}_\tau\}_{\tau\in\{A,B\}}$ *are increasing functions of* q_τ *and* Z_τ*.*

Corollary 2 has the interesting implication that an increase in one entrepreneur's probability of success, q_τ, makes not only that entrepreneur, but also other entrepreneurs, more willing to attempt first-stage discovery of a product-idea. This follows because the strategic complementarity induced by uncertainty aversion. In the absence of uncertainty aversion, an increase in the probability of discovery affects only that entrepreneur. Corollary 2 also shows entrepreneurs are more willing to innovate if her innovation is more related to other entrepreneurs' innovations, that is, r is greater. This happens because greater degree of relatedness increases the probability that both project-ideas are simultaneously successful in the first-stage, increasing the market value of innovation. Finally, Corollary 2 also shows that an increase in productivity of an entrepreneur increases not only that entrepreneur's willingness to innovate, but also makes other entrepreneurs more willing to innovate as well.

3 Acquiring Innovation

We have shown that investors' uncertainty aversion creates externalities across innovations. These externalities are due to endogenous investor sentiment, and create the possibility of value dissipation due to coordination failures. This means there may be gains from internalizing such externalities via acquisitions.

There are two externalities at work in our model. The first externality is due to the valuation spillover discussed in Lemma 2: for any choices of innovation intensities, $\{y_\tau, y_{\tau'}\}$, the two firms are more valuable to uncertainty-averse investors when they are held in the same portfolio than when they are owned separately.

The second externality is due to the strategic complementarity between the choices of innovation intensity y_τ, discussed in Lemma 4: the market value of an individual firm, V_τ^U, is an increasing function of the innovation intensity chosen by both firms, $\{y_\tau, y_{\tau'}\}$, through its effect on investor sentiment. When a firm chooses their own optimal level of innovation intensity, they ignore the positive externality that choice has on the other firm's valuation.

We extend our analysis by examining the effect of the strategic complementarity between innovation intensities. We modify the basic model as follows. If both entrepreneurs are successful in the first stage, we now allow for the possibility that at the interim date, $t = 1$, both entrepreneurs merge their firms in a new firm.[23] After the merger, the entrepreneurs jointly determine the innovation intensity, y_τ, for both innovation processes $\tau \in \{A, B\}$. After the selection of the innovation intensities y_τ, the merged firm will again sell its equity in the public equity market.[24]

After the merger of the first-stage innovations, the problem of the merged firm is to maximize the combined value of the two innovation projects. By identical reasoning to the proof of Lemma 3, the merged firm will value the projects at $V_\tau = p\left(\theta_\tau^I\right) y_\tau$, for $\tau \in \{A, B\}$, where $\overrightarrow{\theta}^I$ is the investors' assessment when the merged firm is sold. Thus, the merged firm's objective is

$$\max_{\{y_A, y_B\}} \quad \mathcal{U}^M = p\left(\theta_A^I\right) y_A + p\left(\theta_B^I\right) y_B - c_A\left(y_A\right) - c_B\left(y_B\right).$$

If investors are uncertainty neutral, $\theta_\tau^I = \theta_T$, so the choice of y_A and y_B are independent of each other. In this case, the merged firm solves the same problem as the original entrepreneurs (9): $U^M = U_A^S + U_B^S$. This implies that the optimal levels of innovation intensity chosen by the merged firm are again given by (10), that is, the values the entrepreneurs would choose if the two firms were independent. Thus, if investors are uncertainty neutral, the merger does not add value.

In contrast, if investors are uncertainty averse, $\overrightarrow{\theta}^I = \overrightarrow{\theta}^a$ which, from (7), depends on the choice of both y_A and y_B. As shown in Lemma 4, for interior assessments (which we will show is the case in equilibrium), we now have that $V_A = V_B = p\left(\theta_T\right) y_A^{\frac{1}{2}} y_B^{\frac{1}{2}}$. The maximization problem of the merged firm becomes

$$\max_{y_A y_B} \quad \mathcal{U}^M = 2p\left(\theta_T\right) y_A^{\frac{1}{2}} y_B^{\frac{1}{2}} - \frac{1}{Z_A\left(1+\gamma\right)} y_A^{1+\gamma} - \frac{1}{Z_B\left(1+\gamma\right)} y_B^{1+\gamma}.$$

Theorem 4 *If investors are uncertainty averse, the merged firm will implement greater innovation intensity in both projects*

$$y_\tau^M \equiv \left[p\left(\theta_T\right) Z_{\tau'}^{\frac{1}{2\gamma+2}} Z_\tau^{\frac{2\gamma+1}{2\gamma+2}}\right]^{\frac{1}{\gamma}} > y_\tau^{U,SS} \quad ,$$

[23] Alternatively, the acquisition may be initiated by a third firm which may acquire both entrepreneurs' innovations.

[24] The two innovations processes may be sold to the public equity market either as a single multi-divisional firm, or as two independent firms. If the two innovations are sold in two separate firms, from Lemma 3, investors will optimally invest in both firms.

and will have a greater value than these firms would have as a stand-alone:

$$V^M = 2\left[p\left(\theta_T\right)\right]^{\frac{1+\gamma}{\gamma}}\left[Z_A Z_B\right]^{\frac{1}{2\gamma}} > V_A^{U,SS} + V_B^{U,SS}.$$

Theorem 4 shows that a merger can add value to the innovative process by merging both firms from the original entrepreneurs and then choosing an innovation intensity at both firms that is greater than the one that the entrepreneurs would chose individually. Because of the positive externality between investment levels y_τ, inefficiently low levels of investment occur when each entrepreneur maximizes his own payoff. By merging, the post-acquisition firm internalizes the spillover effects of investment, leading to greater firm valuation.

We now examine the impact of the possibility of a merger at the interim date $t = 1$ on the entrepreneurs' ex-ante incentives to innovate, that is, to sustain at $t = 0$ the discovery cost k_τ. The initial decision to innovate by an entrepreneur will depend on the terms at which she anticipates the merger will take place. The acquisition price will depend on the allocation of the surplus generated by the acquisition. Allocation of the synergies created in the merger occurs through bargaining, and we will assume that the two entrepreneurs will split the surplus equally. Thus, if both innovations are successful in the first stage, entrepreneur τ earns

$$\Upsilon_\tau = \mathcal{U}_\tau^{U,SS} + \frac{1}{2}\left(\mathcal{U}^M - \mathcal{U}_A^{U,SS} - \mathcal{U}_B^{U,SS}\right).$$

The incentives to pay the initial discover cost are discussed in the following.

Theorem 5 *For low levels of discover cost, $k_\tau \leq \underline{K}_\tau$, the entrepreneur always innovates. For high levels of discovery cost, $k_\tau \geq \bar{K}_\tau$, the entrepreneur never innovates. For intermediate levels of the discovery cost, $k_\tau \in \left(\underline{K}_\tau, \bar{K}_\tau\right)$, the entrepreneur is willing to innovate only if the other entrepreneur innovates. If both entrepreneurs have intermediate levels of discovery cost, there are multiple equilibria, one where both entrepreneurs innovate and one where neither innovate. The innovation equilibrium dominates the no-innovation equilibrium. Finally $\underline{K}_\tau = \underline{k}_\tau < \bar{k}_\tau < \bar{K}_\tau$: the possibility of a merger induces entrepreneurs to innovate more ex-ante.*

Theorems 4 and 5 have the interesting implication that an active M&A market promotes innovative activity and leads to greater innovation rates, stronger investor sentiment, and higher firm valuations. Synergies created in the merger are a direct consequence of endogenous investor sentiment due to uncertainty aversion. A merger allows entrepreneurs to internalize the positive impact that the choice of the innovation intensity in one innovation has on other innovations, and leads to greater innovation rates. Thus, the merger of innovations endogenously promotes stronger investor sentiment and leads to greater valuations.

4 Innovation Waves

In this section we extend our basic model to the case of multiple innovators in the context of a simple dynamic model. We show that entrepreneurs initiate innovation only when there is a sufficiently large number of potentially active innovators. In particular, when the number of potential innovators is low, entrepreneurs do not engage in innovation because they expect weak investor sentiment and, thus, the market for innovation to be "cold," and potential innovations remain latent in the economy. In contrast, when the number of innovators reaches critical mass, a wave of innovations is triggered amid strong investor sentiment.

We modify the basic model as follows. We consider a simple discrete-time dynamic model, where t denotes time. At each date t, a new project-idea arrives with probability π, where each project-idea is owned by a unique entrepreneur. The economy is "bounded" in that at any time only up to N project-ideas can exist in economy. Let $\mathcal{N}_t$ be the set of entrepreneurs endowed with a project-idea at any given time t, and let $\nu_t \equiv |\mathcal{N}_t|$. Different from the basic model, we now assume that an entrepreneur with a project-idea can delay its implementation to a future date. Waiting to implement the project-idea is costly, however: entrepreneurs and investors are impatient and have discount factor δ.

An entrepreneur endowed with a project-idea at time t must decide whether to implement the innovation or delay. We now assume that the decision to initiate innovation is made by the entrepreneur after observing a public signal σ_t which is informative on ν_t. For simplicity, we assume that the public signal is perfectly informative on ν_t: $\sigma_t = \nu_t$.[25] If an entrepreneur decides to innovate at time t, she must pay at that time discovery cost k to implement the first stage of the innovation process. For analytical tractability, we assume the first stage of the innovation process

[25] It is possible, although messy, to extend the model to the case in which the public signal is noisy.

is always successful, if it is implemented, setting $q_n = 1$.[26] At $t+1$, entrepreneurs proceed with the second stage of the innovation process by implementing innovation intensity y_n at cost $c(y_n)$, which is paid for by selling equity to investors. Finally, at time $t+2$ project-ideas have a payoff y_n with probability $p(\theta_{nt})$. For simplicity, we assume entrepreneurs' project ideas have the same productivity, $Z_n = Z$, and that innovation intensity is fixed, $y_n = y$.

The success probability of the second stage of a project implement at date t, p, is uncertain, and depends again on the value of θ_{nt}: $p(\theta_{nt}) = e^{\theta_{nt}-\theta_M}$ with $\theta_{nt} \in [\theta_L, \theta_H] \subseteq [\theta_m, \theta_M]$. For simplicity, we assume uncertainty on p_n is stationary and independent across time.[27] Thus, at any time t, investors are uncertain over $\overrightarrow{\theta} \equiv \{\theta_n\}_{n=1}^N$, and believe $\overrightarrow{\theta} \in \mathcal{C} \subset [\theta_m, \theta_M]^N$ and $\frac{1}{N}\sum_{n=1}^N \theta_n = \theta_T$ for $\theta_T \in (\theta_L, \theta_H)$.

Uncertainty-averse investors form at any time $t+1$ their portfolios of uncertain assets by buying equity from successful entrepreneurs (if any) available at that time. We denote by S_t the set of successful entrepreneurs at $t+1$, and let $s_t \equiv |S_t|$. Given our assumption that all innovations that are undertaken by an entrepreneur have a successful first-stage, S_t is the set of entrepreneurs that initiated their project-ideas at time t. Similar to the basic model, each investor chooses a portfolio of the uncertain assets, $\{\omega_n\}_{n\in S_t}$, given market valuations $\{V_n\}_{n\in S_t}$. By identical reasoning as Lemma 3, investors optimally invest equally in available innovations: $\omega_n = \omega_{n'}$ for all $(n, n') \in S_t$. Furthermore, given investor sentiment, $\overrightarrow{\theta}^a(S_t) = \{\theta_n^a\}_{n=1}^N$, equity is priced at its expected value, $V_n = p_n(\theta_n^a)\, y$ for $n \in S_t$. Investor sentiment depends on the number of entrepreneurs that innovate.

Lemma 6 *There is a threshold $\bar{s} \equiv N\frac{\theta_H-\theta_T}{\theta_H-\theta_L}$ such that if a small number of entrepreneurs innovate, $s_t \leq \bar{s}$, the market will assess all entrepreneurs in the market with very pessimistic sentiment, setting $\theta_{nt} = \theta_L$ for all $n \in S_t$; the value of equity is $V_n = \delta p(\theta_L)y$. If a large number of entrepreneurs innovate, $s_t > \bar{s}$, investors' sentiment satisfies:*

$$\theta_{nt}^a = \theta_H - \frac{N}{s_t}(\theta_H - \theta_T) > \theta_L, \quad for \quad n \in S_t, \tag{20}$$

[26] Our results will go through for the case in which $q_n < 1$.

[27] This assumption rules out, for example, interesting issues such as learning, which can be included in the analysis and we leave for future research.

and the market value of equity is

$$V_n(s_s) = \delta p\,(\theta^a_{nt})\,y. \tag{21}$$

Finally, $V_n(s_t)$ *is increasing in* s_t*.*

Lemma 6 shows investor sentiment at date t depends on the number of successful first-stage innovations, s_t. When the number of entrepreneurs that innovate is small, $s_t \leq \bar{s}$, investor sentiment is weak, and the capital market values innovations very conservatively. In contrast, when a large number of entrepreneurs decide to innovate, $s_t > \bar{s}$, investors have strong sentiment about innovation, leading to greater valuations.

For $s_t > \bar{s}$, from (20) investor sentiment is increasing in the number of projects available at that time, s_t. This happens because, for greater value of s_t, uncertainty-averse investors will be relatively less concerned on each individual project that is actually available at that time, relative to the set of possible projects, including those potentially yet to come.[28] Note also that investor sentiment is negatively affected by the extent of uncertainty in the economy, which we measure by the difference $\alpha = \theta_T - \theta_L$. This happens because, from (20), a greater value of α will make uncertainty-averse investors more pessimistic.

At the beginning of each period t, entrepreneurs endowed with a project-idea must decide whether to pay the discovery cost k and innovate, or to delay to a later period. This decision is made after observing the public signal σ_t which reveals the number of entrepreneurs with a project idea at that time, ν_t.

Theorem 6 *Let* $k \in (\underline{k}_d, \overline{k}_d)$ *(*$\underline{k}_d$ *and* $\overline{k}_d$ *are defined in the Appendix). There is a threshold* $\bar{\nu} > \bar{s}$ *(defined in the Appendix) such that if* $\nu_t \geq \bar{\nu}$*, it is optimal for an entrepreneur with a project-idea to pay the discovery cost* k *and innovate if all other entrepreneurs with project-ideas innovate.*

Lemma 6 and Theorem 6 imply that investor sentiment, market valuations of firm equity, and innovation decisions are endogenous, and depend on the number of innovative firms available on the market. If few entrepreneurs are endowed with a project idea, they rationally anticipate that in the following period investor sentiment will be cold, and correspondingly, market valuations will be

[28] Intuitively, this property can be seen immediately in light of the discussion in Section 1. For given total entropy (i.e. uncertainty), investors must limit the extent of the pessimism that they can have on each individual project.

low. The expectation of "cold equity markets" induce entrepreneurs to delay innovation to a later date. In contrast, when the number of entrepreneurs with an innovation is greater than a certain critical mass, $\bar{\nu}$, entrepreneurs anticipate that, if they innovate, investors will have strong sentiment in the following period, and, correspondingly, market valuations will be high. The expectation of "hot equity markets" will thus induce entrepreneurs to innovate.

From Theorem 6, we know that innovations may occur any time the number of entrepreneurs with a project-ideas exceeds the critical mass $\bar{\nu}$. The following theorem shows that, if the probability that a new project-idea arrives, π, is sufficiently low (or, equivalently, the discount factor δ is sufficiently small) it is best for entrepreneurs with project-ideas to pay the discovery cost k and innovate as soon as their number ν_t exceeds the critical mass $\bar{\nu}$.

Theorem 7 *There a threshold $\bar{\pi}$ (or, equivalently, $\bar{\delta}$) such that if $\pi \leq \bar{\pi}$ (or, equivalently, $\delta \leq \bar{\delta}$) the efficient equilibrium for entrepreneurs is for them to innovate as soon as their number exceeds critical mass, $\nu_t \geq \bar{\nu}$.*

The factors affecting the value of the critical mass $\bar{\nu}$ are characterized in the following corollary.

Corollary 3 *The critical mass $\bar{\nu}$ is increasing in $\{\alpha, N, k\}$, and decreasing in δ.*

The critical mass $\bar{\nu}$ depends positively on the level of uncertainty α, and the number of potential project-ideas, N. This happens because, from (20), a greater value of α and N will make uncertainty-averse investors more pessimistic (all else equal). Thus, a greater number of project ideas is needed to generate a level of investor sentiment sufficiently strong to ignite innovation. Similarly, a greater discovery cost k will require stronger investor sentiment, and thus greater equity valuations, to induce entrepreneurs to pay the initial cost and initiate the innovation process. Finally, a smaller discount factor δ will make entrepreneurs more impatient, so they will require more positive sentiment to be willing to invest, requiring a larger critical mass.

Our model has the following implications for the innovation process in an economy. Theorem 7 implies that innovation activity remains latent in the economy when the number of entrepreneurs with project-ideas is below critical mass. During this time, entrepreneurs delay innovation, the market for entrepreneurial equity is "cold," and dominated by low investor sentiment with a negative outlook. When the number of entrepreneurs with project-ideas reaches critical mass, entrepreneurs expect a substantial improvement in investor sentiment and a "hot" equity market for innovations. The improved expectations on the future market conditions spark an innovation wave that ripples through the economy. In addition, Corollary 3 implies that greater uncertainty, or a greater discovery cost, will lead to less frequent innovation waves, but when the wave takes place it will involve a larger number of innovations and will be characterized by stronger investor sentiment and equity valuations. In addition, if we interpret N as characterizing the complexity on an industry, Corollary 3 implies that less complex industries are characterized by more frequent innovation waves, of smaller intensity, and with less ebullient equity markets. In contrast, more complex industries are characterized by relatively less frequent innovation waves but that, when they occur, are of greater intensity, and with more ebullient equity markets.

5 Process Innovation

An important distinction identified in the literature on innovation is between "product innovation" and "process innovation."[29] Product innovation refers to the generation of a new product that did not exist before. Process innovation is interpreted broadly as involving the improvement of any part of the production process of an existing product, which typically results in efficiency gains due to productivity increases and/or cost reductions.

Innovation considered so far is well suited to describe "product innovation," whereby a firm invest resources, such as R&D, to develop an innovative product. If the R&D is successful, the firm obtains a new product.

In this section we show our analysis extends very easily to the case of process innovation. We model process innovation by assuming that, by paying at $t = 0$ a fixed cost of κ_τ, a firm can increase the productivity of its second-stage innovation process from Z_τ to IZ_τ $(1 < I < \psi)$.[30] In addition, we assume that the first stage of the innovation process is not risky, $q_\tau = 1$, for $\tau \in \{A, B\}$. The rest of the model unfolds as before.

[29] The distinction between process innovation and product innovation goes back at least to Utterback and Abernathy (1975). More recent work includes Klepper (1996), among many others.

[30] This is intentionally the same ψ as before, $\psi = \frac{1}{4}e^{2\alpha(\gamma+1)}\left(1 + \frac{1}{2\gamma}\right)^{2\gamma}$, so that investors always have interior assessments.

Theorem 8 *For low levels of cost, $\kappa_\tau \leq \underline{\kappa}_\tau$, the firm always innovates. For high levels of discovery cost, $\kappa_\tau \geq \bar{\kappa}_\tau$, the firm never innovates. For intermediate levels of the discovery cost, $\kappa_\tau \in (\underline{\kappa}_\tau, \bar{\kappa}_\tau)$, the firm is willing to innovate only if the other firm innovates. If both firms have intermediate levels of discovery cost, there are multiple equilibria, one where both firms innovate and one where neither innovate. The innovation equilibrium dominates the no-innovation equilibrium. There are strategic complementarities in process innovation iff investors are uncertainty averse.*

Similar to the case of product innovation, a firm with low cost of innovation, $\kappa_\tau \leq \underline{\kappa}_\tau$, is willing to implement the process innovation, independent of what the other firm does. A firm with high costs, $\kappa_\tau \geq \bar{\kappa}_\tau$, is never willing to implement process innovation. For intermediate values, $\kappa_\tau \in [\underline{\kappa}_\tau, \bar{\kappa}_\tau]$, a firm is willing to innovate only if the other firm innovates. If both firms have intermediate levels of discovery cost, there are multiple equilibria, generating again an assurance game. The presence of multiple equilibria is again a direct consequence of the strategic complementarities created by investors' aversion to uncertainty. If, on the contrary, investors are uncertainty neutral, $\underline{\kappa}_\tau = \bar{\kappa}_\tau$, and the innovation processes in the two firms are independent from each other.

6 Empirical Implications

Our paper has several novel empirical implications on the relationship between innovation waves, equity valuations in the technology sectors, "hot" IPO markets and M&A activity.

1. Innovation waves. Strategic complementarity between entrepreneurs' innovation decisions in our model creates the possibility of innovation waves. An innovation wave occurs if the number of potential entrepreneurs reaches critical mass. Arrival of innovation opportunities (i.e. project-ideas) in the economy may be random, and it may depend on classic "fundamentals" such as technological advances in certain sectors, say in Information Technologies or Life Sciences. Our paper suggests that such technological advances, while necessary, may not be sufficient to start a wave. Rather, an innovation wave occurs when a critical mass of potential innovators is attained which will spur a "hot" market for innovative companies.

Note that an innovation wave may start in one "sector" and then spill over to other "sectors," even if they are unrelated. This can happen, for example, when a positive shock in the project idea of entrepreneurs in one sector lowers their discovery cost from a high level, $k_\tau > \bar{k}_\tau$, to a low level, $k_\tau < \underline{k}_\tau$, while the other entrepreneur faces a moderate discovery cost, $k_{\tau'} \in \left(\underline{k}_{\tau'}, \bar{k}_{\tau'}\right)$, $\tau \neq \tau'$. If the discovery costs of the first set of entrepreneurs are subject to a shock and decrease to a low level, $k_\tau < \underline{k}_\tau$, it now becomes optimal for them to initiate the innovation process. This decision makes it profitable for other entrepreneurs to innovate as well, in anticipation of the possibility of higher equity prices. Thus, a positive idiosyncratic shock to the technology in one sector spills over to other entrepreneurs, triggering an innovation wave in another sector.[31] Note that the the "contagion" across sectors works through an "equity valuation" channel which is driven by strong investor sentiment, rather than a pure technological channel. Similar results hold for the productivity of innovation, Z_τ, and the probability of success, q_τ.The beneficial spillover effect is more likely to occur the greater the degree of relatedness of the two technologies (the greater the value of r).

2. Innovation waves, investor sentiment, and hot IPO markets. In our model, the market value of an entrepreneur's firm is (weakly) increasing in the number of successful firms in the market. This is because uncertainty-averse investors are more optimistic when they can invest in the equity of a larger set of new firms, leading to higher equity valuations. Innovation waves will be associated with strong investor sentiment toward innovations and, thus, with booms in the equity of technology firms. This also means that innovation waves can be associated with hot IPO markets which are then followed by lower stock returns. Thus, our model can explain the relationship between IPO volume and stock market valuations and the subsequent lower returns documented in the literature (see, for example, Ritter and Welch, 2002, for an extensive survey of the IPO literature).

3. Innovation waves and venture capitalists. An additional implication of our model is a new role for venture capitalists. If discovery costs fall in the intermediate range, $k_\tau \in \left(\underline{k}_\tau, \bar{k}_\tau\right)$, entrepreneurs face an "assurance game" in that each entrepreneur will be willing to incur the discovery cost and innovate only if she is assured that also other entrepreneurs will do the same. Lacking such assurance, entrepreneurs may be confined to the inefficient equilibrium with no innovation. In this setting, a venture capitalist may play a positive role by addressing the coordination failure among entrepreneurs. By investing in several technology firms, the venture capitalist can help coordination

[31] For example, a positive technological shock to, say, LinkedIn may be a boost to Uber, even if no direct technological link is present.

among entrepreneurs and lead to greater innovation. Note that companies in the VC portfolio do not need to have directly related technologies for the VC to have a beneficial role. In addition, as discussed above, coordination among entrepreneurs' innovative activities will be associated with greater equity market valuations. These observations imply that venture capital activity will be associated with innovation waves and greater equity valuations.

4. Venture capital cycles. The innovation cycle discussed in this paper also generates a VC cycle. Periods characterized by strong investor sentiment and a hot IPO market will be also associated with strong VC fundraising and commitment activity. VC investment will be active and at favorable terms for portfolio companies, which are driven by high expected exit multiples. Conversely, in periods where the market is cold, VCs will expect low exit multiples, generating limited fundraising and more unfavorable pricing of equity for portfolio companies.

5. Innovation, investor sentiment and merger activity. Our paper presents a new channel in which merger activity can generate synergies and spur innovation. Synergistic gains are the direct outcome of the beneficial spillover effect of the merger on the expected value of the innovation. In the post-merger firm, innovators choose greater levels of innovation intensity, leading to greater innovation rates for the merged firms. Our model also predicts that merger activities involving innovative firms will be associated with strong investor sentiment and greater firm valuations.

6. Related Technologies. Our paper shows that entrepreneurs have an incentive to invest in related technologies. Formally, entrepreneurs' utility is increasing in project relatedness, r. Investors value innovation more when they can simultaneously invest in other technologies, so entrepreneurs have an incentive to develop innovation which is more likely to succeed at the same time that other innovations succeed. Further, this mitigates the coordination failure, because it makes innovation more attractive to the other entrepreneur as well.

7. Incubators. Our model also provides a new motivation for technological incubators. Incubators allow entrepreneurs to meet each other, and coordinate innovation decisions. Our model made the standard assumption of full information, but this may not hold in practice. By creating an incubator, entrepreneurs can meet each other, perhaps overcoming the coordination failure.

7 Conclusion

In this paper, we show that uncertainty aversion generates innovation waves. Uncertainty aversion causes investors to treat different uncertain lotteries as complements, a property we refer to as uncertainty hedging. Uncertainty hedging by investors produces strategic complementarity in entrepreneurial behavior, producing innovation waves. Specifically, when one entrepreneur has a successful first-stage project, equity valuation, entrepreneur utility, and the intensity of innovation increase for other entrepreneurs. Thus, entrepreneurs are more willing to innovate if they expect other entrepreneurs are going to innovate as well, resulting in multiple equilibria. Our model can thus explain why there are some periods when investment in innovation is "hot," and investors are more willing to invest in risky investment projects tainted by significant uncertainty. Finally, we argue that mergers can add value because the positive spillover effects of innovation due to uncertainty hedging. Thus, our model predicts simultaneous innovation waves, merger waves, and positive investor sentiment in "hot" equity markets.

References

[1] Acemoglu, D. and Zilibotti, F. (1997) "Was Prometheus Unbound by Chance? Risk, Diversification, and Growth," *Journal of Political Economy*, **105**: 709-751.

[2] Aghion, P., Bloom, N., Blundell, R., Griffith, R., and Howitt, P. (2005) "Competition and Innovation: An Inverted U-Shaped Relationship," *Quarterly Journal of Economics* **120**: 701-728.

[3] Aghion, P. and Howitt, P. (1992) "A Model of Growth through Creative Destruction," *Econometrica*, **60**: 323-351.

[4] Anderson, E., Ghysels, E., and Juergens, J. (2009) "The Impact of Risk and Uncertainty on Expected Returns," *Journal of Financial Economics*, **94**: 233-263.

[5] Aramonte, S. (2016) "Firm-level R&D After Periods of Intense Technological Innovation: The Role of Investor Sentiment," SSRN Working Paper 2324958.

[6] Baker, M. and Wurgler, J. (2007) "Investor Sentiment in the Stock Market," *Journal of Economic Perspectives*, **21**: 129-151.

[7] Barilla, F., Hansen, L., and Sargent, T. (2009) "Doubts or Variability?" *Journal of Economic Theory,* **144:** 2388–2418.

[8] Bena, J. and Li, K. (2014) "Corporate Innovations and Mergers and Acquisitions," *Journal of Finance*, **69**: 1923-1960.

[9] Bernstein, S. (2015) "Does Going Public Affect Innovation?" *Journal of Finance*, **70**: 1365-1403.

[10] Booth, B. (2016) "This Time May Be Different," *Nature Biotechnology*, **34**: 25-30.

[11] Bossaerts, P., Ghirardato, P., Guarnaschelli, S., and Zame, W. (2010) "Ambiguity in Asset Markets: Theory and Experiment," *Review of Financial Studies*, **23**: 1325-1359.

[12] Boyarchenko, N. (2012) "Ambiguity Shifts and the 2007-2008 Financial Crisis," *Journal of Monetary Economics*, **59**: 493-507.

[13] Byun, S. (2014) "Incentivizing Innovation Under Ambiguity," University of Mississippi Working Paper.

[14] Caskey, J. (2009) "Information in Equity Markets with Ambiguity-Averse Investors," *Review of Financial Studies*, **22**: 3595-3627.

[15] Chemmanur, T. and Fulghieri, P. (2014) "Entrepreneurial Finance and Innovation: An Introduction and Agenda for Future Research," *Review of Financial Studies*, **27**: 1-19.

[16] Chew, S., Ratchford, M., and Sagi, J. (2013) "You Need to Recognize Ambiguity to Avoid It," SSRN Working Paper 2340543.

[17] Cover, T. and Thomas, J. (2006) *Elements of Information Theory* (second edition). Hoboken, New Jersey: Wiley-Interscience, John Wiley & Sons, Inc.

[18] de Finetti, B. (1974) *Theory of Probability: A Critical Introductory Treatment.* John Wiley & Sons Ltd.

[19] Dicks, D. and Fulghieri, P. (2016) "Ambiguity, Disagreement, and Allocation of Control in Firms," ECGI - Paper 396/2013, UNC Kenan-Flagler Research Paper No. 2357599, SSRN Working Paper 2357599.

[20] Dicks, D. and Fulghieri, P. (2017) "Uncertainty Aversion and Systemic Risk," *Journal of Political Economy,* forthcoming.

[21] Dimmock, S., Kouwenberg, R., Mitchell, O., and Peijnenburg, K. (2016) "Ambiguity Aversion and Household Portfolio Choice Puzzles: Empirical Evidence," *Journal of Financial Economics*, 119: 559-577.

[22] Drechsler, I. (2013) "Uncertainty, Time-Varying Fear, and Asset Prices," *Journal of Finance*, **68**: 1843-1889.

[23] Easley, D. and O'Hara, M. (2009) "Ambiguity and Nonparticipation: The Role of Regulation," *Review of Financial Studies*, **22**: 1817-1843.

[24] Easley, D. and O'Hara, M. (2010) "Microstructure and Ambiguity," *Journal of Finance*, **65**: 1817-1846.

[25] Entezarkheir, M. and Moshiri, S. (2016) "Is Innovation a Factor in Merger Decisions? Evidence from a Panel of U.S. Firms," SSRN Working Paper 2808059.

[26] Epstein, L. and Schneider, M. (2007) "Learning under Ambiguity," *Review of Economic Studies,* **74:** 1275-1303.

[27] Epstein, L. and Schneider, M. (2008) "Ambiguity, Information Quality and Asset Pricing," *Journal of Finance*, **63**, 197–228.

[28] Epstein, L. and Schneider, M. (2010) "Ambiguity and Asset Markets," *Annual Review of Financial Economics*, **2**: 315-346.

[29] Etner, J., Jeleva, M., and Tallon, J. (2012) "Decision Theory under Ambiguity," *Journal of Economic Surveys*, **26**: 234-270.

[30] Fagerberg, J., Mowery, D., and Nelson, R. (2005) *The Oxford Handbook of Innovation.* New York: Oxford University Press.

[31] Gallant, R., Jahan-Parvar, M., and Liu, H. (2015) "Measuring Ambiguity Aversion," SSRN Working Paper 2507306.

[32] Garlappi, L., Giammarino, R., and Lazrak, A. (2016) "Ambiguity and the Corporation: Group Decisions and Underinvestment," *Journal of Financial Economics*, forthcoming.

[33] Gilboa, I. and Schmeidler, D. (1989) "Maxmin Expected Utility with a Non-Unique Prior," *Journal of Mathematical Economics*, **18**: 141-153.

[34] Gompers, P. and Lerner, J. (2000) "Money Chasing Deals? The Impact of Fund Inflows on Private Equity Valuations," *Journal of Financial Economics*, **55**: 281-325.

[35] Gompers, P., Kovner, A., Lerner, J., and Scharfstein, D. (2008) "Venture Capital Investment Cycles: The Impact of Public Markets." *Journal of Financial Economics*, **87**: 1-23.

[36] Hansen, L. and Sargent, T. (2001) "Robust Control and Model Uncertainty," *American Economic Review*, **91**: 60-66.

[37] Hansen, L. and Sargent, T. (2007) "Recursive Robust Estimation and Control without Commitment," *Journal of Economic Theory*, **136**: 1-27.

[38] Hansen, L. and Sargent, T. (2008) *Robustness.* Princeton University Press.

[39] Hart, O. and Holmstrom, B. (2010) "A Theory of Firm Scope," *Quarterly Journal of Economics,* **125**: 483-513.

[40] Illeditsch, P. (2011) "Ambiguous Information, Portfolio Inertia, and Excess Volatility," *Journal of Finance*, **66**: 2213-2247.

[41] Jahan-Parver, M. and Liu, H. (2014) "Ambiguity Aversion and Asset Prices in Production Economies," *Review of Financial Studies*, **27**: 3060-97.

[42] Jeong, D., Kim, H., and Park, J. (2015) "Does Ambiguity Matter: Estimating Asset Pricing Models with a Multiple-Priors Recursive Utility," *Journal of Financial Economics*, **115**: 361-382.

[43] Jovanovic, B. and Rousseau, P. (2001) "Why Wait? A Century of Life Before IPO," *American Economic Review* **91**: 336-341.

[44] Keynes, J. (1936) *The General Theory of Employment, Interest and Money.* Palgrave Macmillan.

[45] Kleinknecht, A. (1987) *Innovation Patterns in Crisis and Prosperity.* Palgrave Macmillan.

[46] Klepper, S. (1996) "Entry, Exit, Growth, and Innovation over the Product Life Cycle," *American Economic Review*, **86**: 562-583.

[47] Klibanoff, P., Marinacci, M., and Mukerji, S. (2005) "A Smooth Model of Decision Making under Ambiguity," *Econometrica*, **73**: 1849–1892.

[48] Knight, F. (1921) *Risk, Uncertainty, and Profit.* Boston: Houghton Mifflin.

[49] Kuznets, S. (1940) "Schumpeter's Business Cycles" *American Economic Review* **30**: 257-271.

[50] Maccheroni, F., Marinacci, M., and Rustichini, A. (2006) "Ambiguity Aversion, Robustness, and the Variational Representation of Preferences," *Econometrica*, **74**: 1447-1498.

[51] Maenhout, P. (2004) "Robust Portfolio Rules and Asset Pricing," *Review of Financial Studies*, **17**: 951-983.

[52] Maksimovic, V. and Phillips, G. (2001) "The Market for Corporate Assets: Who Engages in Mergers and Asset Sales and Are There Efficiency Gains?" *Journal of Finance*, **56**: 2019-2065.

[53] Mele, A. and Sangiorgi, F. (2015) "Uncertainty, Information Acquisition and Price Swings in Asset Markets," *Review of Economic Studies*, **82**: 1533-1567.

[54] Miao, J. and Rivera A. (2016) "Robust Contracts in Continuous Time," *Econometrica*, **84**: 1405-1440.

[55] Mukerji, S. and Tallon, J. (2001) "Ambiguity Aversion and Incompleteness of Financial Markets," *Review of Economic Studies*, **68**: 883-904.

[56] Nanda, R., and Rhodes-Kropf, M. (2013) "Investment Cycles and Startup Innovation," *Journal of Financial Economics* **110**: 403–418.

[57] Nanda, R. and Rhodes-Kropf, M. (2016) "Financing Risk and Innovation," *Management Science*, forthcoming.

[58] Nicholas, T. (2008) "Does Innovation Cause Stock Market Runups? Evidence from the Great Crash." *American Economic Review*, **98**: 1370-96.

[59] Pastor, L. and Veronesi, P. (2005) "Rational IPO Waves." *Journal of Finance*, **60**: 1713-1757.

[60] Pastor, L. and Veronesi, P. (2009) "Technological Revolutions and Stock Prices," *American Economic Review* **99**: 1451-83.

[61] Perez, C. (2002) *Technological Revolutions and Financial Capital,* Edward Elgar.

[62] Phillips, G. and Zhdanov, A. (2013) "R&D and the Incentives from Mergers and Acquisition Activity," *Review of Financial Studies*, **26**: 34-78.

[63] Ritter, J. and Welch, I. (2002) "A Review of IPO Activity, Pricing, and Allocations," *Journal of Finance*, **57**: 1795-1828.

[64] Rhodes-Kropf, M. and Viswanathan, S. (2004) "Market Valuation and Merger Waves," *Journal of Finance*, **59**: 2685-2718.

[65] Scharfstein, D. and Stein, J. (1990) "Herd Behavior and Investment," *American Economic Review*, **80**: 465-479.

[66] Schmookler, J. (1966) *Invention and Economic Growth.* Cambridge, Mass.: Harvard University Press.

[67] Schumpeter, J. (1939) *Business Cycles: A Theoretical, Historical, and Statistical Analysis of the Capitalist Process.* Mansfield Centre, Connecticut: Martino Pub.

[68] Schumpeter, J. (1942) *Capitalism, Socialism and Democracy.* Floyd, Virginia: Impact Books.

[69] Shleifer. A. (1986) "Implementation Cycles," *Journal of Political Economy*, **94**: 1163-1190.

[70] Seru, A. (2014) "Firm Boundaries Matter: Evidence from Conglomerates and R&D Activity," *Journal of Financial Economics*, **111**: 381-405.

[71] Sevilir, M. and Tian, X. (2012) "Acquiring Innovation," SSRN Working Paper 1731722.

[72] Shiller, B. (2000) *Irrational Exuberance*. Princeton, NJ: Princeton University Press.

[73] Siniscalchi, M. (2011) "Dynamic Choice under Ambiguity," *Theoretical Economics*, **6**: 379–421.

[74] Skiadas, C. (2003) "Robust Control and Recursive Utility," *Finance and Stochastics*, **7**: 475-489.

[75] Solow, R. (1957). "Technical Change and the Aggregate Production Function," *Review of Economics and Statistics*, **39**: 312-320.

[76] Strzalecki, T. (2011) "Axiomatic Foundations of Multiplier Preferences," *Econometrica* **79**: 47-73.

[77] Utterback, J. M. and Abernathy, W. J. (1975) "A Dynamic Model of Process and Product Innovation" in *OMEGA, The International Journal of Management Science*, **3**: 639–656.

A Appendix: Proofs

Proof of Lemma 1. Let $x = \{x_A, x_B\}$ be a vector of indicator variables for success of type A and B assets: $x \in \{0,1\}^2$. If the probability of success is $p = \{p_A, p_B\}$ the probability of x is $p_A^{x_A} p_B^{x_B} (1-p_A)^{1-x_A} (1-p_B)^{1-x_B}$. Thus, the relative entropy of p w.r.t. $\hat{p}$ is

$$R(p|\hat{p}) = \sum_{x\in\{0,1\}^2} p_A^{x_A} p_B^{x_B} (1-p_A)^{1-x_A} (1-p_B)^{1-x_B} \ln \frac{p_A^{x_A} p_B^{x_B} (1-p_A)^{1-x_A} (1-p_B)^{1-x_B}}{\hat{p}_A^{x_A} \hat{p}_B^{x_B} (1-\hat{p}_A)^{1-x_A} (1-\hat{p}_B)^{1-x_B}}.$$

Because the log of a product is the sum of the logs, and probabilities sum to one, we can express this as

$$R(p|\hat{p}) = R(p_A|\hat{p}_A) + R(p_B|\hat{p}_B)$$

where $R(p_\tau|\hat{p}_\tau) = p_\tau \ln \frac{p_\tau}{\hat{p}_\tau} + (1-p_\tau)\ln\frac{1-p_\tau}{1-\hat{p}_\tau}$. Because $\frac{\partial^2 R}{\partial p_\tau^2} = \frac{\hat{p}_\tau}{p_\tau} + \frac{1-\hat{p}_\tau}{1-p_\tau}$, $R(p_\tau|\hat{p}_\tau)$ is strictly convex in p_τ. Thus, $R(p|\hat{p})$ is strictly convex in $p = \{p_A, p_B\}$. Also, $\lim_{p_\tau\to 0^+} R(p_\tau|\hat{p}_\tau) = \ln\frac{1}{1-\hat{p}_\tau}$ and $\lim_{p_\tau\to 1^-} R(p_\tau|\hat{p}_\tau) = \ln\frac{1}{\hat{p}_\tau}$. Define $\underline{\eta}(\hat{p}) = \min_{\chi\in Q} \ln\frac{1}{\chi}$, where $Q = \{\hat{p}_A, 1-\hat{p}_A, \hat{p}_B, 1-\hat{p}_B\}$. Therefore, if $\eta < \underline{\eta}(\hat{p})$, $\mathcal{M}$, as the lower level set of a strictly convex function, is strictly convex. Note this generalizes: Theorem 2.5.3 of Cover and Thomas (2006) shows relative entropy is additively separable in independent variables, and Theorem 2.7.2 shows it is strictly convex.

Suppose an investor receives y_A if innovation A is successful and y_B if innovation B is successful, both of which are strictly positive. It can be quickly verified that R achieves a minimum of zero at $p = \hat{p}$ and that R is strictly convex in both arguments (most importantly p here). This implies that $\frac{\partial R}{\partial p_\tau} < 0$ for $p_\tau < \hat{p}_\tau$ and $\frac{\partial R}{\partial p_\tau} > 0$ for $p_\tau > \hat{p}_\tau$. The worst-case scenario solves

$$\min \quad \{p_A y_A + p_B y_B\}$$
$$R(p|\hat{p}) \le \eta$$

Let λ be the multiplier for the constraint, and L be the Lagrangian function. Thus, $L = -(p_A y_A + p_B y_B) - \lambda(R(p|\hat{p}) - \eta)$, so $\frac{dL}{dp_\tau} = -y_\tau - \lambda\frac{\partial R}{\partial p_\tau}$. At the worst-case scenario, $\frac{dL}{dp_\tau} = 0$. Because $y_\tau > 0$, it must be that $\lambda\frac{\partial R}{\partial p_\tau} < 0$. This requires not only that the constraint binds, $\lambda > 0$, but also that p_τ is on the decreasing portion of R, or equivalently, that $p_\tau < \hat{p}_\tau$. If the investor has strictly positive exposure to only one uncertain lottery, but not the other, say $y_\tau > 0$ but $y_{\tau'} = 0$, the worst-case scenario involves choosing the worst possible value of p_τ, $R(p_\tau|\hat{p}_\tau) = \eta$ for $p_\tau < \hat{p}_\tau$, and setting $p_{\tau'} = \hat{p}_{\tau'}$. Finally, if $y_A = y_B = 0$, the claim holds WLOG. ■

Proof of Lemma 2. Investor's worst-case scenario solves $U(\Pi) = \min u(\theta;\Pi)$ s.t. $\frac{1}{2}(\theta_A + \theta_B) = \theta_T$. Let L be the Lagrangian for the minimization problem, and λ be the multiplier on the constraint. Thus, $\frac{\partial L}{\partial \theta_\tau} = -e^{\theta_\tau - \theta_M}\omega_\tau y_\tau + \frac{\lambda}{2}$. Because u is strictly convex in θ, FOCs are sufficient for a minimum. Setting $\frac{\partial L}{\partial \theta_\tau}|_{\theta_\tau = \check{\theta}_\tau} = 0$, and substituting into $\frac{1}{2}(\theta_A + \theta_B) = \theta_T$, this implies

$$\check{\theta}_\tau^a = \theta_T + \frac{1}{2}\ln\frac{\omega_{\tau'} y_{\tau'}}{\omega_\tau y_\tau}$$

Thus, if $\check{\theta}^a_\tau(\Pi) \in [\theta_L, \theta_H]$, $\theta^a_\tau = \check{\theta}^a_\tau$. If $\check{\theta}^a_\tau < \theta_L$, $\frac{\partial L}{\partial \theta_\tau} < 0$ for all $\theta_\tau \in [\theta_L, \theta_H]$, so $\theta^a_\tau = \theta_L$. If $\check{\theta}^a_\tau > \theta_H$, $\frac{\partial L}{\partial \theta_\tau} > 0$ for all $\theta_\tau \in [\theta_L, \theta_H]$, so $\theta^a_\tau = \theta_H$. Therefore, (7) corresponds to the worst-case scenario for an investor with portfolio Π. ■

Proof of Lemma 3. Each investor's objective function is $U(\Pi) = \min_{\theta \in C} u(\theta; \Pi)$ where $u(\theta; \Pi) = e^{\theta_A - \theta_M}\omega_A y_A + e^{\theta_B - \theta_M}\omega_B y_B + w_0 - \omega_A V_A - \omega_B V_B$. Thus, for $\tau \in \{A, B\}$,

$$\frac{dU}{d\omega_\tau} = \frac{\partial u}{\partial \omega_\tau} + \frac{\partial u}{\partial \theta_A}\frac{d\theta_A}{d\omega_\tau} + \frac{\partial u}{\partial \theta_B}\frac{d\theta_B}{d\omega_\tau}.$$

If investors are uncertainty-neutral, they believe $\theta_\tau = \theta_T$, so the second term disappears (θ_τ is constant). If investors are uncertainty averse, $\overrightarrow{\theta}^a$ solves the minimization problem. For interior solutions, by Lemma 2, $\frac{\partial u}{\partial \theta_A} = \frac{\partial u}{\partial \theta_B} = \frac{\lambda}{2}$, so the last two terms sum to $\frac{\lambda}{2}\frac{\partial(\theta_A + \theta_B)}{\partial \omega_\tau}$, which is zero because $\theta_A + \theta_B$ is constant. For corner solutions, $\frac{\partial \theta_A}{\partial \omega_\tau} = \frac{\partial \theta_B}{\partial \omega_\tau} = 0$. Therefore, $\frac{dU}{d\omega_\tau} = \frac{\partial u}{\partial \omega_\tau}$ for $\tau \in \{A, B\}$. Note $\frac{\partial u}{\partial \omega_\tau} = p(\theta^a_\tau) y_\tau - V_\tau$. Thus, market clearing requires that $V_\tau = p(\theta^a_\tau) y_\tau$, and that all investors have the same θ^a_τ. By Lemma 2, this implies $\frac{\omega_A}{\omega_B}$ is identical for all investors. Note that it is WLOG optimal for all investors to set $\omega_A = \omega_B = 1$. ■

Proof of Lemma 4. From Lemma 2, $\theta^a_\tau(\Pi) = \theta_L$ iff $\check{\theta}^a_\tau(\Pi) \leq \theta_L$ iff $y_\tau \geq e^{2\alpha} y_{\tau'}$. Thus, if $y_\tau \geq e^{2\alpha} y_{\tau'}$, $V_\tau = p(\theta_L) y_\tau$ and $V_{\tau'} = p(\theta_H) y_{\tau'}$. The $\theta^a_\tau(\Pi) = \theta_H$ case is symmetric. Finally, from Lemma 2, $\theta^a_\tau(\Pi) \in (\theta_L, \theta_H)$ iff $\check{\theta}^a_\tau(\Pi) \in (\theta_L, \theta_H)$ iff $y_\tau \in \left(e^{-2\alpha} y_{\tau'}, e^{2\alpha} y_{\tau'}\right)$. Because $\theta^a_\tau(\Pi) = \check{\theta}^a_\tau(\Pi)$, $p(\theta^a(\Pi)) = e^{(\theta_T - \theta_M)} y_{\tau'}^{\frac{1}{2}} y_\tau^{-\frac{1}{2}}$, which implies the market values entrepreneur τ's firm at $V_\tau = e^{\theta_T - \theta_M} y_{\tau'}^{\frac{1}{2}} y_\tau^{\frac{1}{2}}$. There is strategic complementarity in production because $\frac{\partial V_\tau}{\partial y_{\tau'}} \geq 0$ for $\tau' \neq \tau$, with strict inequality for $y_\tau \in \left(e^{-2\alpha} y_{\tau'}, e^{2\alpha} y_{\tau'}\right)$. ■

Proof of Lemma 5. Because only entrepreneur τ has a successful first-stage project-idea, so $y_{\tau'} = 0$. By Lemma 2, $\theta^a_\tau = \theta_L$. By Lemma 3, $V_A = p(\theta_L) y_A$, so the entrepreneur's payoff is

$$\mathcal{U}_\tau = p(\theta_L) y_\tau - \frac{1}{Z_\tau(1+\gamma)} y_\tau^{1+\gamma}.$$

Note that $\frac{\partial \mathcal{U}_\tau}{\partial y_\tau} = p(\theta_L) - \frac{1}{Z_\tau} y_\tau^\gamma$, and $\frac{\partial^2 \mathcal{U}_\tau}{\partial y_\tau^2} = -\frac{\gamma}{Z_\tau} y_\tau^{\gamma-1} < 0$, so FOCs are sufficient for a maximum. Thus, entrepreneur τ selects $y_\tau^{U,SF} = [p(\theta_L) Z_\tau]^{\frac{1}{\gamma}}$, sells for $V_\tau^{U,SF} = [p(\theta_L)]^{\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}}$, and earns continuation payoff $\mathcal{U}_\tau^{U,SF} = [p(\theta_L)]^{\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma}$. ■

Proof of Theorem 2. When both entrepreneurs have successful first-stage projects, they select innovation intensity y_τ to maximize $\mathcal{U}_\tau^U = V_\tau(\Pi) - \frac{1}{Z_\tau(1+\gamma)} y_\tau^{1+\gamma}$, where $V_\tau(\Pi)$ is given in Lemma 4. For $y_\tau < e^{-2\alpha} y_{\tau'}$, $\frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau} = p(\theta_H) - \frac{1}{Z_\tau} y_\tau^\gamma$. For $y_\tau \in \left(e^{-2\alpha} y_{\tau'}, e^{2\alpha} y_{\tau'}\right)$, $\frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau} = \frac{1}{2} p(\theta_\tau) y_\tau^{-\frac{1}{2}} y_{\tau'}^{\frac{1}{2}} - \frac{1}{Z_\tau} y_\tau^\gamma$. For $y_\tau > e^{2\alpha} y_{\tau'}$, $\frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau} = p(\theta_L) - \frac{1}{Z_\tau} y_\tau^\gamma$. Thus, $\lim_{y_\tau \uparrow e^{-2\alpha} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau} = p(\theta_H) - \frac{1}{Z_\tau}\left[e^{-2\alpha} y_{\tau'}\right]^\gamma > \frac{1}{2} p(\theta_H) - \frac{1}{Z_\tau}\left[e^{-2\alpha} y_{\tau'}\right]^\gamma = \lim_{y_\tau \downarrow e^{-2\alpha} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau}$, but $\lim_{y_\tau \uparrow e^{2\alpha} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau} = \frac{1}{2} p(\theta_L) - \frac{1}{Z_\tau}\left[e^{2\alpha} y_{\tau'}\right]^\gamma < p(\theta_L) - \frac{1}{Z_\tau}\left[e^{2\alpha} y_{\tau'}\right]^\gamma = \lim_{y_\tau \downarrow e^{2\alpha} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau}$. Therefore, any critical point $y_\tau \leq e^{-2\alpha} y_{\tau'}$ is a global maximum, but a critical point in $\left(e^{-2\alpha} y_{\tau'}, e^{2\alpha} y_{\tau'}\right)$ could be just a local maximum, and must be compared to the critical point $y_\tau \geq e^{2\alpha} y_{\tau'}$.[32]

We will now solve for the best-response function. It is optimal to select $y_\tau < e^{-2\alpha} y_{\tau'}$ only if $y_\tau = [p(\theta_H) Z_\tau]^{\frac{1}{\gamma}} < e^{-2\alpha} y_{\tau'}$. Therefore, for $y_{\tau'} > \bar{y}_\tau \equiv e^{\alpha\left(2+\frac{1}{\gamma}\right)} [p(\theta_T) Z_\tau]^{\frac{1}{\gamma}}$, $y_\tau = [p(\theta_H) Z_\tau]^{\frac{1}{\gamma}}$. It is optimal to select $y_\tau = e^{-2\alpha} y_{\tau'}$ only if $\lim_{y_\tau \uparrow e^{-2\alpha} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau} \geq 0 \geq \lim_{y_\tau \downarrow e^{-2\alpha} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau}$, which holds if $y_{\tau'} \in \left[e^{\alpha\left(2+\frac{1}{\gamma}\right)}\left[\frac{1}{2} p(\theta_T) Z_\tau\right]^{\frac{1}{\gamma}}, e^{\alpha\left(2+\frac{1}{\gamma}\right)} [p(\theta_T) Z_\tau]^{\frac{1}{\gamma}}\right]$. The optimal $y_\tau > e^{2\alpha} y_{\tau'}$ is $y_\tau^{U,SF}$ from Lemma 5, which provides utility $\mathcal{U}_\tau^{U,SF} = [p(\theta_L)]^{\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma}$. For $y_\tau \in \left(e^{-2\alpha} y_{\tau'}, e^{2\alpha} y_{\tau'}\right)$ to be optimal, it must not only be locally optimal, but also must provide greater utility than $\mathcal{U}_\tau^{U,SF}$.[33] Note that y_τ is a critical point, $\frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau} = 0$, iff $y_\tau = \left[\frac{Z_\tau}{2} p(\theta_T) y_{\tau'}^{\frac{1}{2}}\right]^{\frac{1}{\gamma+\frac{1}{2}}}$, which provides utility $\mathcal{U}_\tau^{U,SS} =$

[32] For $y_\tau \in \left(e^{-2\alpha} y_{\tau'}, e^{2\alpha} y_{\tau'}\right)$, $\frac{\partial^2 \mathcal{U}_\tau^U}{\partial y_\tau^2} = -\frac{1}{4} p(\theta_T) y_\tau^{-\frac{3}{2}} y_{\tau'}^{\frac{1}{2}} - \frac{\gamma}{Z_\tau} y_\tau^\gamma$, and for $y_\tau \notin \left(e^{-2\alpha} y_{\tau'}, e^{2\alpha} y_{\tau'}\right)$, $\frac{\partial^2 \mathcal{U}_\tau^U}{\partial y_\tau^2} = -\frac{\gamma}{Z_\tau} y_\tau^{\gamma-1}$. Because $\lim_{y_\tau \uparrow e^{-2\alpha} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau} > \lim_{y_\tau \downarrow e^{-2\alpha} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau}$ but $\lim_{y_\tau \uparrow e^{2\alpha} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau} < \lim_{y_\tau \downarrow e^{2\alpha} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau}$, $\mathcal{U}_\tau^U$ is concave everywhere except at $y_\tau = e^{2\alpha} y_{\tau'}$, where it kinks up. Finally, because $p(\theta_H) > p(\theta_L)$, $\lim_{y_\tau \uparrow e^{-2\alpha} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau} > \lim_{y_\tau \downarrow e^{2\alpha} y_{\tau'}} \frac{\partial \mathcal{U}_\tau^U}{\partial y_\tau}$, any critical point $y_\tau \leq e^{-2\alpha} y_{\tau'}$ will be a global maximum.

[33] This assumes that $y_\tau^{U,SF} > e^{2\alpha} y_{\tau'}$. If not, selecting $y_\tau^{U,SF}$ would give entrepreneur τ a strictly larger payoff than

$[p(\theta_T)]^{\frac{2\gamma+2}{2\gamma+1}} Z_\tau^{\frac{1}{2\gamma+1}} \left[\frac{1}{2}\right]^{\frac{1}{2\gamma+1}} \left[\frac{2\gamma+1}{2\gamma+2}\right] y_{\tau'}^{\frac{1+\gamma}{2\gamma+1}}$. Note $U_\tau^{U,SS} > U_\tau^{U,SF}$ iff $y_{\tau'} > \underline{y}_{\tau'} \equiv e^{-\alpha\frac{2\gamma+1}{\gamma}} [p(\theta_T)]^{\frac{1}{\gamma}} Z_\tau^{\frac{1}{\gamma}} 2^{\frac{1}{\gamma+1}} \left[\frac{2\gamma}{2\gamma+1}\right]^{\frac{2\gamma+1}{\gamma+1}}$. Therefore, if $y_{\tau'} < \underline{y}_{\tau'}$, $y_\tau = [p(\theta_L) Z_\tau]^{\frac{1}{\gamma}}$, while if $y_{\tau'} > \underline{y}_{\tau'}$, $y_\tau = \left[\frac{Z_\tau}{2} p(\theta_T) y_{\tau'}^{\frac{1}{2}}\right]^{\frac{1}{\gamma+\frac{1}{2}}}$.

Restricting attention to pure strategy equilibria, it must be either the investors have interior beliefs, $\frac{y_{\tau'}}{y_\tau} \in (e^{-2\alpha}, e^{2\alpha})$, or corner beliefs, $\frac{y_{\tau'}}{y_\tau} \notin (e^{-2\alpha}, e^{2\alpha})$. Both entrepreneurs select innovation optimally. Because there is a kink at $y_{\tau'} = e^{2\alpha} y_\tau$, it can never be that $\frac{y_{\tau'}}{y_\tau} = e^{2\alpha}$ in equilibrium. Suppose to the contrary that investors have corner beliefs in equilibrium, so one entrepreneur selects $y_{\tau'} > e^{2\alpha} y_\tau$. In that case, $y_{\tau'} = e^{-\frac{\alpha}{\gamma}} [p(\theta_T)]^{\frac{1}{\gamma}} Z_{\tau'}^{\frac{1}{\gamma}}$, $y_\tau = e^{\frac{\alpha}{\gamma}} [p(\theta_T)]^{\frac{1}{\gamma}} Z_\tau^{\frac{1}{\gamma}}$, so $y_{\tau'} > e^{2\alpha} y_\tau$ only if $\frac{Z_{\tau'}}{Z_\tau} > e^{2\alpha(\gamma+1)} > \psi = \frac{1}{4} e^{2\alpha(\gamma+1)} \left(1 + \frac{1}{2\gamma}\right)^{2\gamma}$ because $\left(1 + \frac{1}{2\gamma}\right)^{2\gamma} \in (1, e)$ for all $\gamma > 0$ and $e < 4$. Alternatively, if investors have interior beliefs in equilibrium, $y_\tau = \left[\frac{Z_\tau}{2} p(\theta_T) y_{\tau'}^{\frac{1}{2}}\right]^{\frac{1}{\gamma+\frac{1}{2}}}$ for both entrepreneurs, which implies $y_\tau^{U,SS} = \left[\frac{1}{2} p(\theta_T) Z_\tau^{\frac{2\gamma+1}{2\gamma+2}} Z_{\tau'}^{\frac{1}{2\gamma+2}}\right]^{\frac{1}{\gamma}}$. This is better for entrepreneur τ than $y_\tau^{U,SF}$ only if $y_{\tau'}^{U,SS} > e^{-\alpha\frac{2\gamma+1}{\gamma}} [p(\theta_T)]^{\frac{1}{\gamma}} Z_\tau^{\frac{1}{\gamma}} 2^{\frac{1}{\gamma+1}} \left[\frac{2\gamma}{2\gamma+1}\right]^{\frac{2\gamma+1}{\gamma+1}}$, which holds iff $\frac{Z_{\tau'}}{Z_\tau} > e^{-2\alpha(\gamma+1)} 4 \left[\frac{2\gamma}{2\gamma+1}\right]^{2\gamma} = \frac{1}{\psi}$, which holds by assumption. Therefore, when $\frac{Z_A}{Z_B} \in \left(\frac{1}{\psi}, \psi\right)$, both entrepreneurs select innovation intensity so that investors have interior beliefs in equilibrium,[34] selecting according to best-response function $y_\tau = \left[\frac{Z_\tau}{2} p(\theta_T) y_{\tau'}^{\frac{1}{2}}\right]^{\frac{1}{\gamma+\frac{1}{2}}}$, leading to equilibrium innovation $y_\tau^{U,SS} = \left[\frac{1}{2} p(\theta_T) Z_\tau^{\frac{2\gamma+1}{2\gamma+2}} Z_{\tau'}^{\frac{1}{2\gamma+2}}\right]^{\frac{1}{\gamma}}$. Because the market price is $V_\tau^{U,SS} = p(\theta_T) y_\tau^{\frac{1}{2}} y_{\tau'}^{\frac{1}{2}}$, $V_\tau^{U,SS} = 2^{-\frac{1}{\gamma}} [p(\theta_T)]^{\frac{\gamma+1}{\gamma}} [Z_\tau Z_{\tau'}]^{\frac{1}{2\gamma}}$. Similarly, entrepreneur τ earns continuation utility $\mathcal{U}_\tau^{U,SS} = V_\tau^{U,SS} - \frac{1}{Z_\tau(1+\gamma)} y_\tau^{1+\gamma}$, which can be expressed as $\mathcal{U}_\tau^{U,SS} = 2^{-\frac{1}{\gamma}} [p(\theta_T)]^{\frac{\gamma+1}{\gamma}} Z_\tau^{\frac{1}{2\gamma}} Z_{\tau'}^{\frac{1}{2\gamma}} \frac{2\gamma+1}{2\gamma+2}$, for $\tau \in \{A, B\}$ and $\tau' \neq \tau$. Thus, there are strategic complementarities in production and profit. ■

Proof of Corollary 1. Theorem 2 showed $\mathcal{U}_\tau^{U,SS} > \mathcal{U}_\tau^{U,SF}$ when $\frac{Z_{\tau'}}{Z_\tau} \in \left(\frac{1}{\psi}, \psi\right)$. $V_\tau^{U,SS} = 2^{-\frac{1}{\gamma}} [p(\theta_T)]^{\frac{\gamma+1}{\gamma}} [Z_\tau Z_{\tau'}]^{\frac{1}{2\gamma}}$ and $V_\tau^{U,SF} = [p(\theta_L)]^{\frac{\gamma+1}{\gamma}} Z_\tau^{\frac{1}{\gamma}}$, so $V_\tau^{U,SS} > V_\tau^{U,SF}$ iff $\frac{Z_{\tau'}}{Z_\tau} > 4e^{-2\alpha(1+\gamma)}$. Define $\psi_1 = \frac{1}{4} e^{2\alpha(1+\gamma)}$. Because $y_\tau^{U,SS} = \left[\frac{1}{2} p(\theta_T) Z_\tau^{\frac{2\gamma+1}{2\gamma+2}} Z_{\tau'}^{\frac{1}{2\gamma+2}}\right]^{\frac{1}{\gamma}}$ and $y_\tau^{U,SF} = [p(\theta_L) Z_\tau]^{\frac{1}{\gamma}}$, $y_\tau^{U,SS} > y_\tau^{U,SF}$ iff $\frac{Z_{\tau'}}{Z_\tau} > 4^{\gamma+1} e^{-2\alpha(1+\gamma)}$. Thus, define $\psi_2 = \left[\frac{1}{4}\right]^{\gamma+1} e^{2\alpha(1+\gamma)}$. Because $\gamma > 0$, $\psi_2 < \psi_1 < \psi$. ■

Proof of Theorem 3. If only one entrepreneur innovates, he earns payoff $E\mathcal{U}_\tau^{U,N} = q_\tau \mathcal{U}_\tau^{U,SF} - k_\tau$ (Lemma 5). Thus, if an entrepreneur does not expect the other entrepreneur to innovate, he will innovate iff $k_\tau \leq \underline{k}_\tau \equiv q_\tau \mathcal{U}_\tau^{U,SF}$. Conversely, if the other entrepreneur innovates, the entrepreneur earns payoff $E\mathcal{U}_\tau^{U,I} = (q_\tau q_{\tau'} + r) \mathcal{U}_\tau^{U,SS} + [q_\tau(1 - q_{\tau'}) - r] \mathcal{U}_\tau^{U,SF} - k_\tau$ if he innovates as well. Thus, if the other entrepreneur innovates, he innovates iff $k_\tau \leq \bar{k}_\tau \equiv (q_\tau q_{\tau'} + r) \mathcal{U}_\tau^{U,SS} + [q_\tau(1 - q_{\tau'}) - r] \mathcal{U}_\tau^{U,SF}$. By Corollary 1, $\mathcal{U}_\tau^{U,SF} < \mathcal{U}_\tau^{U,SS}$, so $\underline{k}_\tau < \bar{k}_\tau$. If both equilibria exist, $k_\tau \in [\underline{k}_\tau, \bar{k}_\tau]$ for both entrepreneurs. In the no-innovation equilibrium, entrepreneurs earn zero. In the innovation equilibrium, entrepreneurs earn $E\mathcal{U}_\tau^{U,I} \geq 0$ (strict inequality if $k_\tau < \bar{k}_\tau$). Firms are priced so investors are indifferent. Everyone is better off, so the innovation equilibrium dominates the other. ■

Proof of Corollary 2. Comparative Statics follow immediately from inspection of the expressions for $\underline{k}_\tau$ and $\bar{k}_\tau$ from Theorem 3, and because $\mathcal{U}_\tau^{U,SS}$ is increasing in Z_τ and $Z_{\tau'}$, and $\mathcal{U}_\tau^{U,SF}$ is increasing in Z_τ. ■

Proof of Theorem 4. The merged firm seeks to maximize the combined value of the two projects. By identical reasoning to Lemma 3, $V_\tau = p(\theta_\tau^I) y_\tau$, where $\overrightarrow{\theta}^I$ is the market assessment at $t = 1$ on θ. Thus, the merged firm's objective is $\mathcal{U}^M = p(\theta_A^I) y_A + (\theta_B^I) y_B - c_A(y_A) - c_B(y_B)$. Because investors are uncertainty averse, $\overrightarrow{\theta}^I = \overrightarrow{\theta}^a$,

---

$\mathcal{U}_\tau^{U,SF}$, but a smaller payoff than $y_\tau = \left[\frac{Z_\tau}{2} p(\theta_\tau) y_{\tau'}^{\frac{1}{2}}\right]^{\frac{1}{\gamma+\frac{1}{2}}}$ because $\mathcal{U}_\tau$ is strictly concave on $(e^{-2\alpha} y_{\tau'}, e^{2\alpha} y_{\tau'})$.

[34] If $\frac{Z_\tau}{Z_{\tau'}} \in \left(\psi, e^{2\alpha(\gamma+1)}\right)$, the equilibrium is a mixed strategy equilibrium, where one firm selects $y_{\tau'} = e^{-\alpha\frac{2\gamma+1}{\gamma}} [p(\theta_T)]^{\frac{1}{\gamma}} Z_\tau^{\frac{1}{\gamma}} 2^{\frac{1}{\gamma+1}} \left[\frac{2\gamma}{2\gamma+1}\right]^{\frac{2\gamma+1}{\gamma+1}}$ and the other randomizes between $[p(\theta_L) Z_\tau]^{\frac{1}{\gamma}}$ and $\left[\frac{Z_\tau}{2} p(\theta_T) y_{\tau'}^{\frac{1}{2}}\right]^{\frac{1}{\gamma+\frac{1}{2}}}$.

which depend on the choice of y_A and y_B. As shown in Lemma 4, $V_A = V_B = p(\theta_T)\, y_A^{\frac{1}{2}} y_B^{\frac{1}{2}}$, so the objective function of the merged firm becomes

$$\mathcal{U}^M = 2p(\theta_T)\, y_A^{\frac{1}{2}} y_B^{\frac{1}{2}} - \frac{1}{Z_A(1+\gamma)} y_A^{1+\gamma} - \frac{1}{Z_B(1+\gamma)} y_B^{1+\gamma}.$$

Because $\frac{\partial \mathcal{U}^M}{\partial y_\tau} = p(\theta_T)\, y_\tau^{-\frac{1}{2}} y_{\tau'}^{\frac{1}{2}} - \frac{1}{Z_\tau} y_\tau^\gamma$, for $\tau \in \{A, B\}$ and $\tau' \neq \tau$, this implies[35] $y_\tau = \left[p(\theta_T)\, Z_\tau y_{\tau'}^{\frac{1}{2}}\right]^{\frac{1}{\gamma+\frac{1}{2}}}$, so

$$y_\tau^M = \left[p(\theta_T)\, Z_{\tau'}^{\frac{1}{2\gamma+2}} Z_\tau^{\frac{2\gamma+1}{2\gamma+2}}\right]^{\frac{1}{\gamma}}.$$

Thus, $V_A^M = V_B^M = [p(\theta_T)]^{\frac{1+\gamma}{\gamma}} [Z_{\tau'} Z_\tau]^{\frac{1}{2\gamma}}$. ■

Proof of Theorem 5. If entrepreneur τ does not expect entrepreneur τ' to innovate, he innovates iff $k_\tau \leq \underline{K}_\tau \equiv q_\tau \mathcal{U}_\tau^{U,SF}$, the same cutoff as without the possibility of a merger. However, if entrepreneur τ expects entrepreneur τ' to innovate, he innovates iff $k_\tau \leq \bar{K}_\tau \equiv (q_\tau q_{\tau'} + r)\Upsilon_\tau + [q_\tau(1-q_{\tau'}) - r]\, \mathcal{U}_\tau^{U,SF}$. Because $\Upsilon_\tau = \mathcal{U}_\tau^{U,SS} + \frac{1}{2}\left(\mathcal{U}^M - \mathcal{U}_A^{U,SS} - \mathcal{U}_B^{U,SS}\right)$, if $\Upsilon_\tau > \mathcal{U}_\tau^{U,SS}$, $\bar{K}_\tau > \bar{k}_\tau$, so the cutoff will be larger when mergers are possible, resulting in more innovation. Thus, it is sufficient to show that $\mathcal{U}^M > \mathcal{U}_A^{U,SS} + \mathcal{U}_B^{U,SS}$.

Because $V^M = V_A^M + V_B^M$, the merged firm earns utility

$$\mathcal{U}^M = 2\frac{\gamma}{1+\gamma} [p(\theta_T)]^{\frac{1+\gamma}{\gamma}} [Z_A Z_B]^{\frac{1}{2\gamma}}.$$

Each entrepreneur could earn utility $\mathcal{U}_\tau^{U,SS} = \frac{1}{2^{\frac{1}{\gamma}}} [p(\theta_T)]^{\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{2\gamma}} Z_{\tau'}^{\frac{1}{2\gamma}} \frac{2\gamma+1}{2\gamma+2}$ if they did not merge, so

$$\mathcal{U}_A^{U,SS} + \mathcal{U}_B^{U,SS} = \mathcal{U}^M \frac{1}{2^{\frac{1}{\gamma}}} \frac{2\gamma+1}{2\gamma}$$

Because[36] $\frac{1}{2^{\frac{1}{\gamma}}} \frac{2\gamma+1}{2\gamma} \in (0,1)$ for all $\gamma \in (0,\infty)$, the merger adds value: $\mathcal{U}^M > \mathcal{U}_A^{U,SS} + \mathcal{U}_B^{U,SS}$, so $\Upsilon_\tau > \mathcal{U}_\tau^{U,SS}$. ■

Proof of Lemma 6. At $t+1$, entrepreneurs in S_t chose to implement their project-ideas and thus have a successful first-stage innovation. Only implemented projects can be traded, so investors choose portfolio weights $\{\omega_n\}_{n \in S_t}$ to maximize their minimum expected payoff, $\min_{\overrightarrow{\theta} \in C} u\left(\overrightarrow{\theta}\right)$, where $u\left(\overrightarrow{\theta}\right) = \sum_{n \in S_t} \omega_n [\delta p_n(\theta_{nt})\, y_n - V_n] + w_0$. By identical proof to Lemma 3, in equilibrium, $\omega_n = 1$ for all $n \in S_t$ and $V_n = \delta p_n(\theta_{nt}^a)\, y_n$. Recall that $\overrightarrow{\theta}$ is in C iff $\sum_{n=1}^N \theta_{nt} = N\theta_T$ and $\theta_{nt} \in [\theta_L, \theta_H]$ for all n. Let L be the Lagrangian function for the minimization problem, let λ be the multiplier for the sum, and let γ_{nL} and γ_{nH} be the constraints that $\theta_{nt} \geq \theta_L$ and $\theta_{nt} \leq \theta_H$ respectively. Thus, $\frac{\partial L}{\partial \theta_{nt}} = -e^{\theta_{nt} - \theta_M} y_n + \lambda + \gamma_{nL} - \gamma_{nH}$. For $n \in S_t$, $y_n = y > 0$, while for $n \notin S_t$, $y_n = 0$. Thus, symmetry of the FOCs implies that WLOG that there will be symmetry in the worst-case scenarios: θ_{nt} is constant for all $n \in S_t$ and θ_{nt} is constant for all $n \notin S_t$.

If $\lambda = 0$, then $\gamma_{nL} > 0$ for $n \in S$, so $\theta_{nt} = \theta_L$ (the market has negative sentiment toward all implemented projects). WLOG, for $n \notin S_t$, $\theta_n = \frac{N\theta_T - s\theta_L}{N-s}$. This is feasible if $\frac{N\theta_T - s\theta_L}{N-s} < \theta_H$, or equivalently, if $s \leq \bar{s} \equiv N\frac{\theta_H - \theta_T}{\theta_H - \theta_L}$. If $\lambda > 0$, then $\gamma_{nH} > 0$ for all $n \notin S_t$, so $\theta_{nt} = \theta_H$. Substituting into $\sum_{n=1}^N \theta_{nt} = N\theta_T$, this implies that $s\theta_{nt}^a + (N-s)\,\theta_H = N\theta_T$, or equivalently, that $\theta_{nt}^a = \theta_H - \frac{N}{s}(\theta_H - \theta_T)$. Note that this is feasible, $\theta_{nt}^a \geq \theta_L$, iff $s \geq \bar{s}$. Market valuation follows by substitution, and is increasing in s because θ_{nt}^a is increasing in s. ■

Proof of Theorem 6. Suppose that $k \in \left(\underline{k}_d, \overline{k}_d\right)$, where $\underline{k}_d = \delta^2 e^{\theta_L - \theta_M} y - c(y)$ and $\overline{k}_d = \delta^2 e^{\theta_T - \theta_M} y - c(y)$. Suppose an entrepreneur believes that all other entrepreneurs are going to innovate, and there are $s-1$ other entrepreneurs with project ideas. Entrepreneurs with a project-idea earn 0 if they do not innovate, while they earn

---

[35] Because the cost functions are convex, the problem is globally concave, so first-order conditions are sufficient.

[36] Define $x = \frac{1}{\gamma}$, and $f(x) = 2^{-x-1}(2+x)$: $f'(x) = 2^{-x-1}[1 - (2+x)\ln 2]$, which is strictly negative because $2\ln 2 > 1$, $\lim_{x \to 0^+} f(x) = 1$, and $\lim_{x \to +\infty} f(x) = 0$. Therefore, $\frac{1}{2^{\frac{1}{\gamma}}} \frac{2\gamma+1}{2\gamma} \in (0,1)$ for all $\gamma \in (0,\infty)$.

$U(s)$ if they do innovate. If $s \le \bar{s}$, $U(s) = \delta^2 e^{\theta_L - \theta_M} y - c(y) - k$. For $s > \bar{s}$, $U(s) = \delta^2 e^{\theta_H - \frac{N}{s}(\theta_H - \theta_T) - \theta_M} y - c(y) - k$, which is increasing in s, and $U(N) = \delta^2 e^{\theta_T - \theta_M} y - c(y) - k$. Because $k \in \left(\underline{k}_d, \overline{k}_d\right)$, $U(\lfloor \bar{s} \rfloor) < 0$ and $U(N) > 0$. U is strictly increasing in s for $s \in (\bar{s}, N)$, so define $\bar{\nu} \equiv \min\{s | U(s) > 0, s \in \mathbb{N}\}$. Because $U(\lfloor \bar{s} \rfloor) < 0$, $\bar{\nu} > \bar{s}$. Thus, if an entrepreneur believes all other entrepreneurs are going to innovate, and $\nu_t \ge \bar{\nu}$, it is optimal for her to innovate as well. However, if she does not believe other entrepreneurs to innovate, it is suboptimal for her to innovate as well. Also, there are multiple threshold equilibria – it is also an equilibrium for entrepreneurs to wait until $\bar{\nu} + k$ project-ideas exist, for $\bar{\nu} \le \bar{\nu} + k \le N$ where $k \in \mathbb{N}$. ■

Proof of Theorem 7. Suppose that there are currently $\tilde{\nu} \ge \bar{\nu}$ projects for entrepreneurs to initiate. If entrepreneurs innovate, they receive a payoff of $U(\tilde{\nu})$. If they wait for another project to arrive, they will receive $U(\tilde{\nu} + 1)$ when the next project arrives, which provides expected value $\sum_{t=1}^{\infty} \pi (1-\pi)^{t-1} \delta^t U(\tilde{v} + 1) = \frac{\pi\delta}{1-(1-\pi)\delta} U(\tilde{\nu} + 1)$. Thus, it is inefficient to wait when at $\tilde{\nu}$ if $\delta \le \frac{U(\tilde{\nu})}{\pi U(\tilde{\nu}+1) + (1-\pi) U(\tilde{\nu})}$, or equivalently, if $\pi \le \frac{U(\tilde{\nu})}{U(\tilde{\nu}+1) - U(\tilde{\nu})} \left(\delta^{-1} - 1\right)$ (this cutoff is well-defined because U is strictly increasing for $s \ge \bar{s}$ and $\tilde{\nu} \ge \bar{\nu} > \bar{s}$) Finally, if this is satisfied for all $\tilde{\nu} \ge \bar{\nu}$, if $\delta \le \min_{\tilde{\nu} \ge \bar{\nu}} \frac{U(\tilde{\nu})}{\pi U(\tilde{\nu}+1) + (1-\pi) U(\tilde{\nu})}$, or equivalently, $\pi \le \min_{\tilde{\nu} \ge \bar{\nu}} \frac{U(\tilde{\nu})}{U(\tilde{\nu}+1) - U(\tilde{\nu})} \left(\delta^{-1} - 1\right)$, it is inefficient to wait for another project idea for all $\tilde{\nu} \ge \bar{\nu}$. Because investors are indifferent, the efficient equilibrium is to invest immediately whenever $\bar{\nu}$ projects are available. ■

Proof of Corollary 3. For $s > \bar{s}$, we can express $U(s) = \delta^2 e^{\theta_H - \frac{N}{s}\alpha - \theta_M} y - c(y) - k$, so $U' > 0$, and $\bar{\nu} \equiv \min\{s | U(s) > 0, s \in \mathbb{N}\}$. Thus, anything that increases U decreases $\bar{\nu}$, and vice versa. $\frac{\partial U}{\partial N} = -\frac{\delta^2}{s} e^{\theta_H - \frac{N}{s}\alpha - \theta_M} y\alpha < 0$, so $\bar{\nu}$ is increasing in N. $\frac{\partial U}{\partial \alpha} = -\frac{N}{s} \delta^2 e^{\theta_H - \frac{N}{s}\alpha - \theta_M} y < 0$, so $\bar{\nu}$ is increasing in α. $\frac{\partial U}{\partial k} = -1$, so $\bar{\nu}$ is increasing in k. $\frac{\partial U}{\partial \delta} = 2\delta e^{\theta_H - \frac{N}{s}(\theta_H - \theta_T) - \theta_M} y > 0$, so $\bar{\nu}$ is decreasing in δ. ■

Proof of Theorem 8. Because $I < \psi$, investors always have interior beliefs. Similar to Theorem 2, if neither firm innovates, firm τ receives utility $\mathcal{U}_\tau^{U,N} = \frac{1}{2^{\frac{1}{\gamma}}} e^{(\theta_T - \theta_M)\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{2\gamma}} Z_{\tau'}^{\frac{1}{2\gamma}} \frac{2\gamma+1}{2\gamma+2}$, while, if either firm τ or firm τ' innovates, $\mathcal{U}_\tau^{U,S} = I^{\frac{1}{2\gamma}} \mathcal{U}_\tau^{U,N}$, and if both firms innovate, $\mathcal{U}_\tau^{U,B} = I^{\frac{1}{\gamma}} \mathcal{U}_\tau^{U,N}$. Thus, if firm τ' does not innovate, firm τ executes process innovation iff $\mathcal{U}_\tau^{U,S} - \kappa_\tau > \mathcal{U}_\tau^{U,N}$, or equivalently, iff $\kappa_\tau < \underline{\kappa}_\tau \equiv \left(I^{\frac{1}{2\gamma}} - 1\right) \mathcal{U}_\tau^{U,N}$. Similarly, if firm τ' innovates, firm τ executes process innovation iff $\mathcal{U}_\tau^{U,B} - \kappa_\tau > \mathcal{U}_\tau^{U,S}$, or equivalently, iff $\kappa_\tau < \bar{\kappa}_\tau \equiv I^{\frac{1}{2\gamma}} \left(I^{\frac{1}{2\gamma}} - 1\right) \mathcal{U}_\tau^{U,N}$. Because $I > 1$, $\underline{\kappa}_\tau < \bar{\kappa}_\tau$. ■

B Appendix: Demand Uncertainty

A key ingredient of our paper is that program (5) is a strictly convex programming problem which generates "interior beliefs" for well-diversified portfolios. In the main body of the paper, the possibility of such interior beliefs is a consequence of (strict) convexity of the relative entropy function $R(\cdot)$, which produces a strictly convex core beliefs set $\mathcal{M}$ (see Figure 1). Thus, no specific parametric restriction on the joint probability p is needed to generate our results. In this appendix, we present an alternative "micro-foundation" of our model where interior beliefs are the outcome of uncertainty about consumer demand. All results in our paper remain qualitatively the same in this specification (details are available on request).

Consider a simple extension of our three-dates model. There are three types of goods: type τ goods, $\tau \in \{A, B\}$, and the numeraire. There are two firms, each specializing in the production of goods of type τ. At $t = 0$, entrepreneurs decide whether to pay the discovery cost to innovate. If successful, at $t = 1$, each entrepreneur will select the optimal investment into the project, y_τ, which is financed by issuing equity to uncertainty-averse investors. The investment decision is made under uncertainty on the exact size of the demand for each product (as described below). At $t = 2$, consumer demand is revealed and production decisions of firms are made. If successful, entrepreneurs will be monopolists in their innovative good market. For tractability, we assume that entrepreneur τ has production costs $c_\tau(Q_\tau) = K_\tau Q_\tau$, and that the intermediate investment y_τ lowers, at a cost $\xi(y_\tau) = \frac{\kappa}{2} y_\tau^2$, the per-unit production cost: $K_\tau = K_0 - K_1 y_\tau$.

There are two types of consumers, type A and type B, with a total mass of 1. Consumers value both goods, as well as the numeraire, but each consumer values one good more that the other, which determines their type. The price of the numeraire is fixed to 1, while the price of type τ good, P_τ, is determined in equilibrium. For simplicity, we assume quadratic utility for each type of consumer. Thus

$$U^\tau(q_\tau^\tau, q_{\tau'}^\tau) = (D + \Delta)\, q_\tau^\tau - \frac{\beta}{2}(q_\tau^\tau)^2 + D q_{\tau'}^\tau - \frac{\beta}{2}(q_{\tau'}^\tau)^2 + w - P_\tau q_\tau^\tau - P_{\tau'} q_{\tau'}^\tau,$$

where D, Δ, and β are strictly positive parameters. For simplicity, we assume that w and D large enough so that consumers (in equilibrium) always consume a positive amount of all goods available in the market. It is easy to verify that the consumer τ's demand function for good τ is $q_\tau^\tau = \frac{1}{\beta}(D + \Delta - P_\tau)$, and for good τ' is $q_{\tau'}^\tau = \frac{1}{\beta}(D - P_{\tau'})$. Let $m_\tau \in [m_L, m_H]$ be the proportion of consumers of type τ, with $m_A + m_B = 1$. Market clearing condition for good τ requires that $m_\tau q_\tau^\tau + m_{\tau'} q_\tau^{\tau'} = Q_\tau$, where Q_τ is the output of a firm type τ. Thus, market clearing requires that

$$P_\tau(Q_\tau) = D + m_\tau \Delta - \beta Q_\tau,$$

and the price of type-τ goods is increasing in m_τ. Because producers know m_τ when making their production decisions Q_τ, they maximize

$$\pi_\tau(Q_\tau) = P_\tau(Q_\tau)\, Q_\tau - K_\tau Q_\tau,$$

which gives

$$Q_\tau = \frac{D + m_\tau \Delta - K_\tau}{2\beta}.$$

Letting $\Pi_\tau = \max_{Q_\tau} \pi(Q_\tau)$, we have that entrepreneur τ profits are

$$\Pi_\tau = \frac{[D + m_\tau \Delta - K_\tau]^2}{4\beta}.$$

This implies that, when both entrepreneurs are successful, investors beliefs are determined by solving:

$$\begin{aligned} \min_{\{m_A, m_B\}} \mathcal{U} \quad &\equiv \quad \omega_A \left[\frac{[D + m_A \Delta - K_A(y_A)]^2}{4\beta} - V_A \right] + \omega_B \left[\frac{[D + m_B \Delta - K_B(y_B)]^2}{4\beta} - V_B \right] + \omega_0 \\ s.t. \quad & \quad m_A + m_B = 1, \end{aligned}$$

which is a (strictly) convex programming problem, with the same qualitative properties as (5).

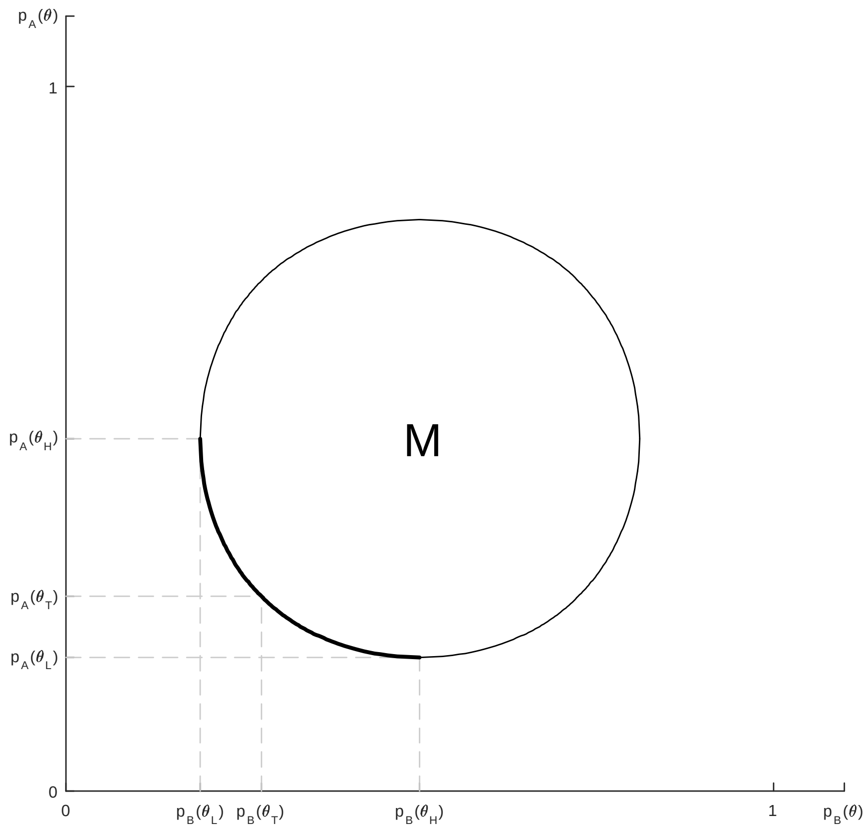

Figure 1: Core Belief Set. This figure represents the core belief set implied by the relative entropy criterion.

This figure shows the core of beliefs when the maximum relative entropy criteria is applied. That is, it shows the set of probability distributions $p = (p_A, p_B)$ that satisfy $\{p|R(p|\hat{p}) \leq \eta\}$ when $\hat{p}_A = \hat{p}_B = \frac{1}{2}$ and $\eta = \frac{3}{10}\ln 2$. If $p_B = \hat{p}_B = \frac{1}{2}$, the relative entropy criteria implies that $p_A \in [0.1893, 0.8107]$. In contrast, if $p_A = p_B$, then the relative entropy criterion implies that $p_A \in [0.2760, 0.7240]$.

This illustrates that an uncertainty-averse investor treats different uncertain lotteries as complements. The lower left boundary, which is darkened, represents the relevant portion of the core beliefs for investors with long positions in both risky assets. Specifically, if an uncertainty-averse investor invested in only one innovative project, he would value it as though it would succeed with probability 18.93%. In contrast, if he could invest equally in both, he would believe that each would succeed with probability 27.60%. Therefore, there will be strategic complementarities in innovation, because entrepreneurs rationally anticipate that they will face more positive sentiment if they both arrive at the same time.