Abstract

We show that ambiguity aversion generates endogenous disagreement between a firm’s insider and outside shareholders. Outsiders are well-diversified, but the insider holds only equity of the firm, leading to endogenous difference of opinion. We show that the strength of the corporate governance system depends on both firm characteristics and the composition of the outsiders’ overall portfolio. A strong governance system is optimal when the value of the firm’s assets in place, relative to the growth opportunity, is sufficiently small or is sufficiently large, suggesting a corporate governance life cycle. In addition, more diversified outsiders (such as generalist mutual funds) prefer stronger governance, while outsiders with a portfolio heavily invested in the same asset class as the firm (such as venture capitalists or private equity investors) are more willing to tolerate a weak governance system, where the portfolio companies’ insiders have more leeway in determining corporate policies. Finally, we find that ambiguity aversion introduces a direct link between the strength of the corporate governance system and firm transparency, whereby firms with weaker governance should also optimally be more opaque.

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1 Introduction

Disagreement adds spice to life. Diversity of opinions makes the decision making process in organizations important and creates the necessity of corporate governance systems. A corporate governance system represents a set of rules for the allocation of control and, thus, of decision making in an organization. Such allocation of decision making is critical when agents have different beliefs, and such difference of beliefs cannot be reconciled contractually.¹

In this paper we develop a model of corporate governance and allocation of control that is based on ambiguity aversion. We argue that ambiguity aversion generates endogenous differences of beliefs among agents in the organization. We show that the difference of beliefs is the outcome of agents’ heterogenous exposure to risk factors affecting their wealth (including, possibly, their human capital). Our stylized model is able to explain the optimality of several commonly observed features in corporate organizations, such as the separation of ownership and control, low level of transparency at weakly governed …rms, and why …rms typically do not report more information than required by the regulatory framework they must adhere to.

The persistence of organizational forms characterized by division of ownership and control is a classic puzzle in corporate …nance. Both Adam Smith² and Berle and Means³ passionately warned about the negative implications of the separation of ownership and control. In a deliberately provocative paper, Jensen (1989) advocates the “eclipse of the public company” and proposes the

¹Disagreements among shareholders, boards, and directors are common events in corporate life. Examples include the ousting of Carly Fiorina from Hewlett and Packard, allegedly “after she and directors disagreed on how to carry out Hewlett’s corporate strategy,” (see “Hewlett-Packard’s Chief Forced Out, Ending Rocky Tenure,” New York Times, February 9, 2005). Similarly, Christopher Galvin was ousted from his position of Chairman and Chief Executive of Motorola because of “disagreements over pace, strategy and progress,” especially concerning the company’s strategy on semiconductors, one of Motorola’s traditional strong products (see “For Motorola, Chief’s Ouster Seen Bringing Strategy Shift,” New York Times, September 22, 2003).
²“The directors of such companies, however, being the managers rather of other people’s money than of their own, it cannot well be expected that they should watch over it with the same anxious vigilance with which the partners in a private copartnery frequently watch over their own. Like the stewards of a rich man, they are apt to consider attention to small matters as not for their master’s honour, and very easily give themselves a dispensation from having it. Negligence and profusion, therefore, must always prevail, more or less, in the management of the a¤airs of such a company.” Book 5, Chapter 1, Part 3, Article 1, v1.107, Smith (1776)
³“In its new aspect the corporation is a means whereby the wealth of innumerable individuals has been concentrated into huge aggregates and whereby control over this wealth has been surrendered to a uni…ed direction. The power attendant upon such concentration has brought forth princes of industry, whose position in the community is yet to be defined. The surrender of control over their wealth by investors has e¤ectively broken the old property relationships and has raised the problem of de…ning these relationship anew. The direction of industry by persons other than those who have ventured their wealth has raised the question of the motive force back of such direction and the effective distribution of the returns from business enterprise.” p. 4, Berle & Means (1967)
“LBO association” (and the avoidance of the separation of ownership and control) as a superior organizational structure. Rappaport (1990), while acknowledging that separation of ownership and control was suboptimal, countered that LBOs may be optimal organization structures temporarily, and defended the “staying power” of public companies. Zingales (2000) argued that allocation of ownership is a relevant issue only under incomplete contracting.\footnote{Unlike Zingales (2000), our results are not due to incomplete contracting.} More recently, models in corporate finance examine the costs and benefits of governance systems to impose discipline on a firm’s insiders (see, for example, Harris and Raviv, 2008 and 2010, and extensive surveys in Shleifer and Vishny, 1997, and Becht, Bolton, and Roell, 2003).

This paper presents a novel source of disagreement and a motivation for the allocation of control that is grounded in decision theory: ambiguity aversion. We present a new tractable model for studying the impact of ambiguity aversion on investment, disclosure, and the allocation of corporate control. The economy is endowed with two classes of risky assets (in addition to the riskless assets) which have a different exposure to the source of uncertainty in the economy. Firm outsiders and the insider are heterogenous in that they hold a different portfolio of the two risky assets. Agents’ heterogeneous portfolios lead to endogenous differences of opinion, which is meaningful for the investment decisions of the firm.

In our model, we assume that the firm is endowed with one type of assets plus a growth opportunity. The growth opportunity can either be an expansion of its current assets in place (that is, a “focused” project), or an investment in the other type of asset (that is, a “diversifying” project). The insider is undiversified (for example, because of her human capital or the presence of an incentive contract) and holds only the firm’s equity, while outsiders are well-diversified. At the outset, outsiders choose the corporate governance system of their firm, which affects the investment decisions of the firm. Outsiders can either select a strong corporate governance system, where they retain the control their firm, or they can select a weak governance system, where investment decisions are delegated to the insider.

We show that the strength of the corporate governance system depends on both firm characteristics and the composition of the outsiders’ overall portfolio. The model rests on the following trade-off. The main benefit of a weak corporate governance system is that, by delegating decision-
making authority to the insider, the outsiders reduce their exposure to information revelation, which is harmful to ambiguity-averse agents. This effect always makes delegation (and corporate opacity) attractive to the outsiders. The second effect results from endogenous disagreement (induced by ambiguity aversion) between insiders and outsiders on the preferred level of investment, which is potentially harmful to outsiders. Delegation of decision making to insiders has the disadvantage of distorting the firm’s investment decisions with respect to the level of investment preferred by outsiders, but it has the advantage of allowing outsiders to avoid ex post underinvestment due to ambiguity aversion.

We find that a strong corporate governance system is optimal when the value of the firm’s assets in place, relative to the growth opportunity, is either sufficiently small or sufficiently large. This property suggests a corporate governance life cycle, whereby stronger governance is optimal for young and mature firms, while weaker governance is optimal for firms at the intermediate stage of their development. In addition, a weaker corporate governance system is also optimal (all else equal) for more productive firms, while a stronger corporate governance is optimal for less productive ones.

We also find that more diversified outsiders prefer stronger governance, while outsiders with a portfolio more heavily invested in the same asset class as the firm tolerate weaker governance systems. If outsiders are institutional investors – such as a mutual fund, or a venture capital or a private equity fund - this property implies that generalist funds should impose strong governance systems on their portfolio companies, while specialized funds are more willing to tolerate weak governance systems, where the portfolio companies’ insiders have more leeway in determining corporate policies.

We also find that ambiguity aversion introduces a direct link between the strength of the corporate governance system and firm transparency. Because of ambiguity aversion, outsiders prefer (all else equal) to have less, rather than more, information on the true state of the firm. This means that outsiders prefer the firm to be less transparent, unless they can benefit from the greater transparency by exerting control. Thus, firms with weaker governance should also optimally be more opaque.

Finally, outsiders and insiders have endogenous difference of beliefs and, thus, different desired
investments. Insiders are more pessimistic than the outsiders on the “focused” project, and more optimistic on the “diversifying” project, if the value of the assets in place (with respect to the growth opportunity) is sufficiently large, and more pessimistic than the outsiders otherwise. This means that an ambiguity-averse insider will overinvest in diversifying projects at value firms, but underinvest at growth firms.

This paper is related to several strands of literature. The first one is the emerging literature in corporate finance focusing on the effect of disagreement on firms’ corporate governance and financing strategy. Van den Steen (2004) and (2010) examine the impact of disagreement on incentives and organization design. Boot, Gopalan, and Thakor (2006) and Boot, Gopalan, and Thakor (2008) examine a firm’s choice between private and public ownership as a trade-off between managerial autonomy and liquidity. Boot and Thakor (2011) argue that potential disagreement with new investors causes the initial shareholders to prefer weak corporate governance (that is, “soft claims” that allows managerial discretion). Bayar, Chemmanur, and Liu (2011) examine the impact of heterogeneous beliefs on equity carve-outs. Harris and Raviv (1993) examine the impact of disagreement on the volume of trading and the reaction to public announcements.

A second related strand of literature focuses on the determinants of a firm’s disclosure policy, with and without disagreement. This includes Boot and Thakor (2001), Fishman and Hagerty (2003), Ferreira and Rezende (2007), and Kogan et al. (2009) among many others. Closer to our work, Thakor (2013) examines the optimal disclosure policy in the presence of disagreement.

A common feature of these papers is that disagreement is exogenously given: it derives from heterogeneous priors among a firm’s stakeholders (that is, it is a “primitive” of the model). In our paper, disagreement is endogenous – it derives from ambiguity aversion. Thus, our paper provides the heterogenous priors approach with an explicit decision-theoretic foundation that is based on ambiguity aversion. Making beliefs endogeneous allows us to link the source of the disagreement between insiders and outsiders to the fundamentals of the economy, namely endowments and firm characteristics. Our approach allows us to generate comparative statics results. For example, we derive endogenously a corporate governance life cycle, whereby the strength of the corporate governance system changes over time as firm characteristics evolve through its life-cycle. Similarly, our model generates predictions of the preferred corporate governance system depending on the
composition of the outsiders’ portfolio.

An exception to the literature mentioned above is Garlappi, Giammarino, and Lazrak (2013). They consider a model with ambiguity-averse agents, where disagreement is driven by (exogenous) differences in the agents’ “core beliefs” sets. In our paper, rather, agents share at the outset the same “core beliefs”; we derive agents’ disagreement endogenously as the outcome of differences in their exposure to uncertainty (due to heterogeneous portfolios), and we study its implications on corporate governance and corporate disclosure policy.


Our paper has two main limitations that can provide fruitful avenues for future research. First, we take the insider’s lack of diversification as given. Insiders may be undiversified for a number of important reasons. For example, lack of diversification may be due to the presence of firm-specific human capital or it be the outcome of an incentive contract due to moral hazard. A second limitation is that, in our model, agents are risk neutral. The presence of risk aversion would provide an additional source of disagreement between insiders and outsiders that would have to be addressed with optimal contracts (see Ross, 1973). A more general model that explicitly considers ambiguity aversion with either moral hazard or risk aversion, however, will have to incorporate the effects discussed in this paper as drivers of optimal contracts. For example, our model suggests that, because of ambiguity aversion, insiders should work under contracts that optimally generate some exposure to industry wide (or even economy wide) shocks, in order to reduce the extent of the disagreement with the outsiders.\textsuperscript{5}

\textsuperscript{5}See Gopalan, Milbourn, and Song (2010).
The paper is organized as follows. In Section 2 we introduce ambiguity aversion and outline its main features. In Section 3, we describe the basic model. In Section 4, we derive the paper’s main results. In Section 5, we present the paper’s empirical implications. Section 6 concludes. All proofs are in the Appendix.

2 Models of Ambiguity Aversion

Most economic models assume that all agents know the distribution of all possible outcomes. There is no distinction between the known-unknown and the unknown-unknown. However, the Ellsberg paradox\(^6\) shows that this assumption is not warranted. This introductory section briefly describes how various models have accounted for risk and ambiguity.

In traditional models, economic agents maximize their Subjective Expected Utility (SEU). Given a von-Neumann Morgenstern (vNM) utility function \(u\) and a wealth distribution \(\mu\), each player maximizes

\[
U^e = E_{\mu}[u(w)].
\]  

(1)

One limitation of the SEU approach is that it cannot account for aversion to “ambiguity”. In the SEU framework, economic agents merely average over the possible probabilities. Under ambiguity aversion, a player does not know the true prior, but only knows that the prior is from a certain set \(\mathcal{M}\).

A common way for modeling ambiguity aversion is minimum expected utility (MEU), promoted in Epstein and Schneider (2011). In this framework, economic agents maximize

\[
U^a = \min_{\mu \in \mathcal{M}} E_{\mu}[u(w)].
\]  

(2)

\(^6\)A good illustration of the Ellsberg paradox is actually from Keynes (1921). There are two urns. Urn K has 50 red balls and 50 blue balls. Urn U has 100 balls, but the subject is not told how many of them are red (all balls are either red or blue). The subject will be given $100 if the color of their choice is drawn, and the subject can choose which Urn is drawn from. Subjects typically prefer urn K, revealing aversion to ambiguity (this preference is shown to be strict if the subject receives $101 from selecting Urn U but $100 from Urn K being drawn). To see this, suppose the subject believes that the probability of drawing Blue is \(p\). If \(p < \frac{1}{2}\), the subject prefers to draw Red from Urn U. If \(p > \frac{1}{2}\), the subject prefers to draw Blue from Urn U. If \(p = \frac{1}{2}\), the subject is indifferent. Because subjects strictly prefer to draw from Urn K, such behavior cannot be consistent with a single prior on Urn U. This paradox provides the motivation for the use of multiple priors.
As shown in Gilboa and Schmeidler (1989), the MEU approach is a consequence of replacing the Sure-Thing Principle of Anscombe and Aumann (1963),\(^7\) that
\[
f \sim g \Rightarrow \alpha f + (1 - \alpha) g \sim f,
\]
with the Uncertainty Aversion Axiom on lotteries, which requires that for non-comonotonic\(^8\) lotteries \(f\) and \(g\), and \(\alpha \in (0, 1)\),
\[
f \sim g \Rightarrow \alpha f + (1 - \alpha) g \succ f.
\]
This assumption captures the intuition that economic agents prefer risk to ambiguity – they prefer known probabilities to unknown. Gilboa and Schmeidler (1989) shows that replacing the Sure-Thing Principle with the Uncertainty Aversion Axiom, but keeping the other Savage assumptions, results in MEU preferences. MEU has the intuitive feature that a player first calculates expected utility with respect to each prior, and then takes the worst-case scenario over all possible priors. In other words, the agent follows the maxim “Average over what you know, then worry about what you don’t know.”

Another approach, developed by Klibanoff, Marinacci, and Mukerji (2005), which we will refer to as KMM, assumes agents optimize
\[
U^{KMM} = E_{\nu} [\phi (E_\mu [u(w)])],
\]
where \(\nu\) is a prior over the set of priors. The advantage of this approach is that if \(\phi\) is smooth, then \(U^{KMM}\) is smooth as well, allowing the use of first order techniques. KMM shows that SEU is equivalent to linear \(\phi\), while ambiguity averse behavior is equivalent to concave \(\phi\), and ambiguity seeking behavior is equivalent to convex \(\phi\). In addition, MEU is a limiting case of concave \(\phi\) (let \(\phi(u) = -\frac{1}{\alpha} e^{-\alpha u}\) and take \(\alpha \to \infty\)).

In this paper, we will use the MEU approach with recursively defined utilities, as described in
\(^7\)Anscombe and Aumann (1963) is an extention of the Savage (1972) framework: the Anscombe and Aumann framework has both objective and subjective probabilities, while the Savage framework has only subjective probabilities.
\(^8\)Two lotteries are comonotonic if they are multiples of each other. Thus, loosely speaking, an ambiguity-averse agent is indifferent between two comonotonic lotteries only if they are basically the same lottery.
Epstein and Schnieder (2011). Formally, we will model sophisticated ambiguity-averse economic agents with consistent planning. In this setting agents are sophisticated in that they correctly anticipate their future ambiguity aversion. Consistent planning accounts for the fact that agents take into account how they will actually behave in the future.\footnote{Siniscalchi (2011) describes this framework as preferences over trees.} In the context of our model, there will be an initial contracting phase (when control is allocated) at $t = 0$. Information is revealed and an action is taken at $t = 1$. All payoffs are determined at $t = 2$. Players at the initial contracting phase, $t = 0$, will correctly anticipate behavior at the interim stage, $t = 1$. Our results are smooth (a.e.) because we explore a setting where we can apply a minimax theorem.

An important property of this class of models is that information harms an ambiguity averse agent if the agent does not use the information. This property is a direct consequence of the minimization operator, and can be stated as follows. Suppose that each prior, $\mu (\cdot)$, is a joint distribution of wealth, $w$, and a signal of wealth, $s \in S$. Further, following Epstein and Schneider (2011), let $\mu (\cdot; s)$ be the condition distribution of $w$ given $s$. Note, by the Law of Iterated Expectations, for all $\mu$.

$$U (\mu) = E_s E_{\mu(\cdot; s)} [u (w)].$$

We have the following:

\textbf{Lemma 1} \textit{Information harms an ambiguity averse agent if the agent does not use it:}

$$\min_{\mu \in \mathcal{M}} E_s E_{\mu(\cdot; s)} [u (w)] \geq E_s \min_{\mu(\cdot; s) \in \mathcal{M}_s} E_{\mu(\cdot; s)} [u (w)]. \quad (5)$$

The property (5) implies that information harms ambiguity averse agents because the minimization can be more fine-tuned when agents anticipate having more information.\footnote{Note that property (5) mirrors the well-known feature that a portfolio of options is worth more than an option on a portfolio and, thus, that writing a portfolio of options is more costly than writing an option on a portfolio. By similar intuition, more information harms an ambiguity-averse agent since the minimization process can be more fine-tuned.} A potential offsetting advantage of learning the signal $s$ may come in cases where the agent’s utility depends also on a specific action (possibly chosen by the agent) which affects the distribution of the agent’s final payoff, $w$. In this case, the agent may find it desirable to choose the action only after learning the realization of the signal $s$ so as to be able to condition the choice of the action to the observed
11 This property plays an important role in this paper.

3 The Model

We consider the problem of the allocation of control between the outsiders of a firm (the “shareholders”), denoted by $S$, and its insider (“the manager”), denoted by $M$. We assume that at the outset the firm outsiders own a fixed fraction $1 - \alpha$ of the firm equity, and the insider retains the residual fraction $\alpha$ for herself. For example, the insider may be an entrepreneur who, after founding the firm, has divested a fraction $1 - \alpha$ of its equity to outside investors to raise capital in earlier financing rounds. In turn, the outsiders could be private equity investors, such as VCs, or a group of dispersed shareholders. We assume that the outside investors behave as a single block and, for brevity, they will be referred to as the “outsiders.”

We study a simple two-periods model with three dates, $t \in \{0, 1, 2\}$. At the beginning of the first period, $t = 0$, the firm outsiders must choose the governance structure, $\delta$, of the firm. Specifically, the outsiders must decide whether to retain control of the firm’s decisions for themselves, denoted by $\delta = r$ (“retention”) or to delegate control to the firm’s insiders, the management, denoted by $\delta = d$ (“delegation”). Note that the outsiders can implement retention of control in a number of ways, for example, by having a management-independent board of directors that responds to them, rather than one dominated by the insiders. More generally, outsiders’ retention of control can be implemented by setting up a strong corporate governance system. Thus, we can interpret retention as a “strong” corporate governance system, and delegation as a “weak” corporate governance system. The outsiders allocate control to maximize their ex-ante utility, as described below.

We assume that the economy is endowed by three (classes of) assets: a riskless asset which will serve as our numeraire, and two types of risky assets: type-$A$ and type-$B$ assets. Type-$\tau$ assets, with $\tau \in \{A, B\}$, are risky in that they generate at the end of the second period, $t = 2$, a random payoff denominated in terms of the riskless asset. Specifically, a unit of type-$\tau$ asset produces at $t = 2$ a payoff $H$ (success) with probability $p_\tau$, and a payoff $L$ (failure) with probability $1 - p_\tau$. For
notational simplicity, we normalize these payoffs to $H = 1$ and $L = 0$.

In our most general model, we will assume that both outsiders and the insider are ambiguity averse, that is they are MEU agents. In particular, we assume that ambiguity-averse agents hold multiple prior beliefs on the success probability of the risky assets. Following Epstein and Schneider (2011), we model ambiguity aversion by assuming that the success probability of an asset of type $\tau$ depends on the value of an underlying parameter $\theta$, and is denoted by $p_\tau(\theta)$. Ambiguity averse agents treat the parameter $\theta$ as ambiguous, and believe that $\theta \in C \equiv \left[\hat{\theta}_0, \hat{\theta}_1\right] \subset [\theta_0, \theta_1]$, where $C$ represents the set of “core beliefs.” In contrast, a SEU agent has a unique prior on the success probability $p_\tau$ of the risky assets. For SEU agents the set of core beliefs $C$ is a singleton, which we denote as $C^e = \{\theta^e\}$. We will also assume that $\theta^e = \frac{1}{2}(\theta_0 + \theta_1) \in \left[\hat{\theta}_0, \hat{\theta}_1\right]$. As will become apparent below, this assumption ensures that an SEU agent has the same beliefs (defined below) as a well-diversified MEU agent.

The parameter $\theta$ affects the two assets in our economy differently. For analytical tractability, we assume that $p_A(\theta) = e^{\theta_0 - \hat{\theta}_1}$ and $p_B(\theta) = e^{\theta_0 - \theta}$. In this specification, increasing the value of the parameter $\theta$ increases the success probability of type $A$ assets and decreases the success probability of type $B$ assets. This means that a greater value of $\theta$ is “favorable” for asset $A$ and “unfavorable” for asset $B$. Also, for a given value of the parameter $\theta$, the probabilities distributions $p_\tau(\theta)$, $\tau \in \{A, B\}$, are independent.\(^{12}\)

We assume that at the beginning of the first period, $t = 0$, the firm is endowed with $V_A$ units of type-$A$ assets and $V_0$ units of the riskless asset. At the interim date, $t = 1$, the firm has access to a new investment opportunity. The type of investment opportunity which becomes available to the firm is random, and is not known at the outset to both outsiders and the insider. We assume that the investment opportunity can either be a project in the same type of assets currently owned by the firm, type-$A$ assets, or an investment in type-$B$ assets. We will denote these investment opportunities respectively as the focused and the diversified project.

The characteristics of the investment opportunity depend on the state of the world, $\omega_\tau$, realized at $t = 1$, with $\tau \in \{A, B\}$. Specifically, in state $\omega_\tau$ the firm can acquire $I_\tau$ units of type $\tau$ assets at the cost of $c(I_\tau)$ units of the riskless asset. We assume that the firm is not cash-constrained in

\(^{12}\)Our model can easily be extended to the case where, given $\theta$, the realization of the asset payoffs at the end of the period are correlated.
that it has a sufficient amount of riskless assets to be able to implement the desired investment $I_\tau$ in the risky asset $\tau \in \{A, B\}$.\footnote{We leave for future research the important question of raising capital under ambiguity aversion.} We also assume that $c(0) = 0$, $0 \leq c'(0) < e^{\theta_0 - \theta_1}$, $c'(I_\tau) > 0$ for $I_\tau > 0$, and $c''(I_\tau) > 0$.\footnote{This restriction on the cost function $c(\cdot)$ captures the notion that the investment project is characterized by diminishing marginal returns, and it ensures that a small positive investment is always optimal.} The cost function $c(\cdot)$ is the same for type-$A$ and type-$B$ projects.

To derive comparative statics, we will assume $c(I) = \frac{1}{Z(1+\gamma)} I^{1+\gamma}$ where $\gamma > 0$.\footnote{Our paper’s main results do not depend on this specification of the firm’s cost function.} The parameter $Z$ affects the cost of acquiring the risky assets and will be interpreted as characterizing the firm’s “productivity.” For simplicity, we assume that both agents believe that these states are equally likely, i.e., $\Pr\{\omega_\tau\} = 1/2$. We also assume that there is no ambiguity on the states $\omega_\tau$: both the insider and the outsiders have a single common prior on the probability of the intermediate states $\omega_\tau$.

The choice of allocation of control is important because the party who is allocated control will choose the level of investment $I_\tau$ to maximize his/her expected utility. The choice of the investment level, $I_\tau$, is made by the party in control after observation of the realization of $\omega_\tau$, that is after the nature of the investment project available to the firm becomes known (i.e., whether the firm has access to a type-$A$ or type-$B$ investment project). The realization of the state of the world $\omega_\tau$ is always observable by the insiders, and it is observable by the outsiders if they are in control at the time the investment is made.

Finally, we assume that at the end of the period, $t = 2$, agent consume their holdings of the riskless asset. We assume that agents are endowed with vNM utility functions, $u(\cdot)$, which are linear in the riskless asset. Following Epstein and Schneider (2011), this means that they are risk-neutral MEU or SEU agents. The model timeline is summarized in Figure 1.

### 3.1 Endogenous Beliefs

A critical feature of our model is that ambiguity aversion endogenously generates differences of opinion in an economy populated by heterogeneous agents, even when agents have identical core prior beliefs. This happens because, within a MEU framework, an agent’s beliefs are determined by the solution to that agent’s expected utility minimization problem. This means that agent heterogeneity (for example, in their endowments) will generate different solutions to the minimization
problem and, thus, different beliefs. Therefore, differences of beliefs will emerge endogenously in our model. As we will show in Section 4, these differences are meaningful and impact the firm’s investment decision. Thus, the ex-ante allocation of control is meaningful due to endogenous belief heterogeneity.

Consider an agent endowed with a portfolio \( \Pi \equiv \{ \bar{w}_A, \bar{w}_B, \bar{w}_0 \} \), where \( \bar{w}_\tau, \tau \in \{ A, B \} \) represents the overall units of the risky asset type \( \tau \) owned by the agent, and \( \bar{w}_0 \) represents the units of riskless asset in the agent’s portfolio. For a given value of the parameter \( \theta \), this portfolio provides the agent with an expected utility of

\[
E \left[ u (\bar{w}_A, \bar{w}_B, \bar{w}_0) ; \theta \right] = e^{\theta - \theta_1} \bar{w}_A + e^{\theta_0} \bar{w}_B + \bar{w}_0
\]  

(6)

An ambiguity-averse agent fears the worst possible outcome of \( \theta \), and has a MEU given by:

\[
U (\bar{w}_A, \bar{w}_B, \bar{w}_0) \equiv \min_{\theta \in \mathcal{C}} E \left[ u (\bar{w}_A, \bar{w}_B, \bar{w}_0) ; \theta \right]
\]  

(7)

Thus, an ambiguity-averse agent’s beliefs, denoted as \( \theta^a \), are determined by the value that minimizes the agent’s expected utility, that is by

\[
\theta^a (\Pi) \equiv \arg \min_{\theta \in \mathcal{C}} E \left[ u (\bar{w}_A, \bar{w}_B, \bar{w}_0) ; \theta \right].
\]  

(8)
It is clear from (6) and (8) that the agent’s belief \( \theta^a \) depends on the amount of asset \( A \) and asset \( B \) in his overall portfolio \( \Pi \). The solution to problem (7) is characterized in the following lemma.

**Lemma 2** Let  
\[
\tilde{\theta}^a(\Pi) \equiv \frac{1}{2} (\theta_0 + \theta_1) + \frac{1}{2} \ln \frac{\bar{w}_B}{\bar{w}_A}.
\]  

The beliefs held by an ambiguity averse agent, \( \theta^a(\Pi) \), are given by  
\[
\theta^a(\Pi) = \begin{cases} 
\hat{\theta}_0 & \tilde{\theta}^a(\Pi) \leq \hat{\theta}_0 \\
\tilde{\theta}^a(\Pi) & \tilde{\theta}^a(\Pi) \in (\hat{\theta}_0, \hat{\theta}_1) \\
\hat{\theta}_1 & \tilde{\theta}^a(\Pi) \geq \hat{\theta}_1 
\end{cases}.
\]  

We will refer to \( \tilde{\theta}^a(\Pi) \) as the “portfolio-distorted” beliefs. We will say that the agent has interior beliefs when \( \theta^a \in (\hat{\theta}_0, \hat{\theta}_1) \). In this case, the agent’s beliefs are equal to the portfolio-distorted beliefs, that is \( \theta^a(\Pi) = \tilde{\theta}^a(\Pi) \). It is important to note that the beliefs of an ambiguity-averse agent depend essentially on the composition of his portfolio \( \Pi \), that is, on his endowment. In particular, in our specification, the portfolio-distorted beliefs \( \tilde{\theta}^a(\Pi) \) differ from the SEU beliefs \( \theta^e = \frac{1}{2} (\theta_0 + \theta_1) \) by a term that depends on the degree of heterogeneity of the agent’s endowment, \( \bar{w}_B/\bar{w}_A \). The following corollary can be immediately verified.

**Corollary 1** Holding type B assets constant, when the agent has a larger position in type A assets, the agent is more pessimistic about type A assets and more optimistic about type B assets. Furthermore, the agent has scale-invariant beliefs that depend only on the ratio \( \bar{w}_B/\bar{w}_A \).

Corollary 1 shows that when an agent has relatively greater endowment of asset \( A \), the agent will be relatively more concerned about the priors that are less favorable to that asset. Thus, the agent will shift his beliefs toward the lower end of the core beliefs set \( C \), and he will give more weight to the states of nature that are less favorable for asset \( A \). In other words, the agent will be more “pessimistic” about the future value of (or the return on) that asset. Correspondingly, the agent will become more “optimistic” with respect to the other asset, asset \( B \). Note also that portfolio-distorted beliefs \( \tilde{\theta}^a \) remain the same when the ratio of the endowment in the two types of assets is constant, that is, \( \tilde{\theta}^a \) is scale invariant (formally, \( \tilde{\theta}^a \) is homogeneous of degree zero in the
endowments $\tilde{w}_B$ and $\tilde{w}_A$). Note that, from (9), the assumption that $\theta^c = \frac{1}{2} (\theta_0 + \theta_1)$ guarantees that the beliefs held by a SEU agent coincide with the beliefs of a well-diversified MEU agent, for whom $\tilde{w}_A = \tilde{w}_B$.

An important property of the MEU approach is that even if agents are endowed with vNM utility functions that are linear in wealth, they display decreasing marginal utility in the value of any single asset in their portfolio, when the amount of all other assets remain constant. This happens because of the negative impact on an agent’s beliefs that is due to the increase in the endowment of any specific asset, when the endowment of all other assets remains the same.

**Lemma 3** Holding the position in the other asset type constant, an agent has decreasing marginal utility from a particular type of asset, $d^2U_{d(w)} \leq 0$, for $\tau \in \{A, B\}$. For interior beliefs, this inequality is strict.

This property plays an important role in the investment choice problem that we consider below.

### 4 Ambiguity and Allocation of Control

We can now consider the allocation of control decision, $\delta \in \{r, d\}$, faced by the outsiders at the beginning of the game. The outsiders can retain control over the selection of level of investment, $\delta = r$, or they can delegate control to the insider, $\delta = d$. In this section, we will show that the optimal level of investment chosen by an ambiguity-averse agent depends on the composition of her portfolio. This makes the allocation of control critical.

As a reference point, we start our discussion by considering the benchmark case where there is no separation of ownership and control: the agent making the decision has full ownership of the firm. We assume that, in addition to the full ownership of the firm, the owner is also endowed with other resources (outside the firm) denoted by $\{w_A, w_B, w_0\}$. Thus, in state $\omega_A$, that is for investment projects involving type-A assets, the overall owner’s portfolio after the investment is made becomes $\Pi(I_A, 0) \equiv \{w_A + V_A + I_A, w_B, w_0 + V_0 - c(I_A)\}$. Similarly, in state $\omega_B$, that is for investment projects involving type-B assets, after the investment is made the owner’s portfolio becomes $\Pi(0, I_B) \equiv \{w_A + V_A, w_B + I_B, w_0 + V_0 - c(I_B)\}$.
In state $\omega_A$, an ambiguity-averse owner chooses at $t = 1$ the optimal investment in a type-$A$ project by maximizing the minimum expected utility function, $U_1$, given by

$$U_1(\Pi(A,0)) \equiv \min_{\theta \in C : \theta \in [\theta_0, \theta_1]} \mathbb{E}[u(\Pi(A,0)) ; \theta], \quad (11)$$

where

$$\mathbb{E}[u(\Pi(A,0)) ; \theta] = e^{\theta - \theta_1}(w_A + V_A + I_A) + e^{\theta_0 - \theta}w_B + w_0 + V_0 - c(I_A). \quad (12)$$

Note that the owner’s beliefs, $\theta^a(\Pi(A,0))$, are given by the solution to the minimum expected utility problem (11) as

$$\theta^a(\Pi(A,0)) = \arg \min_{\theta \in C : \theta \in [\theta_0, \theta_1]} \{\mathbb{E}[u(\Pi(A,0)) ; \theta]\}. \quad (13)$$

By Lemma 2, the insider’s beliefs, $\theta^a(\Pi(A,0))$, are given by

$$\theta^a(\Pi(A,0)) = \begin{cases} \hat{\theta}_0 & \hat{\theta}^a(\Pi(A,0)) \leq \hat{\theta}_0 \\ \hat{\theta}^a(\Pi(A,0)) & \hat{\theta}^a(\Pi(A,0)) \in (\hat{\theta}_0, \hat{\theta}_1) \\ \hat{\theta}_1 & \hat{\theta}^a(\Pi(A,0)) \geq \hat{\theta}_1 \end{cases}. \quad (14)$$

where the portfolio-distorted beliefs are

$$\hat{\theta}^a(\Pi(A,0)) = \theta^e + \frac{1}{2} \ln \left[ \frac{w_B}{w_A + V_A + I_A} \right], \quad (15)$$

since $\theta^e \equiv \frac{1}{2}(\theta_0 + \theta_1)$. The corresponding optimal investment level $I^a_A$ is determined as

$$I^a_A = \arg \max_{I_A} U_1(\Pi^a(A,0)). \quad (16)$$

where $U_1(\Pi^a(A,0)) = \mathbb{E}[u(\Pi^a(A,0)) ; \theta^a(\Pi^a(A,0))]$. Similarly, in state $\omega_B$, the availability of an investment project involving type-$B$ assets leads the owner-manager to an investment level of $I^a_B$, where the portfolio-distorted beliefs are now given by

$$\hat{\theta}^a(\Pi(0, I_B)) = \theta^e + \frac{1}{2} \ln \left[ \frac{w_B + I_B}{w_A + V_A} \right]. \quad (17)$$
Note that when deciding the optimal level of investment, $I^a$, the MEU owner is sophisticated in that he anticipates the impact of his investment choice on his own beliefs, $\theta^a$. This implies that the agent has “no regret” in the sense that the agent will not change beliefs after the investment $I^a$ is made (and, thus, it remains optimal after it is implemented). It also means that the optimal level of investment will be determined by two effects. The first effect is the traditional “marginal cost” effect that is due to the convexity of the cost function, $c(I^a)$. The second effect is a “pessimism” effect due to ambiguity aversion: by increasing investment in the type-$\tau$ asset, from (14) and (16) the owner will change his beliefs in a way that he will become more pessimistic about that asset. Thus, the owner will limit the investment in those assets. These considerations lead to the following theorem.

**Theorem 1** The optimal levels of investment $\{I^a_A, I^a_B\}$ depend on the owner’s pre-existing portfolio. An increase of $w_\tau$ will lead to a decrease of $I^a_\tau$ and an increase of $I^{S,a}_\tau^a$, $\tau' \neq \tau$.

Theorem 1 implies that, under ambiguity aversion, the optimal investment in a project depends on the decision-maker’s overall endowment. Specifically, the optimal investment in any given project is a decreasing function of the owner’s initial endowment in the same asset, and an increasing function of the endowment of the other asset. This property makes investment projects effectively complementary. We will denote this complementarity as the “spillover effect” of ambiguity. This spillover effect happens because, from Corollary 1, we know that an increase in the endowment in asset $A$ makes an ambiguity-averse agent relatively more pessimistic about asset $A$ and more optimistic about asset $B$, resulting in a decrease in $I^a_A$ and an increase in $I^a_B$. Symmetric results hold for an increase of type-$B$ assets. Thus, ambiguity-aversion creates a spillover effect among desired investment levels which is driven by beliefs.

Theorem 1 also has the interesting implication that, under ambiguity aversion, beliefs depend on the composition of the owner’s overall portfolio. Thus, ambiguity aversion generates a possible disagreement among economic agents on the firm’s optimal investment policy. The disagreement is induced by the difference of beliefs that characterizes ambiguity-averse agents, and it makes the allocation of control within a firm important.

For the remainder of the paper, we consider the case where the firm outsiders own a fraction
(1 - α) of the firm, in addition to an endowment of resources external to the firm, \(\{w_A, w_B, w_0\}\), while the remaining fraction \(α\) is owned by the firm insider. Thus, the outsiders’ initial overall portfolio is now given by \(Π^S \equiv \{w_A + (1 - α)V_A, w_B, w_0 + (1 - α)V_0\}\). Furthermore, we will discuss the special case in which the outsiders are truly diversified in that their overall endowment of assets of type-\(A\) and type-\(B\) are the same; in this case \(w_B = w_A + (1 - α)V_A\). In contrast, we assume that the insider is not well diversified: her entire wealth consists of the fraction \(α\) of the firm’s stock. This assumption captures the notion that the insider’s wealth is intimately tied to the firm, possibly because of her undiversified human capital. Thus, the insider’s portfolio is given by \(Π^M \equiv \{αV_A, 0, αV_0\}\).16

We now study four possible scenarios in which the insider and outsiders can, in turn, be MEU or SEU maximizers, and we solve for the optimal allocation of control in each case.

4.1 Expected Utility Outsiders, Expected Utility Insider

As a benchmark, we begin with the simplest (and least interesting) case in which both parties are SEU maximizers and share a common belief \(C = \{θ^e\}\), with \(θ^e = \frac{1}{2}(θ_0 + θ_1)\). In this case, both parties chooses the same investment levels for either a focused or a diversifying project and, thus, the allocation of control is irrelevant. In addition, as we will show in Theorem 4 of Section 4.3, the assumption that \(θ^e \equiv \frac{1}{2}(θ_0 + θ_1)\) and that the outsiders’ portfolio is well diversified together imply that the investment preferred ex-ante by ambiguity averse outsiders is equal to the optimal level of investment for a SEU agent.

A focused project allows the firm to make an investment in type-\(A\) assets. If the firm invests \(I_A\), the outsiders’ portfolio is \(Π^S (I_A, 0) \equiv \{w_A + (1 - α)[V_A + I_A], w_B, (1 - α)[V_0 - c(I_A)]\}\), while the insider’s portfolio is \(Π^M (I_A, 0) \equiv \{α[V_A + I_A], 0, α[V_0 - c(I_A)]\}\). Thus, with an investment level of \(I_A\), the outsiders’ expected utility is

\[
E[u(Π^S (I_A, 0)) ; θ^e] = e^{θ_0 - θ_1} [w_A + (1 - α)[V_A + I_A]] + e^{θ_0 - θ_1} w_B + (1 - α) [V_0 - c(I_A)],
\]

16In addition, the insider may operate under an incentive contract (not modeled here) that exposes her more heavily to the firm’s assets. All we need for our results to follow, however, is a mismatch between the asset composition of the firm and its shareholders’ overall portfolio.
while the insider’s expected utility is
\[
\mathbb{E} \left[ u \left( \Pi^M (IA, 0) ; \theta^e \right) \right] = e^{\theta^e - \theta_1} \alpha [VA + IA] + \alpha [V_0 - c (IA)].
\]

Similarly, a diversified project allows the firm to invest in type-B assets. If the firm invests \( I_B \), the outsiders’ portfolio is \( \Pi^S (0, I_B) = \{ w_A + (1 - \alpha) VA, w_B + (1 - \alpha) IB, (1 - \alpha) [V_0 - c (IB)] \} \), while the insider’s portfolio is \( \Pi^M (0, I_B) = \{ \alpha VA, \alpha IB, \alpha [V_0 - c (IB)] \} \). Thus, with an investment level of \( I_B \), the outsiders’ expected utility is
\[
\mathbb{E} \left[ u \left( \Pi^S (0, I_B) ; \theta^e \right) \right] = e^{\theta^e - \theta_1} [w_A + (1 - \alpha) VA] + e^{\theta_0 - \theta^e} [w_B + (1 - \alpha) IB] + (1 - \alpha) [V_0 - c (IB)]
\]
while the insider’s expected utility is
\[
\mathbb{E} \left[ u \left( \Pi^M (0, I_B) ; \theta^e \right) \right] = e^{\theta^e - \theta_1} \alpha VA + e^{\theta_0 - \theta^e} \alpha IB + \alpha [V_0 - c (IB)].
\]

**Theorem 2** In the absence of ambiguity aversion, the insider and the outsiders choose the same investment level \( I_e = I^e \) for both projects, determined by
\[
c' (I_e) = e^{\frac{1}{2} (\theta_0 - \theta_1)}.
\]
Thus, the initial allocation of control is irrelevant.

When neither party is ambiguity averse, both the insider and outsiders share the same beliefs, \( \theta^e \), and they agree on the optimal level of investment, \( I^e \), for both the focused and the diversified project. Therefore, in the absence of ambiguity aversion, control rights are irrelevant.\(^{17}\)

### 4.2 Expected Utility Outsiders, Ambiguity Averse Insider

We consider now the case in which the outsiders are SEU maximizers, while the insider is a MEU agent. Because the outsiders are SEU agents, they choose an investment level equal to \( I^e \) if they

\(^{17}\)Note that Theorem 2 assumes a common core of beliefs. Since, under Subjective Expected Utility, the core of beliefs is a singleton, this means that Theorem 2 effectively assumes that agents have common beliefs. With exogenous difference of opinion, outsiders believe that the insider will make an investment decision they perceived as inefficient, and they will always retain control.
have control (as in Theorem 2). The MEU insider, however, behaves differently.

If the firm has a focused project, that is in state $\omega_A$, by investing $I_A$ the insider obtains a portfolio $\Pi^M (I_A, 0) = \{ \alpha (V_A + I_A), 0, \alpha [V_0 - c (I_A)] \}$. Given the portfolio $\Pi^M (I_A, 0)$, at $t = 1$ the insider’s minimum expected utility is

$$U^M_1 (\Pi^M (I_A, 0)) \equiv \min_{\theta \in C} \mathbb{E} [u (\Pi^M (I_A, 0)) ; \theta] ,$$

where $\mathbb{E} [u (\Pi^M (I_A, 0)) ; \theta] = e^{\theta - \theta_1} \alpha (V_A + I_A) + \alpha (V_0 - c (I_A))$. Thus, under ambiguity aversion the insider’s beliefs, $\theta^{M,a} (\Pi^M (I_A, 0))$, are determined by minimization of her expected utility, that is

$$\theta^{M,a} (\Pi^M (I_A, 0)) = \arg \min_{\theta \in C} \left\{ e^{\theta - \theta_1} \alpha (V_A + I_A) + \alpha (V_0 - c (I_A)) \right\} .$$

Because the insider holds only type $A$ assets, the beliefs held by the ambiguity-averse insider, $\theta^{M,a} (\Pi^M (I_A, 0))$, are at the lower bound of $C$, the set of core beliefs:

$$\theta^{M,a} (\Pi^M (I_A, 0)) = \hat{\theta}_0 . \tag{17}$$

Since the insider’s portfolio is not well diversified, the insider’s beliefs give maximum weight to the states that are least favorable to the only risky asset in which they have a long position, asset $A$. Thus, they will be concentrated at the lower bound of the core-beliefs set, $\hat{\theta}_0$. Given the insider’s beliefs, $\theta^{M,a} (\Pi^M (I_A, 0)) = \hat{\theta}_0$, the optimal investment $I^{M,a}_A$ is determined by maximizing the insider’s minimum expected utility

$$I^{M,a}_A = \arg \max_{I_A} U^M_1 (\Pi^M (I_A, 0)) , \tag{18}$$

where the insider’s MEU is equal to $U^M_1 (\Pi^M (I_A, 0)) = \mathbb{E} [u (\Pi^M (I_A, 0)) ; \hat{\theta}_0]$. The optimal level of investment, $I^{M,a}_A$, is set by the insider under the “worst-case scenario” belief that the parameter $\theta$ is at the lowest possible level $\hat{\theta}_0$. This property makes the insider very conservative when making focused investments.

Similarly, if the firm has a diversifying project, that is in state $\omega_B$, by investing $I_B$ the insider
obtains a portfolio $\Pi^M (0, I_B) = \{ \alpha V_A, \alpha I_B, \alpha (V_0 - c(I_B)) \}$. Thus, the insider’s minimum expected utility is

$$U^M_1 (\Pi^M (0, I_B)) = \min_\theta \mathbb{E} [u (\Pi^M (0, I_B)) ; \theta] ,$$

where $\mathbb{E} [u (\Pi^M (0, I_B)) ; \theta] = e^{\theta - \theta_1} \alpha V_A + e^{\theta_0 - \theta} \alpha I_B + \alpha [V_0 - c(I_B)]$. Thus, the insider’s beliefs, $\theta^{M,a} (\Pi^M (0, I_B))$, are determined by minimization of her expected utility, that is

$$\theta^{M,a} (\Pi^M (0, I_B)) = \arg \min_{\theta \in C} \left\{ e^{\theta - \theta_1} \alpha V_A + e^{\theta_0 - \theta} \alpha I_B + \alpha [V_0 - c(I_B)] \right\} .$$

By Lemma 2, the beliefs held by an ambiguity-averse insider, $\theta^{M,a} (\Pi^M (0, I_B))$, are given by

$$\theta^{M,a} (\Pi^M (0, I_B)) = \begin{cases} \hat{\theta}_0 & \hat{\theta}^{M,a} (\Pi^M (0, I_B)) \leq \hat{\theta}_0 \\ \tilde{\theta}^{M,a} (\Pi^M (0, I_B)) & \hat{\theta}^{M,a} (\Pi^M (0, I_B)) \in (\hat{\theta}_0, \hat{\theta}_1) \\ \tilde{\theta}_1 & \hat{\theta}^{M,a} (\Pi^M (0, I_B)) \geq \tilde{\theta}_1 \end{cases} . \quad (19)$$

where the insider’s portfolio-distorted beliefs, $\tilde{\theta}^{M,a} (\Pi^M (0, I_B))$, are given by

$$\tilde{\theta}^{M,a} (\Pi^M (0, I_B)) = \theta^c + \frac{1}{2} \ln \left( \frac{I_B}{V_A} \right) . \quad (20)$$

Given the insider’s beliefs, $\theta^{M,a} (\Pi^M (0, I_B))$, the optimal investment $I^{M,a}_B$ chosen by the insider is determined by maximizing the insider’s minimum expected utility,

$$I^{M,a}_B = \arg \max_{I_B} U^M_1 (\Pi^M (0, I_B)) , \quad (21)$$

where the insider’s MEU is equal to $U^M_1 (\Pi^M (0, I_B)) = \mathbb{E} [u (\Pi^M (0, I_B)) ; \theta^{M,a} (\Pi^M (0, I_B))]$. The optimal investment policy of a MEU insider is characterized in the following.

**Theorem 3** If given control, the ambiguity-averse insider underinvests in focused projects relative to the investment desired by the SEU outsiders. Her investment in diversifying projects depends on firm characteristics, and is an increasing function of the value of assets in place, $V_A$: if assets in place are sufficiently large, $V_A > I^c$, the insider overinvests in diversifying projects; otherwise, if $V_A < I^c$, she underinvests in diversifying projects. Thus, SEU outsiders will not delegate control
to an ambiguity-averse insider.

Because the insider holds only type-A assets, a priori, she places a lower value on type-A assets than an SEU investor. Thus, the insider will always underinvest in focused projects (i.e., in type-A assets) relative to the investment that is optimal for SEU outsiders, $I^e$. This means that the extent of underinvestment in the focused project becomes more severe when the lower bound of the core belief set, $C \equiv [\hat{\theta}_0, \hat{\theta}_1]$, is smaller, that is when the worst case scenario $\hat{\theta}_0$ becomes “even worse.”

The level of investment in type-B projects depends on the size of the assets in place, $V_A$, relative to the size of type-B assets that firm will have after the investment is made. In particular, when the firm has a sufficiently large endowment of assets in place, that is, if $V_A > I^e$, the insider will find it desirable to invest relatively more in the diversifying project, when it is available, than SEU outsiders, leading to overinvestment. In contrast, if the size of assets is relatively small, $V_A \leq I^e$, the insider prefers to limit her exposure to type-B assets, and she will underinvest. In either case, the insider’s investment policy will differ from the one preferred by the SEU outsiders, who will always retain control.

**Corollary 2** Outsiders’ loss of welfare from delegating control to the insider is an inverted U-shaped function of $V_A$.

From Theorem 3 we know that it is never optimal to grant control to the insider. This implies that if control is delegated to the insider, it will always have a negative impact on firm value. In addition, the impact is an inverted U-shaped function of $V_A$. This means that the loss of value due to delegation of decision making is greater at the extreme cases, either for very young firms where investment is considerably larger that assets in place, $V_A < I^e$, or for mature firm, where investment is substantially smaller that assets in place, $V_A > I^e$. The intuition for this is simple: the insider invests in focused projects according to the worst-case scenario, $\theta^M = \hat{\theta}_0$, so the negative impact on firm value from a focused project is independent of $V_A$. However, the investment in the diversifying project, $I_B^{M,a}$, is increasing in $V_A$. When $V_A = I^e$, $I_B^{M,a} = I^e$, so the insider chooses the diversifying investment optimally from the outsider’s perspective. Any departure from $V_A = I^e$ results in a greater loss from entrenchment.

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18 In general, distortions become (weakly) larger as the measure of the core beliefs become more “dispersed,” that is, as the measure of set $C$ increases.
4.3 Ambiguity Averse Outsiders, Expected Utility Insider

We now turn to the case where the outsiders are ambiguity averse while the insider is a SEU agent. We will show that, in this case, the ambiguity-averse outsiders find it optimal to delegate decision authority to the SEU insider. There are two reasons why ambiguity-averse outsiders prefer to grant control to an SEU insider. First, well-diversified ambiguity-averse outsiders will ex-post underinvest relative to what they would have wanted to invest ex-ante. This effect is due to the impact of the arrival of the new project on the outsiders’ ex-post beliefs. Second, ambiguity-averse outsiders would prefer not to learn the realization at $t = 1$ of the state of the world $\omega_T$, that is, to learn the kind of projects that becomes available to the firm in the intermediate date. This effect is due to the harmful effect of the arrival of new information on ambiguity-averse agents described in Lemma 1. Thus, delegation of decision making to the insider is ex-ante desirable to the outsiders.

If outsiders maintain control, the optimal levels of investment $I_T$ in state $\omega_T$ is determined in a way similar to Theorem 1. With a focused project, that is, in state $\omega_A$, an investment level of $I_A$ gives the outsiders the portfolio $\Pi^S(I_A, 0)$. Thus, the beliefs held by ambiguity-averse outsiders, $\theta^{S, a}(\Pi^S(I_A, 0))$, are given by (13) where the portfolio-distorted beliefs, $\theta^{S, a}(\Pi^S(I_A, 0))$, are equal to

$$\theta^{S, a}(\Pi^S(I_A, 0)) = \theta^c + \frac{1}{2} \ln \left[ \frac{w_B}{w_A + (1 - \alpha)(V_A + I_A)} \right].$$

The optimal investment $I_A^{S, a}$ is determined by maximizing the outsiders’ minimum expected utility

$$I_A^{S, a} = \arg \max_{I_A} U_1^{S}(\Pi^S(I_A, 0)), \quad (23)$$

where the outsiders’ MEU is equal to

$$U_1^{S}(\Pi^S(I_A, 0)) = \mathbb{E} \left[ u(\Pi^S(I_A, 0)); \theta^{S, a}(\Pi^S(I_A, 0)) \right], \quad (24)$$

where $\mathbb{E} \left[ u(\Pi^S(I_A, 0)); \theta \right] = e^{\theta - \theta_1} [w_A + (1 - \alpha)[V_A + I_A]] + e^{\theta_0 - \theta} w_B + (1 - \alpha)[V_0 - c(I_A)]$. The optimal level of investment for the ambiguity-averse outsiders is determined by the combination of the “marginal cost” and the “pessimism” effects we discussed above. Note that the impact of the “pessimism” effect depends on the outsiders’ overall portfolio. Specifically, when the outsiders
are well-diversified (that is, \( w_B = w_A + (1 - \alpha) V_A \)), it follows that \( w_A + (1 - \alpha) [V_A + I_A] > w_B \) for \( I_A > 0 \). This means that \( \tilde{\theta}^{S,a}(\Pi^S(I_A, 0)) < \theta^e \) and, thus, that the ambiguity-averse outsiders are pessimistic ex-post on type-A assets relative to an SEU agent. This implies that \( I_A^{S,a} < I_A^e \), or equivalently, outsiders underinvest in focused projects.

Similarly, when the firm has a diversifying project, that is in state \( \omega_B \), an investment level of \( I_B \) gives the outsiders the portfolio \( \Pi^S(0, I_B) \). The outsiders’ beliefs, \( \tilde{\theta}^{S,a}(\Pi^S(0, I_B)) \), are given by (13) where the portfolio-distorted beliefs, \( \tilde{\theta}^{S,a}(\Pi^S(0, I_B)) \), now are equal to

\[
\tilde{\theta}^{S,a}(\Pi^S(0, I_B)) = \theta^e + \frac{1}{2} \ln \left[ \frac{w_B + (1 - \alpha) I_B}{w_A + (1 - \alpha) V_A} \right].
\] (25)

Thus the optimal investment level, \( I_B^{S,a} \), is chosen by the outsiders by maximizing their minimum expected utility

\[
I_B^{S,a} = \arg\max_{I_B} U_1^S(\Pi^S(0, I_B)),
\] (26)

where the outsiders’ MEU is now equal to

\[
U_1^S(\Pi^S(0, I_B)) = \mathbb{E}[u(\Pi^S(0, I_B)); \tilde{\theta}^{S,a}(\Pi^S(0, I_B))].
\] (27)

Note \( \mathbb{E}[u(\Pi^S(0, I_B)); \theta] = e^{\theta - \theta_1} [w_A + (1 - \alpha) V_A] + e^{\theta_0 - \theta} [w_B + (1 - \alpha) I_B] + (1 - \alpha) [V_0 - c(I_B)] \).

Again, when outsiders are diversified a priori (that is \( w_B = w_A + (1 - \alpha) V_A \)), we have that \( w_B + (1 - \alpha) I_B > w_A + (1 - \alpha) V_A \) for \( I_B > 0 \). This means that \( \tilde{\theta}^{S,a}(\Pi^S(0, I_B)) > \theta^e \) and, thus, that the ambiguity averse outsiders are pessimistic ex-post on type-B assets relative to an SEU agent. This implies that \( I_B^{S,a} < I_B^e \) and that the outsiders underinvest in the diversifying project as well.

The above discussion implies that, with respect to a SEU agent, well-diversified ambiguity-averse outsiders underinvest in both focused and diversifying projects, leading to the following Lemma.

**Lemma 4** If they have control, well-diversified ambiguity-averse outsiders make the same investment in focused and diversified projects, and underinvest with respect to a SEU agent, \( I_A^{S,a} = I_B^{S,a} < I_B^e \). In addition, underinvestment is more severe when the firm is large (relative to the outsiders’
overall portfolio).

By combining (24) and (27) we obtain that if the outsiders maintain control, that is, $\delta = r$, the ex-ante expected utility that ambiguity-averse outsiders can obtain is given by

$$U_0^{S,r} \left( I_A^{S,a}, I_B^{S,a} \right) = \frac{1}{2} U_1^S \left( \Pi^S \left( I_A^{S,a}, 0 \right) \right) + \frac{1}{2} U_1^S \left( \Pi^S \left( 0, I_B^{S,a} \right) \right)$$

(28)

Note that (28) assumes that the MEU outsiders do not show ambiguity aversion with respect to the state of the world $\omega_r$, and that they are again sophisticated in that they correctly anticipate they future beliefs due their ambiguity aversion.

Consider now the case in which the outsiders delegates control to the SEU insider, $\delta = d$. In this case, from Theorem 2, we know that the insider, who is an SEU agent, will choose the investment level $I_e$ when the state of the world is $\omega_r$. Given the insider’s optimal investment policy, the outsiders ex-ante expected utility is determined as follows. While the outsiders understands that the insider will implement investment of $I_e$ in state $\omega_r$, they will not know which state of the world is realized, and thus, which type of project the firm has actually drawn. In addition, since the outsiders do not display (by our simplifying assumption) ambiguity aversion with respect to this random variable, the outsiders’ expected utility at $t = 1$ is given by

$$\mathbb{E} \left[ u \left( \Pi^S \left( I_e 1_{\tau=A}, I_e 1_{\tau=B} \right) \right) ; \theta \right] = e^{\theta - \theta_1} \left[ w_A + (1 - \alpha) \left( V_A + \frac{1}{2} I_e \right) \right]$$

$$+ e^{\theta_0 - \theta} \left[ w_B + (1 - \alpha) \frac{1}{2} I_e \right] + (1 - \alpha) \left[ V_0 - c(I_e) \right].$$

(29)

where $1_{\tau=A}$ is the indicator variable for the state $\omega_A$ (it equals 1 if the project if focused, and 0 if it is diversifying). Thus, the outsiders’ minimum expected utility is given by

$$U_1^{S,d} \left( \Pi^S \left( I_e 1_{\tau=A}, I_e 1_{\tau=B} \right) \right) = \min_{\theta \in C} \mathbb{E} \left[ u \left( \Pi^S \left( I_e 1_{\tau=A}, I_e 1_{\tau=B} \right) \right) ; \theta \right].$$

Because the outsider learns nothing at $t = 1$, $U_0^{S,d} = U_1^{S,d}$ with probability 1. Thus, the outsider’s payoff under delegation, $\delta = d$, is $U_0^{S,d} (I^e, I^e)$.

To compare the outsiders’ ex-ante payoffs under delegation, $U_0^{S,d} (I^e, I^e)$, and under retention
of control, \( U_0^{S,r}(I_A^{S,a}, I_B^{S,a}) \), as a first step it is helpful to determine the state-dependent level of investment, \( I_\tau \), that is ex-ante optimal for the ambiguity-averse outsiders when the control is allocated to the insider. The expected utility for an ambiguity-averse outsider who is blind to the realization of the state \( \omega_\tau \), when an investment \( I_\tau \) in made in state \( \omega_\tau \), is equal to

\[
E[u(\Pi^S(I_A^{1=\omega_A}, I_B^{1=\omega_B})); \theta] = e^{\theta - \theta_1} \left[ w_A + (1 - \alpha) \left( V_A + \frac{1}{2} I_A \right) \right] + e^{\theta_0 - \theta} \left[ w_B + (1 - \alpha) \frac{1}{2} I_B \right] + (1 - \alpha) \left[ V_0 - \frac{1}{2} c(I_A) - \frac{1}{2} c(I_B) \right].
\]

(30)

Thus, the outsiders’ minimum expected utility is given by

\[
U_1^S(\Pi^S(I_A^{1=\omega_A}, I_B^{1=\omega_B})) = \min_{\theta \in C} E[u(\Pi^S(I_A^{1=\omega_A}, I_B^{1=\omega_B})); \theta].
\]

The optimal levels of investment, \( I_{S*} \), are given by

\[
\{I_{S*}^A, I_{S*}^B\} = \arg \max_{I_A, I_B} U_1^S(\Pi^S(I_A^{1=\omega_A}, I_B^{1=\omega_B}))
\]

and are characterized in the following lemma.

**Lemma 5** If ambiguity-averse outsiders do not have control, their ex-ante optimal investment levels are \( I_{S*}^A = I_{S*}^B = I^e \).

Lemma 5 shows that ambiguity-averse outsiders would like to commit to the level of investment chosen by SEU agents. This result depends on our assumptions that outsiders have a balanced portfolio, \( w_B = w_A + (1 - \alpha) V_A \), and that the SEU beliefs are \( \theta^e = \frac{1}{2} (\theta_0 + \theta_1) \). Lemma 5 leads us to the following theorem, which is the main result for this section.

**Theorem 4** Ambiguity-averse outsiders will optimally delegate control to an expected-utility insider.

In this section, we have shown that ambiguity-averse outsiders have two motivations to grant control to an expected-utility insider. First, outsiders would like to commit to a level of investment that is not optimal ex-post if they maintain control. Because of the impact of ambiguity aversion on
posterior beliefs, outsiders would underinvest ex-post in both types of projects. Second, outsiders would prefer to be blind to the realization of the interim state of the world $\omega_T$ (i.e., the type of project available to the firm), because knowledge of the project type exposes outsiders to additional ambiguity. Granting control to the insider allows outsiders to not see this information and to increase ex-ante expected utility.

**Corollary 3** Delegation is more valuable when the growth options are more valuable.

Corollary 3 follows for two reasons. First, as growth options increase in value, outsiders’ underinvestment worsens. This happens because an increase of the productivity of growth options, $Z$, increases the values of both the investment level of SEU insider, $I^e$, and the investment level of MEU outsiders, $I^{S,a}_T$. However, the positive impact of $Z$ on investment is greater in the case of $I^e$ than $I^{S,a}_T$, making the underinvestment problem of outsiders’ retention of control more severe. The second effect is the adverse impact of information revelation on ambiguity-averse outsiders. Firms with more valuable growth options invest more (greater $I^{S,a}_T$), and the new investment becomes a larger portion of the outsiders’ portfolio. From (22) and (25) it easy to see that greater investment levels leads to greater dispersion of the posteriors, $\tilde{\theta}^{S,a}(\Pi^S(I_A,0))$ and $\tilde{\theta}^{S,a}(\Pi^S(0,I_B))$, which in turn exacerbates the outsiders’ welfare loss due to ambiguity aversion. Together, these properties imply that while MEU outsiders will always grant control to a SEU insider, the value creation from such delegation is an increasing function of the value of firm’s growth options.

We conclude this section by noting that while SEU insiders and well-diversified MEU agree ex-ante on the optimal level of investment in both projects, $I^e$, they will disagree ex-post when they learn the state of the world $\omega_T$. In addition, the MEU outsiders will be ex-post more “pessimistic” than the SEU insider. In our model, the ex-post disagreement between insiders and outsiders derives endogenously from the effect of ambiguity aversion on ex-post beliefs. In this way, this section mirrors results obtained in models with heterogenous priors.\footnote{For example, Boot, Gopalan, and Thakor (2006), Boot, Gopalan, and Thakor (2008), and Boot and Thakor (2011) all share these characteristics.} However, in our model, outsiders are better off by delegating authority SEU insiders, even in face of ex-post disagreement. The value of delegation derives from the combination of time-inconsistency of desired investment levels and the welfare loss of information arrival that characterizes MEU agents.
4.4 Ambiguity Averse Outsiders, Ambiguity Averse Insider

This section focuses on the more interesting case where both insider and outsiders are ambiguity averse, and provides the core results of our paper. The outsiders’ choice of whether to implement a strong corporate governance system, and thus retain control, or to allow for a weak governance system and to delegate decision making to the firm’s insiders is based on the trade-off of two distinct effects.

First is the effect of control on investment. If outsiders retain control, they will ex-post underinvest in both types of projects with respect to the level of investment that they would prefer ex-ante, $I^e$. From Corollary 3, we know that this effect is more severe when the growth options are more valuable. In contrast, if given control, the insider will always underinvest in focused projects, but will either overinvest or underinvest in diversifying projects, depending on the relative size of the existing assets and the value of growth options (Theorem 3). In addition, as in Corollary 2, the loss of value due to delegation is more severe in the youngest firms (small $V_A$ relative to $I^e$) and oldest firms (large $V_A$ relative to $I^e$).

Second is the negative impact of information resolution on ambiguity-averse agents. This effect always makes outsiders prefer to delegate control to the insiders, all else equal. This effect is more severe when the outsiders’ posterior beliefs differ substantially from their prior beliefs. From (22) and (25), it is easy to see that this happens when the level of investment is large relative to relative to the owners’ outside endowment. We now characterize these trade-offs explicitly, and derive comparative statics.

If outsiders retain control, $\delta = r$, they will behave as described in Lemma 4. Thus, their payoff is equal to $U^{S,r}_0 \left(I_A^{S,a}, I_B^{S,a}\right)$, defined in eq. (28). If given control, $\delta = d$, from Theorem 3, the insider will chose a level of investment $\{I_A^{M,a}, I_B^{M,a}\}$ as described in eq. (18) and (21), respectively. Thus, when the insider has control of the firm, the outsiders expected utility is given by

$$E \left[u \left(\Pi^S \left(I_A^{M,a}, I_B^{M,a}\right) \right) ; \theta \right] = e^{\theta_0} \left[w_A + (1 - \alpha) \left(V_A + \frac{1}{2} I_A^{M,a}\right)\right] + e^{\theta_0 - \theta} \left[w_B + (1 - \alpha) \frac{1}{2} I_B^{M,a}\right] + (1 - \alpha) \left[V_0 - \frac{1}{2} c \left(I_A^{M,a}\right) - \frac{1}{2} c \left(I_B^{M,a}\right)\right]. \tag{31}$$

Thus, the outsiders’ ex-ante minimum expected utility and, thus, their payoff under delegation of
control, \( \delta = d \), is given by

\[
U_{0}^{S, d} \left( I_{A}^{M, a}, I_{B}^{M, a} \right) = \min_{\theta \in \mathcal{C}} \mathbb{E} \left[ u \left( \Pi^{S} \left( I_{A}^{M, a} 1_{A}, I_{B}^{M, a} 1_{B} \right) \right) ; \theta \right].
\]

The optimal allocation of control – that is, the strength of the firm's governance system - is determined as follows. When the levels of investment chosen by the insider and the outsiders are the same, outsiders are strictly better off delegating control to the insider than retaining control, i.e., to have a weak rather than a strong governance system. By delegating control to the insider, the outsiders remain blind to the realization of the interim uncertainty, which increases their ex-ante payoff (from Lemma 1). This implies that a strong governance system (that is, retention of control) is optimal when the insider chooses investment levels that are very inefficient with respect to the investment that the outsiders would choose if they retained control. Otherwise, when this inefficiency is not too large, outsiders prefer a weak governance system (that delegating control to the insider) even knowing that the insider will invest inefficiently.

**Theorem 5** There are \((V_{A}, \tilde{V}_{A})\), with \(V_{A} < \tilde{V}_{A}\), such that the outsiders retain control for all \(V_{A} < \tilde{V}_{A}\) and for all \(V_{A} > \tilde{V}_{A}\), and delegate for all \(V_{A} \in (\tilde{V}_{A}, V_{A})\). In addition, there are critical values \(\{Z, \tilde{Z}\}\), with \(Z \leq \tilde{Z}\), and \(\tilde{K} > 0\) such that diversified outsiders (i) retain control for all \(Z < Z\) and delegate control for all \(Z > \tilde{Z}\); (ii) retain control if \(K \geq \tilde{K}\), where \(w_{A} = K - (1 - \alpha) V_{A}\) and \(w_{B} = K\).

For well diversified outsiders, a strong governance system (retention) is optimal for less productive firms (low values of \(Z\)) or when outsiders have a sufficiently large portfolio (a large value of \(K\)). This happens because the realization of the project type (the state \(\omega_{r}\)) does not significantly impact investment and the ex-post ambiguity of their portfolio. In this case, the adverse effect of information revelation on the outsiders, and the efficiency losses due to underinvestment are both small. Thus, outsiders are better-off by establishing a strong governance system and retaining control. Conversely, in more productive firms (large \(Z\)) or when the firm is sufficiently large component of the outsiders’ portfolio, outsiders optimally delegate control to the insiders by implementing a weak governance system.

A strong governance system is also optimal when the value of assets in place, \(V_{A}\), is either
sufficiently small, $V_A \leq V_A$, or sufficiently large, $V_A \geq V_A$, and a weak governance system, where the insider has more freedom to decide the firm’s investment policy is optimal in the intermediate case where $V_A \in (\underline{V}_A, \overline{V}_A)$. This happens because in the latter case, insider’s and outsiders’ optimal investment policies are more closely aligned and, thus, the disagreement between the insider and the outsiders is reduced, making delegation optimal. In contrast, when the the value of assets in place is either sufficiently large, $V_A \geq V_A$, or sufficiently small, $V_A \leq V_A$, the outsiders prefers to retain control in order to select a level of investment better aligned with their ex-ante objectives. The value of assets in place also has an impact on the threshold $\bar{Z} (V_A)$, as follows.

**Corollary 4** $\bar{Z} (V_A)$ is U-shaped.

The effect of strength of the corporate governance system on firm investment is examined in the following corollary.

**Corollary 5** Under retention, investment in diversified and focused projects are balanced. Under delegation, investment in diversified projects will be larger than investment in focused projects.

Corollary 5 has the interesting result that if control is retained by outsiders, the firm has balanced investment $(I^S_A = I^S_B)$, while the insider, granted control, overinvests in diversifying projects $(I^M_B > I^M_A)$. This means that firms endowed with a strong governance system follow a more balanced investment policy than firms endowed with a weak governance system, which overinvest in diversifying projects. Note that these results hold even though governance is optimally chosen.

For the remainder of this section, it is helpful to define the value of delegation as the difference in firm value under delegation and retention: $U_0^{S_d} - U_0^{S_r}$. Note that this difference can also be interpreted as the differential value of firms with weak and strong corporate governance systems, and is characterized in the following.

**Corollary 6** The value of delegation, $U_0^{S_d} - U_0^{S_r}$, is

1. decreasing in outside portfolio size for well diversified portfolios if $\gamma \geq \gamma_\overline{w} = \frac{\bar{b}_1 - \bar{b}_0}{\ln 2}$.
2. decreasing in the outsiders’ endowment in $w_B$;
3. increasing in the value of growth options, $Z$, if $Z$ is large enough.
Figure 2: Indifference Curve for Outside Portfolio. The solid lines plot the indifference curve between delegation and retention for different levels of productivity, $Z$. The dotted line plots $w_B = w_A + (1 - \alpha)V_A$. An increase in $w_B$ causes retention to be more attractive, while an increase in $w_A$ is non-monotonic, favouring first delegation and then retention. When $Z$ is larger, this cutoff increases.

Point 1 of Corollary 6 follows by a similar intuition to Theorem 5: increasing the size of the outside portfolio diminishes the adverse impact of the ambiguity on outsiders. Point 2 follows by the following intuition. First, the value of delegation decreases in the size of the outside portfolio, as shown in Point 1. In addition, increasing $w_B$ also increases the ex-ante disagreement between outsiders and the insider and aggravates the difference of opinion between outsiders and the insider on the desired investment decisions. Both of these effects make retention more attractive.\textsuperscript{20} Finally, Point 3 derives from the fact that the adverse effect of ambiguity-aversion on outsiders increases as the value of the growth options become larger, making delegation more attractive. Conversely, the effect of productivity, $Z$, on the extent of the investment distortions is ambiguous. In the proof of Theorem 5, however, we have shown that when $Z$ is very large, both insider’s and outsiders’ beliefs converge to the worst-case scenario, generating unambiguous comparative statics results only for sufficiently large $Z$.

Figure 2 shows the indifference curve between retention and delegation as a function of the outsiders’ endowment, $w_A$ and $w_B$. First note that, as shown in Point 1 of Corollary 6 when the outsiders have a diversified portfolio (that is, along the dotted line) weak governance is optimal when the outsiders’ portfolio is relatively small, that is, it is closer to the origin, and strong governance is optimal for larger outsiders’ portfolios, that is for values of $w_A$ and $w_B$ further away from the origin on the dotted line. Second, as shown in Point 2 of Corollary 6, retention is more attractive when

\textsuperscript{20}Note that these effects work in opposite directions for $w_A$, so we cannot derive comparative statics for $w_A$. In fact, numerical simulations, reported below, suggest the comparative statics are nonmonotonic in $w_A$. 
Increasing the outside portfolio of outsiders, $K$ (where $w_A = K - (1 - \alpha) V_A$ and $w_B = K$), makes retention more attractive, but increasing the growth options, $Z$, makes delegation more attractive.

$w_B$ is larger. Finally, as anticipated, the effect of $w_A$ is nonmonotonic. Increasing $w_A$ decreases disagreement between the insider and outsiders, making delegation more attractive. This happens because insider’s and outsiders’ portfolios become more similar, decreasing disagreement. Increasing $w_A$ also decreases the importance of new investment to outsiders, reducing the adverse welfare effect of ambiguity and, thus, making retention more attractive (similar to Point 1 of Corollary 6). This happens because the dispersion of ex post beliefs decreases as $w_A$ increases. The disagreement effect dominates for small values of $w_A$, while the portfolio effect dominates for large values of $w_A$, resulting in the inverted U-shaped relationship. Finally, as shown in Point 3 of Corollary 6, delegation is more attractive when $Z$, the value of growth options, is larger.

Figure 3 displays the indifference curve between strong governance (retention) and weak governance (delegation) as a function of the size of the outside portfolio, $K$, (where $w_A = K - (1 - \alpha) V_A$ and $w_B = K$) and the value of growth options, $Z$. Here, the outsiders’ portfolio is assumed to be diversified. As shown in Point 1 of Corollary 6, strong governance (retention) is more attractive as the outside portfolio becomes larger. As shown in Point 3 of Corollary 6, weak governance (delegation) becomes more attractive as the value of growth options increases.

Figure 4 shows the indifference curve between strong governance (retention) and weak governance (delegation) as a function of productivity of growth options, $Z$, and the value of assets in place, $V_A$. As stated in Corollary 4, the relationship between $Z$ and $V_A$ is a inverted U-shaped function. From Point 3 of Corollary 6, we know that as $Z$ gets larger, delegation becomes more attractive. The relationship with $V_A$ is more interesting. As shown in Theorem 5, for a given
Figure 4: Indifference Curve for Assets in Place, $V_A$, and Productivity, $Z$. The solid lines plot the indifference curve between delegation and retention against productivity, $Z$, and assets in place, $V_A$. The dotted line plots a constant level of $Z$. An increase in $Z$ causes delegation to be more attractive, while the relationship with $V_A$ is nonmonotonic, suggesting the life-cycle relationship.

level of $Z$, for small values of $V_A$, retention is optimal, for intermediate values of $V_A$, delegation is optimal, while for larger values of $V_A$, retention is once again optimal.

5 Empirical Implications

We determine the optimal governance structure of firms in the presence of disagreement between firm insiders and outsiders. In our model, disagreement emerges endogenously among ambiguity-averse agents with heterogenous portfolios. In our model, the strength of a firm’s corporate governance system and the allocation of control is made by the firm’s outsiders, who maximize their ex-ante MEU. The optimal allocation of control depends on both firm characteristics and the overall portfolio composition of the firm’s outsiders. In addition, ambiguity aversion endogenously creates a direct link between corporate governance and firm transparency.

The model has the following empirical and policy implications.

1. Corporate governance life-cycle: If the value of a firm’s assets in place increases over a firm’s life cycle (relative to the value of its growth opportunities), Corollary 4 suggests that firms should follow a governance structure life cycle. In particular, in the earlier stages of development, a young, high-growth firm should have a strong governance system; as the firm ages, it should move to a weak corporate governance system where the firm insiders have discretion over investment decisions. Finally, as the firm matures, it should revert back to strong governance system. Because we would expect it to be easy to give control to a CEO but difficult to take back control, this suggests a role
for LBOs as a mechanism for outside investors to regain control of a firm.

2. Corporate governance and market-to-book ratio: Corollary 4 also suggests that a non-monotonic relationship between the strength of a firm’s corporate governance system and its market-to-book ratio. This property may be seen as follows. Taking $V_A$ as a proxy for book value, and the value of the firm to outsiders as market value, Corollary 4 demonstrates that we should observe strong governance systems at the two extremes of growth firms and value firms, but weak governance in the middle of the spectrum.

3. Well-diversified outsiders, where the firm represents a small fraction of their overall portfolio, prefer strong governance system. This result follows from Theorem 5 and suggests that, all else equal, small firms, that are more likely to represent a smaller proportion of the owners' portfolio, should have a strong governance system. Conversely, larger firms, which are more likely to represent a greater proportion of their owners' portfolio, should have weak governance. In addition, firms with well-diversified owners, such as a mutual fund, are more likely to have a strong governance system.

Corollary 6 also suggests that outsiders whose portfolio is focused in sectors different from the firm’s core business prefer a strong governance system, while outsiders whose portfolio has the same focus as the firm’s core business are more likely to prefer a weak governance system. This means that generalist venture capital or private equity funds should impose strong governance systems on their portfolio companies, while specialized funds are more willing to tolerate weak governance systems, where the management of their portfolio companies have more leeway in determining company corporate policies. In addition, Corollary 6 also implies that diversified outsiders implement a strong governance system in firms with less productive growth options, but implement a weak governance system in firms with more productive growth options. This result suggests that, all else equal, more valuable firms and firms with more productive growth options (higher $Z$) should have weak governance. Firms with less productive growth options (smaller $Z$) should have strong governance.

4. A decline in firm productivity leads to stronger corporate governance system. Point 3 in Corollary 6 shows that a weak governance system is more valuable to the outsiders when the firm is more productive, and that the firm should switch to a strong corporate governance system
when productivity decreases. This suggests that a weakening of a firm productivity leads to a strengthening of its corporate governance system. As suggested above, a stronger governance system may be obtained by having the outsiders take over the firm through a LBO. This means that weaker firm performance may lead to going-private transactions.

5. **Weak corporate governance systems should also be less transparent.** Firms with weak corporate governance systems should also be more opaque. If the outsiders retain control, the firm will disclose relevant information (in our model, the project type) so that outsiders can make an informed decision. If outsiders delegate control, they require that the insider do not disclose the type of project. Thus, the model predicts that outsider controlled firms with a strong governance system will be more transparent, while insider controlled firms with a weak governance system will be more opaque.

These observations have implications for the regulation of corporate disclosures. If the government were to implement mandatory disclosure regulation, firms would, in general, be harmed. Mandatory disclosure regulation destroys the benefit of delegation, insulation from ambiguity, so outsiders at all firms would find it optimal to retain control. However, firms that would have found it optimal to delegate control to the insider would be harmed. The harmful effect of mandatory disclosure regulation would be worse if control rights have already been delegated to the insider.

6. **Weak-governance firms overinvest in diversifying projects relative to their investment in focused projects.** **Strong-governance firms implement balanced investment in focused and diversifying projects.** This result, which follows directly from Corollary 5, implies that firms with weak corporate governance systems tend to be more diversified than comparable firms with a stronger governance system. In addition, weak-governance firms diversifying projects underperform ex post focused projects, while in strong-governance firms, ex post performance is similar for focused and diversifying projects. This observation can be seen as follows. A measure of ex-post performance can be obtained by defining \( R(I) = \frac{I}{c(I)} \) as the return on investment for a given project. It is easy to verify that \( R(I) \) is strictly decreasing in \( I \) (from convexity of \( c(I) \)). From Corollary 5, this implies that firms with weak governance systems underperform in their diversifying investments, \( R(I_{B}^{M,a}) < R(I_{A}^{M,a}) \), while firms with strong governance systems have a more uniform performance across divisions.
6 Conclusions

We study a model where agents’ ambiguity aversion generates endogenously differences of opinion between a firm’s insiders and its outsiders. We show that the allocation of control and, thus, the strength of the corporate governance system, depends on firm characteristics and the portfolio composition of both insiders and outsiders. We predict that small firms and less productive firms should have stronger governance, while larger firms and more productive firms should have weaker governance systems. In addition, we predict that firms should display a corporate governance life cycle, where both younger and more mature firms should be characterized by a stronger corporate governance system, while firms at their intermediate deployment stage have weaker governance, where the firm insiders have more discretion over corporate investment decisions. Finally, we argue that weaker governance systems are optimally less transparent.
References


A Appendix: Proofs

Proof of Lemma 1. The claim holds due to a property of the minimum. Suppose we have a set of priors \( \mu(w, s) \in \mathcal{M} \), so that each prior gives the joint distribution of wealth, \( w \), and the signal, \( s \in S \), and all priors share a common support of \( s \). For a given \( s \), for each \( \mu \), define \( \mu(\cdot; s) \) as the conditional distribution of wealth given \( s \) (each conditional distribution exists and is a.s. unique by the standard arguments).\(^{21}\)

For each \( s \), define \( \mathcal{M}_s = \{ \mu(\cdot; s) | \mu(w, s) \in \mathcal{M} \} \). By the definition of \( \mu(\cdot; s) \), for all \( \mu \) and for all \( s \),

\[
E(\mu) = E_s E_{\mu(\cdot; s)}[u(w)]
\]

Define \( \nu = \arg \min_{\mu \in \mathcal{M}} E\mu[u(w)] \), so that \( \nu \) is the worst-case scenario if the agent does not learn \( s \). For each \( s \), define \( \nu(\cdot; s) \) as the conditional distribution of \( \nu \) given \( s \) is observed.

Similarly, define \( \nu(s) = \arg \min_{\mu(\cdot; s) \in \mathcal{M}_s} E\mu(\cdot; s)[u(w)] \) as the worst-case scenario after the agent has learned \( s \). By definition of \( \nu(s) \),

\[
E_{\mu(\cdot; s)}[u(w)] \geq E_{\nu(s)}[u(w)]
\]

for all \( \mu(\cdot; s) \in \mathcal{M}_s \). Because \( \nu(\cdot; s) \in \mathcal{M}_s \), this implies that

\[
E_{\nu(\cdot; s)}[u(w)] \geq E_{\nu(s)}[u(w)]
\]

By the monotonicity of integration, this implies that

\[
E_s E_{\nu(\cdot; s)}[u(w)] \geq E_s E_{\nu(s)}[u(w)]
\]

Because \( E_s E_{\nu(\cdot; s)}[u(w)] = E_s u(w) = \min_{\mu \in \mathcal{M}} E_s E_{\mu(\cdot; s)}[u(w)] \), and \( E_s E_{\nu(s)}[u(w)] = E_s \min_{\mu(\cdot; s) \in \mathcal{M}_s} E_{\mu|s}[u(w)] \), the claim is shown: more information harms an ambiguity-averse agent.

Proof of Lemma 2. The minimization problem is \( \min_{\theta \in C} \mathbb{E}[u(\tilde{w}_A, \tilde{w}_B, \tilde{w}_0) | \theta] \), where

\[
\mathbb{E}[u(\tilde{w}_A, \tilde{w}_B, \tilde{w}_0) | \theta] = e^{\theta_1 \tilde{w}_A} + e^{\theta_0 - \theta} \tilde{w}_B + \tilde{w}_0.
\]

Note that

\[
\frac{\partial \mathbb{E}[u(\theta)]}{\partial \theta} = e^{\theta_1 \tilde{w}_A} - e^{\theta_0 - \theta} \tilde{w}_B, \quad \frac{\partial^2 \mathbb{E}[u(\theta)]}{\partial \theta^2} = e^{\theta_1 \tilde{w}_A} + e^{\theta_0 - \theta} \tilde{w}_B.
\]

Thus, \( \frac{\partial \mathbb{E}[u(\theta)]}{\partial \theta} \geq 0 \) because \( \tilde{w}_A \geq 0 \) and \( \tilde{w}_B \geq 0 \) (usually, one or both of these inequalities will be strict, so \( \frac{\partial^2 \mathbb{E}[u(\theta)]}{\partial \theta^2} > 0 \)). Because the inner problem (minimization) is convex, and \( C \) is closed and connected, the solution is unique and continuous. Interior solutions to the inner problem satisfy \( \frac{\partial \mathbb{E}[u(\theta)]}{\partial \theta} = 0 \), which implies that

\[
e^{\tilde{\theta}_1 \tilde{w}_A} = e^{\theta_0 - \tilde{\theta} \tilde{w}_B}, \quad e^{2 \tilde{\theta}_0 - \tilde{\theta} \tilde{w}_B} = \tilde{w}_B.
\]

This implies that

\[
\tilde{\theta}^a = \frac{1}{2} (\theta_0 + \theta_1) + \frac{1}{2} \ln \left( \frac{\tilde{w}_B}{\tilde{w}_A} \right),
\]

\(^{21}\)In our setting, \( s \) is discrete, so we can express \( \mu(w; s) = \sum_{s} \mu_{(w, s)} \), though our proof still applies with general distributions.
where $\Pi = \{\bar{w}_A, \bar{w}_B, \bar{w}_0\}$ is the portfolio owned by the agent. If $\bar{\theta}^a(\Pi) < \hat{\theta}_0$, $\frac{\partial E[u;\theta]}{\partial \theta}$ > 0 for all $\theta \in [\hat{\theta}_0, \hat{\theta}_1]$, so $\theta^a(\Pi) = \hat{\theta}_0$. Similarly, if $\bar{\theta}^a(\Pi) > \hat{\theta}_1$, $\frac{\partial E[u;\theta]}{\partial \theta}$ < 0 for all $\theta \in [\hat{\theta}_0, \hat{\theta}_1]$, so $\theta^a(\Pi) = \hat{\theta}_1$. Therefore,

$$\theta^a(\Pi) = \begin{cases} \hat{\theta}_0 & \bar{\theta}^a(\Pi) \leq \hat{\theta}_0 \\ \hat{\theta}_1 & \bar{\theta}^a(\Pi) \geq \hat{\theta}_1 \end{cases}$$

are the endogenous beliefs.

**Proof of Lemma 3.** This property may be seen from the minimax theorem, as follows. From (7) we have that

$$\frac{dU}{d\bar{w}_A} = \frac{\partial E(u)}{\partial \bar{w}_A} + \frac{\partial E(u)}{\partial \theta} \frac{d\theta^a(\Pi)}{d\bar{w}_A}.$$  \hspace{1cm} (32)

The second term of (32) is uniformly zero, since for interior solutions we have that $\frac{\partial E(u)}{\partial \theta} = 0$, and for corner solutions we have that $\frac{d\theta^a(\Pi)}{d\bar{w}_A} = 0$. Thus, $\frac{dU}{d\bar{w}_A} = \frac{\partial E(u)}{\partial \bar{w}_A} = \theta^a(\Pi) - \bar{\theta}_1 > 0$, and $\frac{d^2U}{d\bar{w}_A^2} = \theta^a(\Pi) - \bar{\theta}_1 \frac{d\theta^a(\Pi)}{d\bar{w}_A} < 0$, because $\frac{d\theta^a(\Pi)}{d\bar{w}_A} \leq 0$. These inequalities are strict for interior $\bar{\theta}^a$. Similarly, $\frac{dU}{d\bar{w}_B} = \theta^a(\Pi) - \bar{\theta}_1 > 0$ and $\frac{d^2U}{d\bar{w}_B^2} = -\theta^a(\Pi) \frac{d\theta^a(\Pi)}{d\bar{w}_B} \leq 0$, because $\frac{d\theta^a(\Pi)}{d\bar{w}_B} \geq 0$. Again, these inequalities are strict for interior $\bar{\theta}^a(\Pi)$.

**Proof of Theorem 1.** Consider type A projects. Investment is chosen to maximize

$$U_1(\Pi(I_A, 0)) \equiv \min_{\theta \in C} E[u(\Pi(I_A, 0); \theta)],$$

where $\Pi(I_A, 0) = \{w_A + V_A + I_A, w_B, w_0 + V_0 - c(I_A)\}$. Applying the envelope theorem (either $\frac{\partial}{\partial \theta} E[u; \theta^a] = 0$ or $\frac{\partial \theta^a}{\partial \Pi} = 0$)\footnote{With a focused project, $U_1$ and $\theta^a$ are understood to be functions of the portfolio $\Pi^a(I_A, 0) = \{w_A + V_A + I_A, w_B, w_0 + V_0 - c(I_A)\}$. For ease of notation, we will not always write out $U_1(\Pi^a(I_A, 0))$ and $\theta^a(\Pi^a(I_A, 0))$, but that is how $U_1$ and $\theta^a$ should be interpreted. Similarly, for a diversified project, $U_1$ and $\theta^a$ understood to be functions of the portfolio $\Pi^a(0,I_B) = \{w_A + V_A, w_B + I_B, w_0 + V_0 - c(I_B)\}$.}, the benefit of increasing investment is

$$\frac{d}{dI_A} U_1(\Pi(I_A, 0)) = \frac{\partial E[u; \theta^a]}{\partial w_A} - \frac{\partial E[u; \theta^a]}{\partial w_0} c'(I_A) = \theta^a - \bar{\theta}_1 - c'(I_A),$$

because $\frac{\partial E[u; \theta^a]}{\partial w_A} = \theta^a - \bar{\theta}_1$ and $\frac{\partial E[u; \theta^a]}{\partial w_0} = 1$. However, the equilibrium beliefs depend on the level of investment, similar to Lemma 2:

$$\bar{\theta}^a(\Pi(I_A, 0)) = \theta^a + \frac{1}{2} \ln \left[ \frac{w_B}{w_A + I_A} \right]$$

and

$$\theta^a(\Pi(I_A, 0)) = \begin{cases} \hat{\theta}_0 & \bar{\theta}^a(\Pi(I_A, 0)) \leq \hat{\theta}_0 \\ \hat{\theta}_1 & \bar{\theta}^a(\Pi(I_A, 0)) \geq \hat{\theta}_1 \end{cases}.$$
Also, note that
\[
\frac{d^2}{dI_A^2} U_1 = \frac{d}{dI_A} \left[ e^{\theta_0 - \theta_1} - c'(I_A) \right] = e^{\theta_0 - \theta_1} \frac{d\theta_0}{dI_A} - c''(I_A).
\]

Because \(\frac{d\theta_0}{dI_A} \leq 0\) (with strict inequality for interior \(\theta_0\)), and because the cost function is convex, \(c'' > 0\),
\[
\frac{d^2}{dI_A^2} U_1 (\Pi^a (I_A, 0)) < 0,
\]
so first-order conditions are sufficient for a maximum.

Similarly, for a type \(B\) project, investment is chosen to maximize
\[
U_1 (\Pi (0, I_B)) = \min_{\theta \in C} \mathbb{E}[u(\Pi (0, I_B)); \theta],
\]
where \(\Pi (0, I_B) = \{w_A + V_A, w_B + I_B, w_0 + V_0 - c(I_B)\}\). Applying the envelope theorem (either \(\frac{\partial E[u; \theta^a]}{\partial \theta} = 0\) or \(\frac{d\theta_0}{dI_B} = 0\)),
\[
\frac{d}{dI_B} U_1 (\Pi (0, I_B)) = \frac{\partial E[u; \theta^a]}{\partial w_B} - \frac{\partial E[u; \theta^a]}{\partial w_0} c'(I_B) = e^{\theta_0 - \theta^a} - c'(I_B)
\]
because \(\frac{\partial E[u; \theta^a]}{\partial w_B} = e^{\theta_0 - \theta^a}\) and \(\frac{\partial E[u; \theta^a]}{\partial w_0} = 1\), where
\[
\theta^a (\Pi (0, I_B)) = \theta^a + \frac{1}{2} \left[ \frac{w_B + I_B}{w_A} \right]
\]
and
\[
\theta^a (\Pi (0, I_B)) = \begin{cases} 
\hat{\theta}_0 & \theta^a (\Pi (0, I_B)) \leq \hat{\theta}_0 \\
\hat{\theta}_1 & \theta^a (\Pi (0, I_B)) \in (\hat{\theta}_0, \hat{\theta}_1) \\
\hat{\theta}_1 & \theta^a (\Pi (0, I_B)) \geq \hat{\theta}_1 
\end{cases}
\]

Also,
\[
\frac{d^2}{dI_B^2} U_1 = \frac{d}{dI_B} \left[ e^{\theta_0 - \theta_1} - c'(I_B) \right] = -e^{\theta_0 - \theta^a} \frac{d\theta_0}{dI_B} - c''(I_B).
\]

Because \(\frac{d\theta_0}{dI_B} \geq 0\) (with strict inequality for interior \(\theta_0\)), and because the cost function is convex, \(c'' > 0\), if follows that \(\frac{d^2}{dI_B^2} U_1 < 0\), so first-order conditions are sufficient for a maximum.

For comparative statics on \(I_A\), note that
\[
\frac{\partial}{\partial w_A} \left[ \frac{d}{dI_A} U_1 \right] = \frac{\partial}{\partial w_A} \left[ e^{\theta_0 - \theta_1} - c'(I_A) \right] = e^{\theta_0 - \theta_1} \frac{\partial \theta_0}{\partial w_A}.
\]

Because \(\frac{\partial \theta_0}{\partial w_A} \leq 0\), with strict inequality for interior \(\theta_0\), \(\frac{\partial}{\partial w_A} \left[ \frac{d}{dI_A} U_1 \right] \leq 0\), with strict inequality for interior \(\theta_0\). Because \(\frac{d^2}{dI_A^2} U_1 < 0\), it follows that \(\frac{dI_A}{dw_A} \leq 0\), with strict inequality for interior \(\theta^a\). Therefore, optimal investment in a type \(A\) project is decreasing in the portfolio position the player has in type \(A\) assets. Similarly,
\[
\frac{\partial}{\partial w_B} \left[ \frac{d}{dI_A} U_1 \right] = \frac{\partial}{\partial w_B} \left[ e^{\theta_0 - \theta_1} - c'(I_A) \right] = e^{\theta_0 - \theta_1} \frac{\partial \theta_0}{\partial w_B}.
\]

Because \(\frac{\partial \theta_0}{\partial w_B} \geq 0\), \(\frac{\partial}{\partial w_B} \left[ \frac{d}{dI_A} U_1 \right] \geq 0\). Because \(\frac{d^2}{dI_A^2} U_1 < 0\), \(\frac{dI_A}{dw_B} \geq 0\) (similarly, strict inequality for interior
which implies

Thus, the insider chooses

Because \( \frac{\partial \theta^a}{\partial w_A} \leq 0 \), \( \frac{\partial \theta^a}{\partial w_A} \geq \frac{d}{dI_B} \) \( \geq 0 \). Because \( \frac{d^2}{dI_B^2} U_1 < 0 \), \( \frac{dI_B}{dI_A} \geq 0 \) (strict inequality for interior \( \theta^a \)). Therefore, optimal investment in a type \( B \) project is increasing in the portfolio position the player has in type \( A \) assets. Also,

Because \( \frac{\partial \theta^a}{\partial w_B} \geq 0 \), \( \frac{\partial \theta^a}{\partial w_B} \leq \frac{d}{dI_B} \leq 0 \). Because \( \frac{d^2}{dI_B^2} U_1 < 0 \), \( \frac{dI_B}{dI_A} \leq 0 \) (strict inequality for interior \( \theta^a \)).

**Proof of Theorem 2.** First, consider a focused project. If outsiders have control, they choose \( I_B \) to maximize

which implies

Because \( \theta^e = \frac{1}{2} (\theta_0 + \theta_1) \), \( e^{\theta^e - \theta_1} = e^{\frac{1}{2}(\theta_0 - \theta_1)} \). Thus, outsiders set \( I_{A,e}^S \) so that \( c' \left( I_{A,e}^S \right) = e^{\theta^e - \theta_1} \) (second-order conditions are satisfied because \( c \) is convex: \( \frac{d^2}{dI_A^2} \mathbb{E} \left[ u \left( I_{A}^S, 0 \right); \theta^e \right] = - (1 - \alpha) c'' \left( I_{A} \right) \), which is strictly negative because \( c'' > 0 \). If the insider has control, she chooses \( I_A \) to maximize

which implies

Thus, the insider chooses \( I_{A,e}^M \) so that \( c' \left( I_{A,e}^M \right) = e^{\frac{1}{2}(\theta_0 - \theta_1)} \). Therefore, outsiders and the insider will choose the same level of investment for a focused project: \( I_{A,e}^S = I_{A,e}^M \).

Second, consider a diversified project. If outsiders have control, they choose \( I_B \) to maximize

which implies

\[ \mathbb{E} \left[ u \left( I_{A}^S, 0 \right); \theta^e \right] = e^{\theta^e - \theta_1} \left[ w_A + (1 - \alpha) V_A \right] + e^{\theta_0 - \theta} w_B + (1 - \alpha) \left[ V_0 - c(I_A) \right], \]

\[ \mathbb{E} \left[ u \left( I_{B}^S, 0 \right); \theta^e \right] = e^{\theta_0 - \theta} - c' \left( I_B \right) \cdot \frac{d}{dI_B} \]
which implies

$$E \left[ u \left( \Pi^M (0, I_B) \right); \theta^e \right] = e^{\theta^e - \theta_1} \alpha V_A + e^{\theta_0 - \theta^e} \alpha I_B + \alpha [V_0 - c(I_B)],$$

which implies

$$\frac{d}{dI_B} E \left[ u \left( \Pi^M (0, I_B) \right); \theta^e \right] = \alpha \left[ e^{\theta_0 - \theta^e} - c' (I_B) \right].$$

Thus, the insider chooses $I_{B}^{M,e}$ so that $c' \left( I_{B}^{M,e} \right) = e^{\frac{1}{2} (\theta_0 - \theta_1)}$. Therefore, the outsider and insider will choose the same level of investment for a diversified project: $I_{B}^{e} \equiv I_{B}^{S,e} = I_{B}^{M,e}$. Because the same level of investment results independent of who is given control or which project is chosen, $I^{e} \equiv I_{A}^{e} = I_{B}^{e}$, the allocation of control does not matter. □

**Proof of Theorem 3.** In this proof, we will consider optimal behavior by the insider. Because outsiders are not averse to ambiguity, they will behave as in Theorem 2 if they retains control. Further, they will retain control iff the insider acts suboptimally from their perspective.

For focused projects, the insider’s minimum expected utility is

$$U^{M}_{1} \left( \Pi^M (I_A, 0) \right) = \min_{\theta} E \left[ u \left( \Pi^M (I_A, 0) \right); \theta \right],$$

where $E \left[ u \left( \Pi^M (I_A, 0) \right); \theta \right] = e^{\theta - \theta_1} \alpha (V_A + I_A) + \alpha (V_0 - c (I_A)).$ Because she is exposed only to type $A$ assets, her worst-case scenario is $\theta^{M,a} \left( \Pi^M (I_A, 0) \right) = \hat{\theta}_0$ (Lemma 2). Thus, her objective becomes

$$U^{M}_{1} \left( \Pi^M (I_A, 0) \right) = E \left[ u \left( \Pi^M (I_A, 0) \right); \hat{\theta}_0 \right] = e^{\hat{\theta}_0 - \theta_1} \alpha (V_A + I_A) + \alpha (V_0 - c (I_A)),

which implies

$$\frac{d}{dI_A} U^{M}_{1} \left( \Pi^M (I_A, 0) \right) = \alpha \left[ e^{\hat{\theta}_0 - \theta_1} - c' (I_A) \right].$$

Therefore, the insider chooses $I_{A}^{M,a}$ so that $c' \left( I_{A}^{M,a} \right) = e^{\hat{\theta}_0 - \theta_1}$. Because $\hat{\theta}_0 < \theta^e$, $I_{A}^{M,a} < I^e$, so the insider underinvests in focused projects.

For diversifying projects, the insider’s objective is

$$U^{M}_{1} \left( \Pi^M (0, I_B) \right) = \min_{\theta} E \left[ u \left( \Pi^M (0, I_B) \right); \theta \right],$$

where $E \left[ u \left( \Pi^M (0, I_B) \right); \theta \right] = e^{\theta - \theta_1} \alpha V_A + e^{\theta_0 - \theta} \alpha I_B + \alpha [V_0 - c (I_B)].$ For a given choice of $I_B$, she has the portfolio $\Pi^M (0, I_B) = \{ \alpha V_A, \alpha I_B, \alpha (V_0 - c (I_B)) \}$. Thus, her beliefs follow from Lemma 2 for a given level of investment $I_B$. Thus, her endogenous beliefs are given by $\theta^{M,a}$:

$$\theta^{M,a} \left( \Pi^M (0, I_B) \right) = \left\{ \begin{array}{ll}
\hat{\theta}_0 & \text{if } \theta^{M,a} \left( \Pi^M (0, I_B) \right) \leq \hat{\theta}_0 \\
\hat{\theta}_1 & \text{if } \theta^{M,a} \left( \Pi^M (0, I_B) \right) \in (\hat{\theta}_0, \hat{\theta}_1) \\
\hat{\theta}_1 & \text{if } \theta^{M,a} \left( \Pi^M (0, I_B) \right) \geq \hat{\theta}_1
\end{array} \right.$$
Thus, $dI$

The insider optimally chooses $I_B^{M,a}$:

$$\frac{d}{dI_B} U_1^M (\Pi^M (0, I_B)) = \frac{\partial E u}{\partial I_B} + \frac{\partial E u}{\partial \theta} \frac{d \theta^{M,a}}{d I_B}.$$ 

Applying the minimax theorem, (either $\frac{\partial E u}{\partial \theta} = 0$ or $\frac{d \theta^{M,a}}{d I_B} = 0$):

$$\frac{\partial E u}{\partial I_B} = \frac{\partial E u}{\partial w_B} - \frac{\partial E u}{\partial w_0} c' (I_B) = e^{\theta_0 - \theta^{M,a}} - c' (I_B),$$

because $\frac{\partial E u}{\partial w_B} = e^{\theta_0 - \theta^{M,a}}$ and $\frac{\partial E u}{\partial w_0} = 1$. Thus, the insider chooses investment $I_B^{M,a}$ so that

$$c' (I_B^{M,a}) = e^{\theta_0 - \theta^{M,a}} (\Pi^M (0, I_B^{M,a})).$$

Note that the insider may underinvest or overinvest in this situation. Totally differentiating with respect to $V_A$,

$$c'' (I_B^{M,a}) \frac{d I_B^{M,a}}{d V_A} = e^{\theta_0 - \theta^{M,a}} (\Pi^M (0, I_B)) \left[ - \frac{\partial \theta^{M,a}}{\partial I_B} \frac{d I_B^{M,a}}{d V_A} - \frac{\partial \theta^{M,a}}{\partial V_A} \right]$$

For corner $\theta^{M,a}$, $\frac{\partial \theta^{M,a}}{\partial I_B} = \frac{\partial \theta^{M,a}}{\partial V_A} = 0$, so $\frac{d I_B^{M,a}}{d V_A} = 0$. For interior $\theta^{M,a}$, $\theta^{M,a} = \tilde{\theta}^{M,a}$ (Theorem 2), so $\frac{\partial \theta^{M,a}}{\partial I_B} = \frac{1}{2 I_B}$ and $\frac{\partial \theta^{M,a}}{\partial V_A} = -\frac{1}{2 V_A}$, which implies

$$\left[ c'' (I_B^{M,a}) + e^{\theta_0 - \theta^{M,a}} (\Pi^M (0, I_B)) \frac{1}{2 I_B} \right] \frac{d I_B^{M,a}}{d V_A} = e^{\theta_0 - \theta^{M,a}} (\Pi^M (0, I_B)) \frac{1}{2 V_A}.$$ 

Thus, $\frac{d I_B^{M,a}}{d V_A} > 0$ because $c'' > 0$, $e^{\theta_0 - \theta^{M,a}} (\Pi^M (0, I_B)) > 0$, $V_A > 0$, and $I_B > 0$.

The optimal investment under expected utility, $I^e$, satisfies $c' (I^e) = e^{\frac{1}{2} (\theta_0 - \theta_1)}$ (Theorem 2). If $V_A = I^e$, it follows that $I_B^{M,a} = I^e$, because $\tilde{\theta}^{M,a} (\Pi^M (0, I^e)) = \theta^e$, so $c' (I^e) = e^{\theta_0 - \theta^e} = e^{\frac{1}{2} (\theta_0 - \theta_1)}$. Because

$$\frac{d I_B^{M,a}}{d V_A} > 0, I_B^{M,a} > I^e$$

when $V_A > I^e$ and $I_B^{M,a} < I^e$ when $V_A < I^e$. Therefore, the insider overinvests in diversifying projects if $V_A > I^e$ but underinvests if $V_A < I^e$.

Because the insider always underinvests in focused projects, and invests with distortions a.s. in diversifying projects, the SEU outsiders refuse to delegate control to her. ■

**Proof of Corollary 2.** If outsiders delegate control to the insider, their payoff is

$$U_0^{S,d} = e^{\theta_0 - \theta_1} \left[ w_A + (1 - \alpha) \left( V_A + \frac{1}{2} I_A^{M,a} \right) \right] + e^{\theta_0 - \theta^e} \left[ w_B + (1 - \alpha) \frac{1}{2} I_B^{M,a} \right] + (1 - \alpha) \left[ V_0 - \frac{1}{2} c (I_A^{M,a}) - \frac{1}{2} c (I_B^{M,a}) \right].$$

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If they retain control, however, their payoff is

\[ U_0^{S,r} = e^{\theta_e - \theta_1} \left[ w_A + (1 - \alpha) \left( V_A + \frac{1}{2} I_A^e \right) \right] + e^{\theta_0 - \theta_e} \left[ w_B + (1 - \alpha) \frac{1}{2} I_B^e \right] + (1 - \alpha) \left[ V_0 - \frac{1}{2} c(I_A^e) - \frac{1}{2} c(I_B^e) \right]. \]

If the insider is exogenously granted control, the impact on outsider’s utility is \( \Delta = U_0^{S,d} - U_0^{S,r} \), which simplifies to

\[ \Delta = \frac{1}{2} (1 - \alpha) \left[ \left( \rho \left( I_A^{M,a} \right) - \rho (I_A^e) \right) + \left( \rho \left( I_B^{M,a} \right) - \rho (I_B^e) \right) \right] \]

where \( \rho(I) = e^{\frac{1}{2}(\theta_0 - \theta_1)} I - c(I) \), the outsiders’ payoff from investing \( I \) in either project. Note \( \rho'(I) = e^{\frac{1}{2}(\theta_0 - \theta_1)} - c'(I) \) and \( \rho''(I) = -c''(I) \), so \( \rho \) is concave because \( c \) is convex. Because \( c'(I^e) = e^{\frac{1}{2}(\theta_0 - \theta_1)} \), \( I^e \) maximizes \( \rho \), so \( \Delta \) is strictly negative. Further, \( I^e \) does not depend on \( V_A \). \( I_A^{M,a} \) does not depend on \( V_A \), because \( I_A^{M,a} \) satisfies \( c'(I_A^{M,a}) = e^{\theta_0 - \theta_1} \). Thus, \( \left[ \rho \left( I_A^{M,a} \right) - \rho (I_A^e) \right] \) does not depend on \( V_A \). Similarly, \( \rho (I_B^e) \) does not depend on \( V_A \). Theorem 3 showed that \( I_B^{M,a} \) is increasing in \( V_A \), and that \( I_B^{M,a} = I^e \) when \( V_A = I^e \). Thus, an increase in \( V_A \) increases \( \Delta \) when \( V_A < I^e \) but decreases \( \Delta \) when \( V_A > I^e \), resulting in the inverted U-shaped relationship. ■

**Proof of Lemma 4.** With a focused project, an investment level of \( I_A \) provides outsiders with the portfolio \( \Pi^S(I_A,0) = \{ w_A + (1 - \alpha) (V_A + I_A), w_B, (1 - \alpha) [V_0 - c(I_A)] \} \), which provides them with utility

\[ U_1^S(\Pi^S(I_A,0)) = \min_{\theta \in C} \mathbb{E} \left[ u(\Pi^S(I_A,0)) ; \theta \right], \]

where \( E [u(\Pi^S(I_A,0)) ; \theta] = e^{\theta_e - \theta_1} [w_A + (1 - \alpha) (V_A + I_A)] + e^{\theta_0 - \theta} w_B + (1 - \alpha) [V_0 - c(I_A)] \). Beliefs are given by Lemma 2:

\[ \theta^{S,a}(\Pi^S(I_A,0)) = \begin{cases} \hat{\theta}_0 & \hat{\theta}^{S,a}(\Pi^S(I_A,0)) \leq \hat{\theta}_0 \\ \hat{\theta}_1 & \hat{\theta}^{S,a}(\Pi^S(I_A,0)) \in (\hat{\theta}_0, \hat{\theta}_1) \\ \hat{\theta}_2 & \hat{\theta}^{S,a}(\Pi^S(I_A,0)) \geq \hat{\theta}_2 \end{cases} \]

where

\[ \hat{\theta}^{S,a}(\Pi^S(I_A,0)) = \theta^e + \frac{1}{2} \ln \left[ \frac{w_B}{w_A + (1 - \alpha) (V_A + I_A)} \right]. \]

Applying the minimax theorem \( \left( \frac{\partial U^S_{I_A}}{\partial \theta} \right) \times \frac{\partial \theta^{S,a}}{\partial I_A} = 0 \), so \( \frac{dU^S_{I_A}}{dI_A} = \frac{\partial U^S_{I_A}}{\partial I_A} \),

\[ \frac{d}{dI_A} U_1^S(\Pi^S(I_A,0)) = (1 - \alpha) \left[ e^{\theta^{S,a} - \theta_1} - c'(I_A) \right] \]

Thus, outsiders choose \( I_A^{S,a} \) so that \( c'(I_A^{S,a}) = e^{\theta^{S,a}(\Pi^S(I_A,0)) - \theta_1} \). Because \( w_A + (1 - \alpha) V_A = w_B \), it follows that \( w_A + (1 - \alpha) [V_A + I_A^{S,a}] > w_B \) for \( I_A^{S,a} > 0 \). Thus, \( \theta^{S,a} < \theta^e \), which implies \( I_A^{S,a} < I^e \).

With a diversifying project, an investment level of \( I_B \) provides outsiders with the portfolio \( \Pi^S(0, I_B) = \)
\{w_A + (1 - \alpha) V_A, w_B + (1 - \alpha) I_B, (1 - \alpha) [V_0 - c(I_B)]\}, which provides them with utility

\[U_1^S (\Pi^S (0, I_B)) = \min_{\theta \in C} \mathbb{E} \left[ u(\Pi^S (0, I_B)) \right],\]

where \(E \left[u(\Pi^S (0, I_B))\right] = e^{\theta_1} [w_A + (1 - \alpha) V_A] + e^{\theta_0 - \theta} [w_B + (1 - \alpha) I_B] + (1 - \alpha) [V_0 - c(I_B)].\)

Beliefs are given by Lemma 2:

\[
\tilde{\theta}^{S,a} (\Pi^S (0, I_B)) = \left\{ \begin{array}{ll}
\hat{\theta}_0 & \tilde{\theta}^{S,a} (\Pi^S (0, I_B)) \leq \hat{\theta}_0 \\
\hat{\theta}_1 & \tilde{\theta}^{S,a} (\Pi^S (0, I_B)) \in (\hat{\theta}_0, \hat{\theta}_1) \\
\tilde{\theta}^{S,a} (\Pi^S (0, I_B)) \geq \hat{\theta}_1
\end{array} \right.
\]

where

\[
\tilde{\theta}^{S,a} (\Pi^S (0, I_B)) = \theta^e + \frac{1}{2} \ln \left[ \frac{w_B + (1 - \alpha) I_B}{w_A + (1 - \alpha) V_A} \right].
\]

Applying the minimax theorem,

\[
\frac{dU_1^S}{dI_B} = (1 - \alpha) \left[ e^{\theta_0 - \theta^{S,a} (\Pi^S (0, I_B))} - c' (I_B) \right]
\]

This implies that \(I_B\) is chosen so that \(c' (I_B) = e^{\theta_0 - \theta^{S,a} (\Pi^S (0, I_B))}\). Because \(w_B = w_A + (1 - \alpha) V_A\), it follows that \(w_B + (1 - \alpha) I_B > w_A + (1 - \alpha) V_A\) for \(I_B > 0\). Thus, \(\theta^{S,a} (\Pi^S (0, I_B)) > \theta^e\), so \(I_B^{S,a} < I^c\).

Therefore, ambiguity-averse outsiders underinvest in both focused and diversifying projects.

To show that \(I_A^{S,a} = I_B^{S,a}\), note that \(\theta^{S,a} (\Pi^S (I, 0)) - \theta_1 = \theta_0 - \theta^{S,a} (\Pi^S (0, I))\) for all \(I\) because the outsider is diversified a priori, \(w_B = w_A + (1 - \alpha) V_A\). Thus, the pessimism effect is identical for focused and diversifying projects.

Finally, we will show that underinvestment is more severe at large firms (relative to outsiders’ portfolio) by showing the equivalent claim – underinvestment is less severe when outsiders’ portfolio is larger. Let \(K = w_B = w_A + (1 - \alpha) V_A\). Suppose ambiguity-averse outsiders are faced with a focused project: their portfolio-distorted beliefs are given by

\[
\tilde{\theta}^{S,a} (\Pi^S (I_A, 0)) = \theta^e + \frac{1}{2} \ln \left[ \frac{K}{K + (1 - \alpha) I_A} \right].
\]

Focused investment by ambiguity-averse outsiders satisfies \(c' (I_A^{S,a}) = e^{\theta^{S,a} (\Pi^S (I_A^{S,a}, 0)) - \theta_1}\). Totally differentiating with respect to \(K\) and rearranging,

\[
\left[ c'' (I_A^{S,a}) - e^{\theta^{S,a} (\Pi^S (I_A^{S,a}, 0)) - \theta_1} \frac{\partial \theta^{S,a} (\Pi^S (I_A^{S,a}, 0))}{\partial I_A^{S,a}} \right] \frac{dI_A^{S,a}}{dK} = e^{\theta^{S,a} (\Pi^S (I_A^{S,a}, 0)) - \theta_1} \frac{\partial \theta^{S,a} (\Pi^S (I_A^{S,a}, 0))}{\partial K}.
\]

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If \( \theta^{S,a} \) is a corner solution, then
\[
\frac{\partial \theta^{S,a}(\Pi^S(I^{S,a}_A,0))}{\partial K} = \frac{\partial \theta^{S,a}(\Pi^S(I^{S,a}_A,0))}{\partial I^{S,a}_A} = 0,
\]
so \( \frac{dI^{S,a}}{dK} = 0 \). If \( \theta^{S,a} \) is interior, then
\[
\theta^{S,a} = \hat{\theta}^{S,a},
\]
so
\[
\frac{\partial \theta^{S,a}(\Pi^S(I^{S,a}_A,0))}{\partial I^{S,a}_A} = -\frac{1-\alpha}{2K+(1-\alpha)I_A},
\]
and
\[
\frac{\partial \theta^{S,a}(\Pi^S(I^{S,a}_A,0))}{\partial K} = \frac{(1-\alpha)I_A}{2K[1+(1-\alpha)I_A]}.
\]
Because
\[
\frac{\partial \theta^{S,a}(\Pi^S(I^{S,a}_A,0))}{\partial I^{S,a}_A} < 0 < \frac{\partial \theta^{S,a}(\Pi^S(I^{S,a}_A,0))}{\partial K},
\]
this implies that \( \frac{dI^{S,a}}{dK} > 0 \). Note that \( I^{S,a}_A < I^e \), and \( I^e \) does not depend on \( K \), because \( I^e \) satisfies \( c'(I^e) = e^{\frac{1}{2}(\theta_0-\theta_1)} \). Thus, underinvestment is less severe when \( K \) is larger, and underinvestment is more severe when \( K \) is smaller. This is equivalent to the firm size result, because a large firm will be more important to the portfolio of its owners (the diversifying portfolio will be smaller). Identical results hold for diversifying projects, \( \frac{dI^e}{dK} > 0 \), by similar proof. ■

**Proof of Lemma 5.** Suppose outsiders know they will not know which type of project the firm draws, but they anticipate that investment of \( I_A \) and \( I_B \) will be implemented. Thus, their MEU is
\[
U^S_1(\Pi^S(I_A^{1\tau=A},I_B^{1\tau=B}) = \min_{\theta \in C} E \left[ u(\Pi^S(I_A^{1\tau=A},I_B^{1\tau=B})) \right],
\]
where
\[
E \left[ u(\Pi^S(I_A^{1\tau=A},I_B^{1\tau=B})) \right] = \frac{1}{2} E \left[ u(\Pi^S(I_A,0)) \right] + \frac{1}{2} E \left[ u(\Pi^S(0,I_B)) \right]
\]
\[
= e^{\theta-\theta_1} \left[ w_A + (1-\alpha) \left( V_A + \frac{1}{2} I_A \right) \right]
\]
\[
+ e^{\theta_0-\theta} \left[ w_B + (1-\alpha) \frac{1}{2} I_B \right] + (1-\alpha) \left[ V_0 - \frac{1}{2} c(I_A) - \frac{1}{2} c(I_B) \right].
\]
Define
\[
\theta^{S,a}(\Pi^S(I_A^{1\tau=A},I_B^{1\tau=B})) = \arg \min_{\theta \in C} E \left[ u(\Pi^S(I_A^{1\tau=A},I_B^{1\tau=B})) \right] \theta
\]
As shown in Lemma 2,
\[
\theta^{S,a}(\Pi^S(I_A^{1\tau=A},I_B^{1\tau=B})) = \begin{cases} \hat{\theta}_0 & \theta^{S,a}(\Pi^S(I_A^{1\tau=A},I_B^{1\tau=B})) \leq \hat{\theta}_0 \\ \hat{\theta}_1 & \theta^{S,a}(\Pi^S(I_A^{1\tau=A},I_B^{1\tau=B})) \in (\hat{\theta}_0,\hat{\theta}_1) \\ \tilde{\theta}^{S,a}(\Pi^S(I_A^{1\tau=A},I_B^{1\tau=B})) & \theta^{S,a}(\Pi^S(I_A^{1\tau=A},I_B^{1\tau=B})) \geq \tilde{\theta}_1 \end{cases}
\]
where
\[
\tilde{\theta}^{S,a}(\Pi^S(I_A^{1\tau=A},I_B^{1\tau=B})) = \theta^e + \frac{1}{2} \ln \left[ \frac{w_B + (1-\alpha) \frac{1}{2} I_B}{w_A + (1-\alpha) \left( V_A + \frac{1}{2} I_A \right)} \right].
\]
Applying the minimax theorem, their first-order conditions of optimality are
\[
\frac{\partial U^S_1}{\partial I_A} = \frac{1}{2} (1-\alpha) \left[ e^{\theta^{S,a}-\theta_1} - c'(I_A) \right],
\]
\[
\frac{\partial U^S_1}{\partial I_B} = \frac{1}{2} (1-\alpha) \left[ e^{\theta_0-\theta^{S,a}} - c'(I_B) \right].
\]
This implies that investment levels satisfy \( c'(I_A) = e^{\theta^{S,a}-\theta_1} \) and \( c'(I_B) = e^{\theta_0-\theta^{S,a}} \). Suppose \( I_A > I_B \). Because outsiders are assumed to be diversified ex ante \( (w_B = w_A + (1-\alpha)V_A) \), this implies \( w_A + (1-\alpha) \left[ V_A + \frac{1}{2} I_A \right] > w_B + (1-\alpha) \frac{1}{2} I_B \). Therefore, \( \theta^{S,a}(\Pi^S(I_A^{1\tau=A},I_B^{1\tau=B})) < \theta^e \), so
\[ e^{\theta S^a - \theta_1} < e^{\theta_0 - \theta S^a}. \] This implies, however, that \( c'(I_A) < c'(I_B) \), which implies that \( I_A < I_B \). Contradiction. Therefore, \( I_A \leq I_B \). We can show similarly that \( I_B \leq I_A \). Thus, \( I_A = I_B \), which implies \( \theta S^a (\Pi^S(I_A^{1\tau = A}, I_B^{1\tau = B})) = \theta^d \). Therefore, outsiders would like to commit to an efficient level of investment: \( I_A = I^e \) and \( I_B = I^e \). ■

**Proof of Theorem 4.** Lemma 1 shows that exposure to information harms an ambiguity-averse outsider. Lemma 4 demonstrates that an ambiguity-averse outsider will underinvest, both relative to first best and to what they would like to commit to ex ante by Lemma 5. By Theorem 2, a SEU insider chooses investment optimally, setting \( I_A = I_B = I^e \). Thus, the outsider protects herself from ambiguity and achieves efficient investment by delegating control to the insider. ■

**Proof of Corollary 3.** For this proof, we make the simplifying assumption that \( c(I) = \frac{1}{Z(1+\gamma)} I^{1+\gamma} \), so the value of growth options are determined by the variable \( Z \). Outsiders’ payoff from delegation is, from equation 28,

\[
U_0^{S,r}(I_A^{S,a}, I_B^{S,a}) = \frac{1}{2} U_1^S(\Pi^S(I_A^{S,a}, 0)) + \frac{1}{2} U_1^S(\Pi^S(0, I_B^{S,a})).
\]

We can apply the minimax theorem and envelope theorem\(^2^3\), so that the effect of a change in \( Z \) is only the direct effect:

\[
\frac{dU_0^{S,r}(I_A^{S,a}, I_B^{S,a})}{dZ} = \frac{1}{2} \left[ \frac{\partial U_1^S(\Pi^S(I_A^{S,a}, 0))}{\partial Z} + \frac{\partial U_1^S(\Pi^S(0, I_B^{S,a}))}{\partial Z} \right].
\]

Note \( \frac{\partial U_1^S(\Pi^S(I_A^{S,a}, 0))}{\partial Z} = -(1-\alpha) \frac{\partial c(I_A^{S,a})}{\partial I_A^{S,a}} \) and \( \frac{\partial U_1^S(\Pi^S(0, I_B^{S,a}))}{\partial Z} = -(1-\alpha) \frac{\partial c(I_B^{S,a})}{\partial I_B^{S,a}} \). Because \( c(I) = \frac{1}{Z(1+\gamma)} I^{1+\gamma} \), \( \frac{\partial c(I)}{\partial Z} = -\frac{1}{Z^2(1+\gamma)} I^{1+\gamma} = -\frac{1}{Z} c(I) \), so

\[
\frac{dU_0^{S,r}(I_A^{S,a}, I_B^{S,a})}{dZ} = \frac{1}{2} (1-\alpha) \frac{1}{Z} c(I_A^{S,a}) + \frac{1}{2} (1-\alpha) \frac{1}{Z} c(I_B^{S,a}).
\]

Under delegation, the outsiders’ payoff is given by

\[
U_0^{S,d}(I_A^{M,a}, I_B^{M,a}) = \min_{I_A^{M,a}, I_B^{M,a}} \mathbb{E} \left[ u(\Pi^S(I_A^{M,a}, I_B^{M,a}) \theta) \right]
\]

where

\[
\mathbb{E} \left[ u(\Pi^S(I_A^{M,a}, I_B^{M,a}) \theta) \right] = e^{\theta - \theta_1} \left[ w_A + (1-\alpha) \left( V_A + \frac{1}{2} I_A^{M,a} \right) \right] + e^{\theta - \theta} \left[ w_B + (1-\alpha) \frac{1}{2} I_B^{M,a} \right] + (1-\alpha) \left[ V_0 - \frac{1}{2} c(I_A^{M,a}) - \frac{1}{2} c(I_B^{M,a}) \right].
\]

\(^2^3\)Because \( \theta \) is the worst-case scenario, \( \frac{\partial U_1^S}{\partial \theta} = 0 \) (for interior solutions, the first term is zero; for corner solutions, the second term is zero). By the envelope theorem, \( \frac{\partial \Pi^S}{\partial I_A} = 0 \) and \( \frac{\partial \Pi^S}{\partial I_B} = 0 \).
We can apply the minimax theorem but not the envelope theorem\(^{24}\), so
\[
\frac{dU_0^{S,d}(I_A^{M,a}, I_B^{M,a})}{dZ} = \frac{\partial U_0^{S,d}(I_A^{M,a}, I_B^{M,a})}{\partial I_A} \frac{dI_A^{M,a}}{dZ} + \frac{\partial U_0^{S,d}(I_A^{M,a}, I_B^{M,a})}{\partial I_B} \frac{dI_B^{M,a}}{dZ},
\]
where
\[
\frac{\partial U_0^{S,d}(I_A^{M,a}, I_B^{M,a})}{\partial I_A} = -\frac{1}{2} (1 - \alpha) \left[ \frac{\partial c(I_A^{M,a})}{\partial I_A} + \frac{\partial c(I_B^{M,a})}{\partial I_B} \right] = \frac{1}{2} (1 - \alpha) \frac{1}{2} \left[ c(I_A^{M,a}) + c(I_B^{M,a}) \right],
\]
\[
\frac{\partial U_0^{S,d}(I_A^{M,a}, I_B^{M,a})}{\partial I_B} = \frac{1}{2} (1 - \alpha) \left( \psi^{S}_{-\theta_0} - c'(I_A^{M,a}) \right) ,
\]
and
\[
\frac{\partial U_0^{S,d}(I_A^{M,a}, I_B^{M,a})}{\partial I_B} = \frac{1}{2} (1 - \alpha) \left( \psi^{S}_{-\theta_0} - c'(I_B^{M,a}) \right) .
\]

Because the insider is SEU, she sets \(I_A^{M,a} = I_B^{M,a} = I^e\) (Theorem 2). Balanced investment implies the outsider agrees with the insider:
\[
\theta^S = \theta^e = \frac{1}{2} \ln \left( \frac{w_B + (1 - \alpha) I_B^{M,a}}{w_A + (1 - \alpha) I_A^{M,a}} \right) = \theta^e, \text{ because } w_B = w_A + (1 - \alpha) V_A. \text{ Because investment is chosen optimally, } c'(I_A^{M,a}) = e^{\theta^e - \theta_0} \text{ and } c'(I_B^{M,a}) = e^{\theta^0 - \theta^e}, \text{ so }
\]
\[
\frac{\partial U_0^{S,d}(I_A^{M,a}, I_B^{M,a})}{\partial I_A} = \frac{\partial U_0^{S,d}(I_A^{M,a}, I_B^{M,a})}{\partial I_B} \text{ and }
\]
\[
\frac{dU_0^{S,d}(I_A^{M,a}, I_B^{M,a})}{dZ} = \frac{1}{2Z} \left[ c(I_A^{M,a}) + c(I_B^{M,a}) \right].
\]
Therefore, \(dU_0^{S,d}(I_A^{M,a}, I_B^{M,a}) = 0\), and
\[
\frac{dU_0^{S,d}(I_A^{M,a}, I_B^{M,a})}{dZ} = \frac{1}{2Z} \left[ c(I_A^{M,a}) + c(I_B^{M,a}) \right].
\]

The insider is SEU, \(I_A^{M,a} = I_B^{M,a} = I^e\), but outsiders underinvest ex post (Lemma 4), so \(I_A^{S,a} = I_B^{S,a} < I^e\). Thus, \(dU_0^{S,d}(I_A^{M,a}, I_B^{M,a}) > 0\), therefore, as growth options improve \((Z\text{ larger})\), delegation becomes more valuable. ■

**Proof of Theorem 5.** *Delegation optimal when \(V_A \in (V_A, \bar{V}_A)\).* To prove this result, we will show that the benefit of delegation, \(U_0^{S,d} - U_0^{S,r}\), is inverted U-shaped in \(V_A\) (holding \(w_A + (1 - \alpha) V_A\) constant), which immediately implies the result.\(^ {25}\) It is helpful to define \(I_B^{M,a}(V_A)\) as the diversifying investment by the insider when the value of the assets in place is \(V_A\). When \(V_A\) is small, \(V_A \leq V_A^1 = Z^{\frac{1}{2}} e^{2\theta^0 + \frac{1}{2} \theta_0} - \left( 2 + \frac{1}{2} \right) \theta_1\), the insider sets \(I_B^{M,a}\) so that \(c'(I_B^{M,a}) = e^{\theta^0 - \theta_1}\), or equivalently, \(I_B^{M,a} = \left[ Z e^{\theta^0 - \theta_1} \right]^{\frac{1}{2}}\). When \(V_A\) is large, \(V_B \geq V_A^2 = Z^{\frac{1}{2}} e^{2\theta^e + \frac{1}{2} \theta_0 - \left( 2 + \frac{1}{2} \right) \theta_0}\), the insider sets \(I_B^{M,a}\) so that \(c'(I_B^{M,a}) = e^{\theta^0 - \theta_0}\), or equivalently, \(I_B^{M,a} = \left[ Z e^{\theta^0 - \theta_0} \right]^{\frac{1}{2}}\). Therefore, \(I_B^{M,a}(V_A)\) is constant for all \(V_A \leq V_A^1\), and \(I_B^{M,a}(V_A)\) for all \(V_A \geq V_A^2\).

For \(V_A \in (V_A^1, V_A^2)\), \(I_B^{M,a}\) is chosen to that \(c'(I_B^{M,a}) = e^{\theta^0 - \theta^M}\), where \(\theta^M = \theta^e + \frac{1}{2} \ln \left[ \frac{\bar{V}_A}{V_A} \right]\), which implies \(I_B^{M,a} = Z^{\frac{1}{2}} e^{2\theta^0 - \left( \theta^0 - \theta^M \right)} V_A^{\frac{1}{2}}\). Therefore, \(I_B^{M,a}(V_A)\) is strictly increasing in \(V_A\) for \(V_A \in (V_A^1, V_A^2)\).

For this comparative static, we will increase \(V_A\) and decrease \(w_A\) so that \(w_A + (1 - \alpha) V_A\) remains constant.\(^ {26}\) Thus, let \(\tilde{w}_A = w_A - (1 - \alpha) \varepsilon\) and \(\bar{V}_A = V_A + \varepsilon\). From the proof of Corollary 3, \(\frac{\partial U_0^{S,r}}{\partial \varepsilon} = \frac{\partial U_0^{S,r}}{\partial \varepsilon}\), while
\[
\frac{\partial U_0^{S,d}}{\partial \varepsilon} = \frac{\partial U_0^{S,d}}{\partial I_A} \frac{dI_A^{M,a}}{dZ} + \frac{\partial U_0^{S,d}}{\partial I_B} \frac{dI_B^{M,a}}{dZ}. \text{ By construction, } \frac{\partial U_0^{S,d}}{\partial \varepsilon} = \frac{\partial U_0^{S,d}}{\partial \varepsilon} = 0. \text{ The insider chooses }
\]

\(^{24}\)By the minimax theorem, \(\frac{\partial U_0^{S,d}}{\partial \theta^S} = 0\). We cannot apply the Envelope Theorem under delegation, because the insider chooses investment optimally for herself, not the outsiders.

\(^{25}\)The proof does not require that delegation and retention are both optimal for some values of \(V_A\). For example, if other parameters are such that retention is optimal for all \(V_A\) (for example, very large \(K\) or very small \(Z\), the result holds by setting \(V_A = \bar{V}_A\). Alternatively, if other parameters are such that delegation is optimal for all \(V_A\) (for example, very large \(Z\)), the result holds by setting \(V_A = 0\) and \(\bar{V}_A = \infty\).

\(^{26}\)We will show numerically that the value of delegation is nonmonotonic in \(w_A\). See Figure 2.
\( I_A^{M,a} \) such that \( c' \left( I_A^{M,a} \right) = e^{\hat{\theta}_0 - \hat{\theta}_1} \), so \( I_A^{M,a} = \left[ Ze^{\hat{\theta}_0 - \hat{\theta}_1} \right]^{1/2} \), so \( I_A^{M,a} \) does not depend on \( V_A \): \( \frac{dI_A^{M,a}}{d\varepsilon} = 0 \).

As shown above, \( \frac{dI_B^{M,a}}{d\varepsilon} = 0 \) for \( V_A \leq V_A^1 \) and for \( V_A \geq V_A^2 \), but \( \frac{dI_B^{M,a}}{d\varepsilon} > 0 \) for \( V_A \in (V_A^1, V_A^2) \). Thus, \( \frac{dI_B^{M,a}}{d\varepsilon} = \frac{dU_0^{S,d}}{d\varepsilon} \), so \( \frac{dI_B^{M,a}}{d\varepsilon} > 0 \) if \( \frac{dU_0^{S,d}}{d\varepsilon} > 0 \).

\[
\frac{\partial U_0^{S,d}}{\partial I_B} = \frac{1}{2} (1 - \alpha) \left[ e^{\hat{\theta}_0 - \hat{\theta}_1} - c' \left( I_B^{M,a} \right) \right]
\]

The second line follows because \( c' \left( I_B^{M,a} \right) = e^{\hat{\theta}_0 - \hat{\theta}_1} \) because the insider chooses investment optimally according to her beliefs. For \( V_A \leq V_A^1 \), \( I_B^{M,a} = I_A^{M,a} \), so \( \theta^S = \theta^e \). Also, for \( V_A \leq V_A^1 \), \( \theta^M = \hat{\theta}_1 \). Thus, for \( V_A \leq V_A^1 \),

\[
\frac{\partial U_0^{S,d}}{\partial I_B} = \frac{1}{2} (1 - \alpha) \left[ e^{\hat{\theta}_0 - \hat{\theta}_1} - e^{\hat{\theta}_0 - \hat{\theta}_1} \right],
\]

so \( \frac{\partial U_0^{S,d}}{\partial I_B} > 0 \) for all \( V_A \leq V_A^1 \) (because \( e^{\hat{\theta}_0 - \hat{\theta}_1} \) is decreasing in \( \theta \) and \( \hat{\theta}_0 > \hat{\theta}_1 \)). By continuity, \( \frac{\partial U_0^{S,d}}{\partial I_B} > 0 \) when \( V_A = V_A^1 + \delta \) for small positive \( \delta \). For \( V_A \geq V_A^2 \), \( I_B^{M,a} = \left[ Ze^{\hat{\theta}_0 - \hat{\theta}_1} \right]^{1/2} > \left[ Ze^{\hat{\theta}_0 - \hat{\theta}_1} \right]^{1/2} = I_A^{M,a} \), so \( \theta^S > \theta^e \). Also, for \( V_A \geq V_A^2 \), \( \theta^M = \hat{\theta}_0 \). Thus, for \( V_A \geq V_A^2 \),

\[
\frac{\partial U_0^{S,d}}{\partial I_B} = \frac{1}{2} (1 - \alpha) \left[ e^{\hat{\theta}_0 - \hat{\theta}_1} - e^{\hat{\theta}_0 - \hat{\theta}_1} \right] < 0
\]

which follows because \( \theta^S > \theta^e > \hat{\theta}_0 \) and \( e^{\hat{\theta}_0 - \hat{\theta}_1} \) is decreasing in \( \theta \). By continuity, \( \frac{\partial U_0^{S,d}}{\partial I_B} < 0 \) when \( V_A = V_A^2 - \delta \) for small positive \( \delta \). Finally, for \( V_A \in (V_A^1, V_A^2) \), \( I_B^{M,a} (V_A) \) is strictly increasing in \( V_A \).

Outsiders have beliefs

\[
\theta^S = \theta^e + \frac{1}{2} \ln \left[ \frac{w_B + \frac{1}{2} (1 - \alpha) I_B^{M,a} (V_A)}{[w_A + (1 - \alpha) V_A]} + \frac{1}{2} (1 - \alpha) I_A^{M,a} \right],
\]

so \( \theta^S \) is increasing in \( V_A \) (because \( \frac{d}{d\varepsilon} [w_A + (1 - \alpha) V_A] = 0 \), \( \frac{dI_A^{M,a}}{d\varepsilon} = 0 \), and \( \frac{dI_B^{M,a}}{d\varepsilon} \geq 0 \)), which implies \( e^{\hat{\theta}_0 - \hat{\theta}_1} \) is decreasing in \( V_A \). The insider believes \( \theta^M = \theta^e + \frac{1}{2} \ln \left[ I_B^{M,a} (V_A) \right] ; I_B^{M,a} = Z_2^{2+\tau} e^{\frac{2}{2+\tau} \left( \theta^0 - \theta^e \right)} V_A^{2+\tau}, \)

so \( I_B^{M,a} = \left[ Z_2^{2+\tau} e^{\frac{2}{2+\tau} \left( \theta^0 - \theta^e \right)} V_A^{2+\tau} \right]^{1/2} \), which implies \( \frac{I_B^{M,a}}{V_A} \) is decreasing in \( V_A \); thus \( \theta^M \) is decreasing in \( V_A \) and \( e^{\hat{\theta}_0 - \hat{\theta}_1} \) is increasing in \( V_A \). Because \( \frac{\partial U_0^{S,d}}{\partial I_B} = \frac{1}{2} (1 - \alpha) \left[ e^{\hat{\theta}_0 - \hat{\theta}_1} - e^{\hat{\theta}_0 - \hat{\theta}_1} \right] \), this implies that \( \frac{\partial}{\partial V_A} \frac{\partial U_0^{S,d}}{\partial I_B} < 0 \). Thus, there exists a unique \( \tilde{V}_A \) such that \( \frac{\partial U_0^{S,d}}{\partial I_B} > 0 \) for \( V_A < \tilde{V}_A \) and \( \frac{\partial U_0^{S,d}}{\partial I_B} < 0 \) for \( V_A > \tilde{V}_A \). Therefore, \( \frac{dU_0^{S,d}}{d\varepsilon} > \frac{dU_0^{S,r}}{d\varepsilon} \) if \( V_A < \tilde{V}_A \). If \( U_0^{S,d} \mid _{V_A=\tilde{V}_A} > U_0^{S,r} \mid _{V_A=\tilde{V}_A} \), then define \( \tilde{V}_A \) such that \( U_0^{S,d} \mid _{V_A=\tilde{V}_A} = U_0^{S,r} \mid _{V_A=\tilde{V}_A} \) and \( \tilde{V}_A > \tilde{V}_A \). If \( U_0^{S,d} \mid _{V_A=\tilde{V}_A} \leq U_0^{S,r} \mid _{V_A=\tilde{V}_A} \), then define \( \tilde{V}_A = \tilde{V}_A = \tilde{V}_A \) and the claim trivially holds.

**Diversified outsiders retain control if** \( Z \leq Z \). When the project is small, the pessimism effect
disappears, but the insider invests inefficiently, so outsiders retain control. Because $U_{0}^{S,r}(0,0) = U_{0}^{S,d}(0,0)$, to show $U_{0}^{S,r} > U_{0}^{S,d}$ for all $Z \in (0, Z)$, it is sufficient to show that $\frac{dU_{0}^{S,r}}{dZ}|_{Z=\varepsilon} > \frac{dU_{0}^{S,d}}{dZ}|_{Z=\varepsilon}$ for small positive $\varepsilon$. From the proof of Corollary 3, $\frac{dU_{0}^{S,r}}{dZ} = -\frac{\partial U_{0}^{S,r}}{\partial Z}$, and $\frac{dU_{0}^{S,d}}{dZ} = \frac{1}{2} (1 - \alpha) \left[ \frac{1}{Z} c \left( I_{A}^{S,a} \right) + \frac{1}{2} c \left( I_{B}^{S,a} \right) \right]$. Because $c \left( I \right) = \frac{1}{Z(1+\gamma)} I^{1+\gamma}$, $c \left( I \right) = \frac{1}{Z(1+\gamma)} Z^\frac{1}{\gamma} \left[ c \left( I \right) \right]^{1+\gamma \gamma}$. Also, $c \left( I_{A}^{S,a} \right) = e^{\theta_{0} - \theta_{1}}$, and $c \left( I_{B}^{S,a} \right) = e^{\theta_{0} - \theta_{1}}$, so $\frac{\partial U_{0}^{S,r}}{\partial Z} = \frac{1}{2} (1 - \alpha) \left[ \frac{1}{Z} c \left( I_{A}^{S,a} \right) + \frac{1}{2} c \left( I_{B}^{S,a} \right) \right]$. As $Z \rightarrow 0$, $\theta^{S} \left( \Pi^{S} \left( I_{A}^{S,a}, 0 \right) \right) \rightarrow \theta^{e}$, so for sufficiently small $\varepsilon$,

$$\frac{\partial U_{0}^{S,r}}{\partial Z}|_{Z=\varepsilon} = \frac{1 - \alpha}{2 (1 + \gamma)} \varepsilon^{\frac{1}{\gamma} - 1} \left[ e^{\frac{1}{\gamma} (\theta^{e} - \theta_{1})} + e^{\frac{1}{\gamma} (\theta_{0} - \theta_{1})} \right].$$

Similarly, the proof of Corollary 3 shows

$$\frac{dU_{0}^{S,d}}{dZ} = \frac{\partial U_{0}^{S,d}}{\partial Z} + \frac{\partial U_{0}^{S,d}}{\partial I_{A}}|_{I_{A}=I_{A}^{M,a}} \frac{dI_{A}^{M,a}}{dZ} + \frac{\partial U_{0}^{S,d}}{\partial I_{B}}|_{I_{B}=I_{B}^{M,a}} \frac{dI_{B}^{M,a}}{dZ}$$

and $\frac{\partial U_{0}^{S,d}}{\partial Z} = \frac{1}{2} (1 - \alpha) \left[ \frac{1}{Z} c \left( I_{A}^{M,a} \right) + \frac{1}{2} c \left( I_{B}^{M,a} \right) \right]$. Because the insider is not diversified, $c \left( I_{A}^{M,a} \right) = e^{\theta_{0} - \theta_{1}}$, and, for sufficiently small $Z$, $c \left( I_{B}^{M,a} \right) = e^{\theta_{0} - \theta_{1}}$. Thus, for sufficiently small $\varepsilon$,

$$\frac{\partial U_{0}^{S,d}}{\partial Z}|_{Z=\varepsilon} = \frac{1 - \alpha}{2 (1 + \gamma)} \varepsilon^{\frac{1}{\gamma} - 1} \left[ e^{\frac{1}{\gamma} (\theta_{0} - \theta_{1})} + e^{\frac{1}{\gamma} (\theta_{0} - \theta_{1})} \right].$$

For indirect focused-investment effects, $\frac{\partial U_{0}^{S,d}}{\partial I_{A}}|_{I_{A}=I_{A}^{M,a}} = \frac{1}{2} (1 - \alpha) \left[ e^{\theta_{0} - \theta_{1}} - c \left( I_{A}^{M,a} \right) \right]$. Because $c \left( I_{A}^{M,a} \right) = e^{\theta_{0} - \theta_{1}}$, $I_{A}^{M,a} = Z^\frac{1}{\gamma} e^{\frac{1}{\gamma} (\theta_{0} - \theta_{1})}$, so $dI_{A}^{M,a} = \frac{1}{2\gamma} Z^\frac{1}{\gamma} - 1 e^{\frac{1}{\gamma} (\theta_{0} - \theta_{1})}$, and

$$\frac{\partial U_{0}^{S,d}}{\partial I_{A}}|_{I_{A}=I_{A}^{M,a}} \frac{dI_{A}^{M,a}}{dZ} = \frac{1 - \alpha}{2} \frac{1}{\gamma} Z^\frac{1}{\gamma} - 1 e^{\frac{1}{\gamma} (\theta_{0} - \theta_{1})}.$$  

As $Z$ gets small, $\theta^{S}$ approaches $\theta^{e}$, so for sufficiently small $\varepsilon$,

$$\frac{\partial U_{0}^{S,d}}{\partial I_{A}}|_{I_{A}=I_{A}^{M,a}} \frac{dI_{A}^{M,a}}{dZ}|_{Z=\varepsilon} = \frac{1 - \alpha}{2\gamma} \varepsilon^{\frac{1}{\gamma} - 1} \left[ e^{\theta_{0} - \theta_{1}} - e^{\theta_{0} - \theta_{1}} \right] e^{\frac{1}{\gamma} (\theta_{0} - \theta_{1})}.$$

For indirect diversified-investment effects, $\frac{\partial U_{0}^{S,d}}{\partial I_{B}}|_{I_{B}=I_{B}^{M,a}} = \frac{1}{2} (1 - \alpha) \left[ e^{\theta_{0} - \theta_{1}} - c \left( I_{B}^{M,a} \right) \right]$. For sufficiently small $Z$, the insider sets $I_{B}^{M,a} < e^{2(\theta_{0} - \theta_{1}) \Pi^{S}}$, so $c \left( I_{B}^{M,a} \right) = e^{\theta_{0} - \theta_{1}}$, which implies $I_{B}^{M,a} = Z^\frac{1}{\gamma} e^{\frac{1}{\gamma} (\theta_{0} - \theta_{1})}$, so $dI_{B}^{M,a} = \frac{1}{2\gamma} Z^\frac{1}{\gamma} - 1 e^{\frac{1}{\gamma} (\theta_{0} - \theta_{1})}$, and

$$\frac{\partial U_{0}^{S,d}}{\partial I_{B}}|_{I_{B}=I_{B}^{M,a}} \frac{dI_{B}^{M,a}}{dZ} = \frac{1 - \alpha}{2} \frac{1}{\gamma} Z^\frac{1}{\gamma} - 1 e^{\frac{1}{\gamma} (\theta_{0} - \theta_{1})}.$$  

As $Z$ gets small, $\theta^{S}$ approaches $\theta^{e}$, so for sufficiently small $\varepsilon$,

$$\frac{\partial U_{0}^{S,d}}{\partial I_{B}}|_{I_{B}=I_{B}^{M,a}} \frac{dI_{B}^{M,a}}{dZ}|_{Z=\varepsilon} = \frac{1 - \alpha}{2\gamma} \varepsilon^{\frac{1}{\gamma} - 1} \left[ e^{\theta_{0} - \theta_{1}} - e^{\theta_{0} - \theta_{1}} \right] e^{\frac{1}{\gamma} (\theta_{0} - \theta_{1})}.$$
therefore, for sufficiently small positive ε,

\[
\frac{dU_0^{S,r}}{dZ} \big|_{Z=\varepsilon} = \frac{1 - \alpha}{2(1 + \gamma)} \varepsilon^{\frac{1}{\gamma}} - 1 \left[ e^{\frac{1}{\gamma}(0^e-\theta_1)} + e^{\frac{1}{\gamma}(\theta_0-\theta_1)} \right],
\]

and

\[
\frac{dU_0^{S,d}}{dZ} \big|_{Z=\varepsilon} = \frac{1 - \alpha}{2(1 + \gamma)} \varepsilon^{\frac{1}{\gamma}} - 1 \left[ e^{\frac{1}{\gamma}(0^e-\theta_1)} + e^{\frac{1}{\gamma}(\theta_0-\theta_1)} \right] + \frac{1 - \alpha}{2\gamma} \varepsilon^{\frac{1}{\gamma}} - 1 \left[ e^{\theta_0-\theta^e} - e^{\theta_0-\theta_0} \right] e^{\frac{1}{\gamma}(\theta_0-\theta)}.
\]

Define \(\phi(\theta)\) so that

\[
\phi(\theta) = \frac{1}{1 + \gamma} \left[ e^{\frac{1}{\gamma}(\theta^e-\theta_1)} + e^{\frac{1}{\gamma}(\theta_0-\theta)} \right] + \frac{1}{\gamma} \left[ e^{\theta_0-\theta} - e^{\theta_0-\theta_1} \right] e^{\frac{1}{\gamma}(\theta_0-\theta_1)} + \frac{1}{\gamma} \left[ e^{\theta_0-\theta^e} - e^{\theta_0-\theta} \right] e^{\frac{1}{\gamma}(\theta_0-\theta)}.
\]

Note \(\frac{dU_0^{S,r}}{dZ} \big|_{Z=\varepsilon} = \frac{1 - \alpha}{2} \varepsilon^{\frac{1}{\gamma}} - 1 \phi(\theta^e)\) and \(\frac{dU_0^{S,d}}{dZ} \big|_{Z=\varepsilon} = \frac{1 - \alpha}{2} \varepsilon^{\frac{1}{\gamma}} - 1 \phi(\theta_0)\). It is sufficient to show \(\phi(\theta^e) > \phi(\theta_0)\). With a little rearranging, it follows that

\[
\phi'(\theta) = \frac{1}{\gamma} \left[ e^{\theta^e-\theta_1} - e^{\theta_0-\theta_1} \right] e^{\frac{1}{\gamma}(\theta_0-\theta_1)} - \frac{1}{\gamma^2} \left[ e^{\theta_0-\theta^e} - e^{\theta_0-\theta} \right] e^{\frac{1}{\gamma}(\theta_0-\theta)}.
\]

Thus, \(\phi'(\theta) > 0\) for all \(\theta \in \left(\theta_0, \theta^e\right)\), so \(\phi(\theta^e) > \phi(\theta_0)\). Therefore, \(\frac{dU_0^{S,r}}{dZ} \big|_{Z=\varepsilon} > \frac{dU_0^{S,d}}{dZ} \big|_{Z=\varepsilon}\) for all sufficiently small positive \(\varepsilon\). Thus, \(U_0^{S,r} > U_0^{S,d}\) for \(Z\) close to zero: equivalently, there exists \(Z\) such that \(U_0^{S,r} > U_0^{S,d}\) for all \(Z < Z\).

**Outsiders delegate control when \(Z > \bar{Z}\).** We will show that, when growth options are sufficiently large, equilibrium beliefs of outsiders and the insider coincide, so equilibrium investment will be the same. By Lemma 1, outsiders delegate control to the insider. Consider the equilibrium beliefs of the outsider who retains control and is faced with a focused project.

\[
\hat{\theta}^S \left( \Pi^S \left( I_A, 0 \right) \right) = \theta^e + \frac{1}{2} \ln \left[ \frac{w_B}{w_A + (1 - \alpha)(V_A + I_A)} \right].
\]

Note that \(\theta^S = \hat{\theta}_0\) iff \(\hat{\theta}^S \leq \hat{\theta}_0\) iff

\[
I_A \geq \tilde{I}_A^S \triangleq \frac{1}{1 - \alpha} \left[ e^{2(\theta^e-\hat{\theta}_0)} w_B - w_A - (1 - \alpha) V_A \right].
\]

Thus, if outsiders invest sufficiently, they will agree with the insider (because \(\theta^M = \hat{\theta}_0\) as shown in Theorem 3). It is optimal to set investment this large when \(\phi' \left( \tilde{I}_A \right) \leq \phi(\theta^e_1)\). Under specification \(c(I) = \frac{1}{Z(1+\gamma)} I^{(1+\gamma)}\), this is equivalent to \(Z \geq e^{\theta_1-\theta_0} \left\{ \frac{1}{1-\alpha} \left[ e^{2(\theta^e-\hat{\theta}_0)} w_B - w_A - (1 - \alpha) V_A \right] \right\}^\gamma \).
Similarly, with a diversifying project,

$$\hat{\theta}^S(\Pi^S(0, I_B)) = \theta^e + \frac{1}{2} \ln \left[ \frac{w_B + (1 - \alpha) I_B}{w_A + (1 - \alpha) V_A} \right].$$

Note that $\theta^S = \hat{\theta}_1$ iff $\hat{\theta}^S \geq \hat{\theta}_1$ iff

$$I_B \geq \hat{I}_B^S \triangleq \frac{1}{1 - \alpha} \left\{ e^{2(\hat{\theta}_1^0)} [w_A + (1 - \alpha) V_A] - w_B \right\}.$$

This is optimal iff $c' \left( \hat{I}_B \right) \leq e^{\theta_0 - \hat{\theta}_1}$. Under specification $c(I) = \frac{1}{Z(1 + \gamma)} I^{1 + \gamma}$, this is equivalent to

$$Z \geq e^{\hat{\theta}_1 - \theta_0} \left\{ e^{2(\hat{\theta}_1 - \theta^e)} [w_A + (1 - \alpha) V_A] - w_B \right\}^{\gamma}.$$

Under assumptions of symmetry, that $w_B = w_A + (1 - \alpha) V_A$ and $\hat{\theta}_1 - \theta^e = \theta^e - \hat{\theta}_0$, investment cutoffs are the same: $\hat{I}_A = \hat{I}_B$ (so are the cutoffs for $Z$). When growth options are sufficiently large, outsiders invest according to the worst-case scenario for the type of project drawn: $\theta^S(\Pi^S(I_A, 0)) = \hat{\theta}_0$ and $\theta^S(\Pi^S(0, I_B)) = \hat{\theta}_1$ for $I_A \geq \hat{I}_A^S$ and $I_B \geq \hat{I}_B^S$.

The insider always invests in focused projects according to the worst-case scenario: $\theta^M(\Pi^I(I_A, 0)) = \hat{\theta}_0$. Her portfolio-distorted beliefs for the diversifying project are given by $\hat{\theta}^M(\Pi^I(0, I_B)) = \theta^e + \frac{1}{2} \ln \frac{I_B}{V_A}$. $\theta^M = \hat{\theta}_1$ iff $I_B \geq e^{2(\hat{\theta}_1 - \theta^e)} V_A$. It is optimal for her to set $I_B \geq e^{2(\hat{\theta}_1 - \theta^e)} V_A$ iff $c' \left( e^{2(\hat{\theta}_1 - \theta^e)} V_A \right) \leq e^{\theta_0 - \hat{\theta}_1}$, or under our specification, $Z \geq e^{\hat{\theta}_1 - \theta_0} \left( e^{2(\hat{\theta}_1 - \theta^e)} V_A \right)^{\gamma}$. Therefore, when growth options are sufficiently big (in our specification, $Z$ is big enough), the insider will invest the same as the outsider. By Lemma 1, delegation is strictly preferred.

**Diversified outsider retain control if $K \geq \bar{K}$**. When diversified outsiders have a sufficiently large portfolio, they will always want control. To see this, suppose $w_A = K - (1 - \alpha) V_A$ and $w_B = K$. Thus,

$$\hat{\theta}^S(\Pi^S(I_A, 0)) = \theta^e + \frac{1}{2} \ln \left[ \frac{K}{K + (1 - \alpha) I_A} \right];$$

$$\hat{\theta}^S(\Pi^S(0, I_B)) = \theta^e + \frac{1}{2} \ln \left[ \frac{K + (1 - \alpha) I_B}{K} \right];$$

$$\hat{\theta}^S(\Pi^S(I_A \tau = A, I_B \tau = A)) = \theta^e + \frac{1}{2} \ln \left[ \frac{K + \frac{1}{2} (1 - \alpha) I_B}{K + \frac{1}{2} (1 - \alpha) I_A} \right].$$

As $K \to \infty$, all of these converge to $\theta^e$. That is, as $K$ gets large, the outsiders’ worst case scenario converges to $\theta^{S, a} = \theta^e$ for either project. Because the worst-case scenario is not moving around, they do not fear the ambiguity in the limit (Lemma 1 implies strict preference only when the worst-case scenario is not constant).

Thus, when the outsiders’ outside portfolio is sufficiently large, they will always want control. □

**Proof of Corollary 4.** Define $\Delta = U^{S,d}_0 - U^{S,r}_0$ as the value of delegation. Holding everything else constant, define $\bar{Z}(V_A)$ as $Z$ from Theorem 5: $\Delta > 0$ for all $Z > Z(V_A)$ and $\Delta < 0$ for $Z = Z(V_A) - \varepsilon$ for small positive $\varepsilon$. This implies that $\frac{\partial \Delta}{\partial \varepsilon}|_{Z=Z(V_A)} > 0$. By definition of $Z(V_A)$, $\Delta$ is uniformly zero as a function of $V_A$. Totally differentiating $\Delta$ with respect to $V_A$, $\frac{\partial \Delta}{\partial V_A} = \frac{\partial \Delta}{\partial V_A} + \frac{\partial \Delta}{\partial Z} \frac{dZ(V_A)}{dV_A}$. As shown in Theorem

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5. $\frac{\partial \Delta}{\partial V_A} = 0$ for $V_A < V_A^1$, $\frac{\partial \Delta}{\partial V_A} > 0$ for $V_A \in (V_A^1, \hat{V}_A)$, $\frac{\partial \Delta}{\partial V_A} < 0$ for $V_A \in (\hat{V}_A, V_A^2)$, and $\frac{\partial \Delta}{\partial V_A} = 0$ for $V_A > V_A^2$. This implies that $\frac{dZ(V_A)}{dV_A} = 0$ for $V_A < V_A^1$, $\frac{dZ(V_A)}{dV_A} < 0$ for $V_A \in (V_A^1, \hat{V}_A)$, $\frac{dZ(V_A)}{dV_A} > 0$ for $V_A \in (\hat{V}_A, V_A^2)$, and $\frac{dZ(V_A)}{dV_A} = 0$ for $V_A > V_A^2$. Thus, $Z(V_A)$ is U-Shaped in $V_A$. ■

Proof of Corollary 5. The result for outsiders follow from Lemma 4, while the results for the insider follows from Theorem 3. ■

Proof of Corollary 6. The value of delegation is $U_{0}^{S,d} - U_{0}^{S,r}$. To show that Point 1 holds, consider increasing both $w_A$ and $w_B$ by a small amount. From the proof of Corollary 3,

$$
\frac{dU_{0}^{S,d}}{dw_A} = \frac{\partial U_{0}^{S,d}}{\partial w_A} + \frac{\partial U_{0}^{S,d}}{\partial I_A} \frac{dI_A}{dw_A} + \frac{\partial U_{0}^{S,d}}{\partial I_B} \frac{dI_B}{dw_A}.
$$

The outside portfolio of the outsider does not affect the investment decisions of the insider, so

$$
\frac{dI_A}{dw_A} = \frac{dI_B}{dw_A} = 0.
$$

Therefore, $\frac{dU_{0}^{S,d}}{dw_A} = \frac{\partial U_{0}^{S,d}}{\partial w_A}$. Further, $\frac{\partial U_{0}^{S,d}}{\partial w_A} = e^{\theta S}\left(I_A^{M,a} + I_B^{M,a} + 1\right)^{-\theta}$. Similarly,

$$
\frac{dU_{0}^{S,d}}{dw_B} = \frac{\partial U_{0}^{S,d}}{\partial w_B} = e^{\theta S}\left(I_A^{M,a} + I_B^{M,a} + 1\right)^{-\theta} - \theta_1 + e^{\theta S}\left(I_A^{M,a} + I_B^{M,a} + 1\right)^{-\theta_1}.
$$

Under retention, because utility is defined recursively,

$$
U_{0}^{S,r} = \frac{1}{2} \left[ U_{1}^{S,a}\left( I_A^{S,a} + I_B^{S,a} \right) + U_{1}^{S,a}\left( I_A^{S,a} + I_B^{S,a} \right) \right].
$$

From the proof of Corollary 3, $\frac{dU_{1}^{S,a}}{dw_A} = \frac{\partial U_{1}^{S,a}}{\partial w_A}$; $\frac{\partial U_{1}^{S,a}}{\partial w_A} = e^{\theta S}\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta_1}$ and $\frac{\partial U_{1}^{S,a}}{\partial w_B} = e^{\theta S}\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta_1}$. So,

$$
\frac{dU_{0}^{S,r}}{dw_A} = \frac{1}{2} \left[ e^{\theta S}\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta_1} + e^{\theta S}\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta_1} \right].
$$

Similarly,

$$
\frac{dU_{0}^{S,r}}{dw_B} = \frac{1}{2} \left[ e^{\theta S}\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta_1} + e^{\theta S}\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta_1} \right].
$$

Thus, the total impact of increasing $w_A$ and $w_B$ each a little is

$$
\frac{dU_{0}^{S,d}}{dw_A} + \frac{dU_{0}^{S,d}}{dw_B} = \frac{1}{2} \left[ e^{\theta S}\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta_1} + e^{\theta S}\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta_1} \right] + \frac{1}{2} \left[ e^{\theta S}\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta_1} + e^{\theta S}\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta_1} \right].
$$

Define $g(\theta) = e^{\theta_1} + e^{\theta_0}$. Thus,

$$
\frac{dU_{0}^{S,d}}{dw_A} + \frac{dU_{0}^{S,d}}{dw_B} = g\left(\theta S\left(I_A^{M,a} + I_B^{M,a} + 1\right)^{-\theta} \right) + \frac{dU_{0}^{S,r}}{dw_A} + \frac{dU_{0}^{S,r}}{dw_B} = \frac{1}{2} \left[ g\left(\theta S\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta} \right) + g\left(\theta S\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta} \right) \right].
$$

It can be easily verified that $g$ is symmetric around $\theta$. By Lemma 2 (setting $\hat{w}_A = \hat{w}_B = 1$), $g$ is convex and achieves its minimum at $\theta = \theta^e$. Because outsiders are diversified a priori, $w_A = w_A + (1 - \alpha) V_A$, $I_A^{S,a} = I_B^{S,a}$ by Corollary 5. It can quickly be verified that $\theta^e - \theta^S\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta_1} = \theta^S\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta_1} - \theta^e$ (because the core of beliefs is symmetric: $\theta_1 - \theta^e = \theta^e - \theta_0$). By symmetry of $g$, this implies $g\left(\theta S\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta} \right) = g\left(\theta S\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta} \right)$. Therefore, $\frac{dU_{0}^{S,d}}{dw_A} + \frac{dU_{0}^{S,d}}{dw_B} = g\left(\theta S\left(I_A^{M,a} + I_B^{M,a} + 1\right)^{-\theta} \right)$ and $\frac{dU_{0}^{S,d}}{dw_A} + \frac{dU_{0}^{S,d}}{dw_B} = g\left(\theta S\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta} \right)$. By Corollary 5, $I_A^{M,a} \geq I_A^{M,a}$, so $\theta^S\left(I_A^{M,a} + I_B^{M,a} + 1\right)^{-\theta} \geq \theta^e$. Further, $I_B^{S,a} > 0$ so $\theta^S\left(I_A^{S,a} + I_B^{S,a} + 1\right)^{-\theta} \geq \theta^e$. Because $g$ achieves its minimum at $\theta^e$, $g$ is increasing in $\theta$ for $\theta \geq \theta^e$. Therefore, $\frac{dU_{0}^{S,d}}{dw_A} + \frac{dU_{0}^{S,d}}{dw_B} \geq \frac{dU_{0}^{S,d}}{dw_A} + \frac{dU_{0}^{S,d}}{dw_B}$. Thus, $Z(V_A)$ is U-Shaped in $V_A$. ■
\[ g\left(\theta^S \left(\Pi^S \left(I_{A}^{M,a}1_A, I_{B}^{M,a}1_B\right)\right)\right) \text{ which holds iff } \tilde{\theta}^S \left(\Pi^S \left(0, I_{B}^{S,a}\right)\right) \geq \tilde{\theta}^S \left(\Pi^S \left(I_{A}^{M,a}1_A, I_{B}^{M,a}1_B\right)\right). \]

Recall that

\[ \tilde{\theta}^S \left(\Pi^S \left(0, I_{B}^{S,a}\right)\right) = \theta^e + \frac{1}{2} \ln \left[ \frac{w_B + (1 - \alpha) I_{B}^{S,a}}{w_A + (1 - \alpha) V_A} \right] \]

and

\[ \tilde{\theta}^S \left(\Pi^S \left(I_{A}^{M,a}1_A, I_{B}^{M,a}1_B\right)\right) = \theta^e + \frac{1}{2} \ln \left[ \frac{w_B + (1 - \alpha) \frac{1}{2} I_{B}^{M,a}}{w_A + (1 - \alpha) \left(V_A + \frac{1}{2} I_{B}^{M,a}\right)} \right]. \]

Thus, \( \tilde{\theta}^S \left(\Pi^S \left(0, I_{B}^{S,a}\right)\right) \geq \tilde{\theta}^S \left(\Pi^S \left(I_{A}^{M,a}1_A, I_{B}^{M,a}1_B\right)\right) \text{ iff} \)

\[ \frac{w_B + (1 - \alpha) \frac{1}{2} I_{B}^{M,a}}{w_A + (1 - \alpha) \left(V_A + \frac{1}{2} I_{B}^{M,a}\right)} \leq \frac{w_B + (1 - \alpha) I_{B}^{S,a}}{w_A + (1 - \alpha) V_A}. \]

After some algebra, it can be shown that this holds iff

\[ I_{B}^{M,a} \leq 2I_{B}^{S,a} + \frac{w_B + (1 - \alpha) I_{B}^{S,a}}{w_A + (1 - \alpha) V_A} I_{A}^{M,a}. \] (33)

In numerical simulations, we have never found an occasion when (33) failed to hold. Note that (33) is satisfied if \( I_{B}^{M,a} \leq 2I_{B}^{S,a} \), which is guaranteed to hold if \( \gamma \geq \gamma \equiv \frac{\theta_1 - \theta_0}{\ln 2} \), where \( \gamma \) controls the curvature of the cost function: \( c(I) = \frac{1}{2(1 + \gamma)} I^{1+\gamma} \). Thus, Point 1 is proven: increasing the outside portfolio size makes retention more attractive. By having a larger outside portfolio, ambiguity is lessened.

For Point 2, the claim is that the value of delegation, \( U_{0}^{S,d} - U_{0}^{S,r} \), is decreasing in \( w_B \), or equivalently, that \( \frac{dt_{0}^{S,r}}{dw_B} > \frac{dt_{0}^{S,d}}{dw_B} \). The claim will be shown by proving that \( \frac{dt_{0}^{S,r}}{dw_B} > e^{\theta_0 - \theta^e} \geq \frac{dt_{0}^{S,d}}{dw_B} \). The impact of \( w_B \) on the payoff under retention is \( \frac{dU_{0}^{S,r}}{dw_B} = \frac{1}{2} \left[ e^{\theta_0 - \theta^e \left(\Pi^S \left(I_{B}^{S,a},0\right)\right)} + e^{\theta_0 - \theta^e \left(\Pi^S \left(0,I_{B}^{S,a}\right)\right)} \right] \) (from above). Note that \( e^{\theta_0 - \theta^e} \) is decreasing and convex in \( \theta \). Because \( \theta^S \left(\Pi^S \left(I_{A}^{S,a},0\right)\right) \leq \theta^e \leq \theta^S \left(\Pi^S \left(0,I_{B}^{S,a}\right)\right) \) and \( \theta^e - \theta^S \left(\Pi^S \left(0,I_{B}^{S,a}\right)\right) = \theta^S \left(\Pi^S \left(0,I_{B}^{S,a}\right)\right) - \theta^e \), or equivalently,

\[ \frac{1}{2} \left[ \theta^S \left(\Pi^S \left(0,I_{B}^{S,a}\right)\right) + \theta^S \left(\Pi^S \left(I_{A}^{S,a},0\right)\right) \right] = \theta^e. \]

Thus, \( \frac{dt_{0}^{S,r}}{dw_B} > e^{\theta_0 - \theta^e} \) by convexity of \( e^{\theta_0 - \theta^e} \). The impact of \( w_B \) on the payoff under delegation is \( \frac{dt_{0}^{S,d}}{dw_B} = e^{\theta_0 - \theta^S \left(\Pi^S \left(I_{A}^{M,a}1_A, I_{B}^{M,a}1_B\right)\right)} \) (from above). By the Corollary 5, \( I_{B}^{M,a} \geq I_{A}^{M,a} \), which implies that \( \theta^S \left(\Pi^S \left(I_{A}^{M,a}1_A, I_{B}^{M,a}1_B\right)\right) \geq \theta^e \) by Corollary 1. Thus, \( \frac{dt_{0}^{S,d}}{dw_B} \leq e^{\theta_0 - \theta^e} \) because \( e^{\theta_0 - \theta^e} \) is decreasing in \( \theta \). Therefore, \( \frac{dt_{0}^{S,r}}{dw_B} > e^{\theta_0 - \theta^e} \geq \frac{dt_{0}^{S,d}}{dw_B} \), so \( \frac{dt_{0}^{S,r}}{dw_B} > \frac{dt_{0}^{S,d}}{dw_B} \).

For Point 3, under retention, \( U_{0}^{S,r} = \frac{1}{2} U_{1}^{S,r} + \frac{1}{2} I_{B}^{S,a} \). By the proof of Corollary 3, \( \frac{dU_{0}^{S,r}}{dz} = \frac{\partial U_{0}^{S,r}}{\partial I_{A}^{S,a}} \) and \( \frac{dU_{0}^{S,r}}{dz} = \frac{\partial U_{0}^{S,r}}{\partial I_{B}^{S,a}} \). Also, \( \frac{\partial U_{0}^{S,r}}{\partial I_{A}^{S,a}} = (1 - \alpha) \frac{1}{2} c \left(I_{A}^{S,a}\right) \) and \( \frac{\partial U_{0}^{S,r}}{\partial I_{B}^{S,a}} = (1 - \alpha) \frac{1}{2} c \left(I_{B}^{S,a}\right) \) because \( \frac{\partial c}{\partial z} c(I) = -\frac{1}{2} c(I) \). Therefore, the total impact of a change in \( Z \) on the outsiders’ payoff is

\[ \frac{dU_{0}^{S,r}}{dz} = (1 - \alpha) \frac{1}{2} \left[ c\left(I_{A}^{S,a}\right) + c\left(I_{B}^{S,a}\right) \right]. \]
Under delegation, by the proof of Corollary 3,
\[
dU_0^{S,d} \frac{dZ}{dZ} = \frac{\partial U_0^{S,d}}{\partial Z} + \frac{\partial U_0^{S,d}}{\partial I_A} \frac{dI_A^{M,a}}{dZ} + \frac{\partial U_0^{S,d}}{\partial I_B} \frac{dI_B^{M,a}}{dZ}.
\]

The insider chooses investment optimally, \( c' \left( I_A^{M,a} \right) = e^{\hat{\theta}_0 - \hat{\theta}_1} \), so
\[
\frac{\partial U_0^{S,d}}{\partial I_A} \bigg|_{I_A = I_A^{M,a}} = -\frac{1}{2} (1 - \alpha) \left[ e^{\theta_S - \hat{\theta}_1} - e^{\hat{\theta}_0 - \hat{\theta}_1} \right].
\]
Because \( \theta_S \geq \hat{\theta}_0 \) (usually with strict inequality),
\[
\frac{\partial U_0^{S,d}}{\partial I_A} \bigg|_{I_A = I_A^{M,a}} = \frac{1}{2} (1 - \alpha) \left[ e^{\theta_S - \hat{\theta}_1} - e^{\hat{\theta}_0 - \hat{\theta}_1} \right] > 0.
\]
Further, \( I_A^{M,a} = \left( Z e^{\hat{\theta}_0 - \hat{\theta}_1} \right) \), so \( \frac{dI_A^{M,a}}{dZ} = \frac{1}{\gamma} e^{\hat{\theta}_0 - \hat{\theta}_1} Z^{\gamma - 1} < 0 \). Therefore,
\[
\frac{\partial U_0^{S,d}}{\partial I_A} \frac{dI_A^{M,a}}{dZ} > 0.
\]
Similarly, \( \frac{\partial U_0^{S,d}}{\partial I_B} \bigg|_{I_B = I_B^{M,a}} = \frac{1}{2} (1 - \alpha) \left[ e^{\theta_S - \hat{\theta}_1} - e^{\hat{\theta}_0 - \hat{\theta}_1} \right] \). We cannot sign \( \frac{\partial U_0^{S,d}}{\partial I_B} \), but we can say that for \( Z \) big enough, it is strictly positive (see below).

It can be quickly verified that \( I_B^{M,a} \) is increasing in \( Z \), which implies \( \theta^M \) is increasing in \( Z \). As \( Z \) goes to zero, investment goes to zero, so \( \theta^S \left( \Pi^S \left( I_A^{M,a}, I_B^{M,a} \right) \right) \) approaches \( \theta^e \). For small \( Z \), 
\[
\theta^M \left( \Pi^M \left( 0, I_B^{M,a} \right) \right) = \hat{\theta}_0 \text{ (type B assets are an insignificant portion of her portfolio)}.
\]
Thus, for small values of \( Z \), \( e^{\theta_0 - \theta^S} < e^{\theta_0 - \theta^M} \), so \( \frac{\partial U_0^{S,d}}{\partial I_B} < 0 \). When \( Z \) gets big enough, \( I_A^{M,a} \geq e^{2(\hat{\theta}_1 - \theta^e)} V_A \), so \( \theta^M \left( \Pi^M \left( 0, I_B^{M,a} \right) \right) = \hat{\theta}_1 \), and \( I_B^{M,a} = I_B^{M,a} \), because \( c' \left( I_A^{M,a} \right) = e^{\theta_0 - \theta^S} - e^{\theta_0 - \theta^M} \). We cannot sign \( \frac{\partial U_0^{S,d}}{\partial I_B} \), but for sufficiently large \( Z \), \( Z \geq e^{\hat{\theta}_1(\gamma - 1) - \theta_0(\gamma - 1) - \theta_1 V_A^1} \). Thus,
\[
\theta^S \left( \Pi^S \left( I_A^{M,a}, I_B^{M,a} \right) \right) = \theta^e \text{ and } \theta^M \left( \Pi^M \left( 0, I_B^{M,a} \right) \right) = \hat{\theta}_1 \text{ for } Z \geq e^{\hat{\theta}_1(\gamma - 1) - \theta_0(\gamma - 1) - \theta_1 V_A^1}.
\]

Therefore, \( \frac{\partial U_0^{S,d}}{\partial I_B} > 0 \) and, for sufficiently large \( Z \), \( \frac{\partial U_0^{S,d}}{\partial I_A} > 0 \). Further,
\[
\frac{\partial U_0^{S,d}}{\partial Z} = \frac{1}{2} \left[ c \left( I_A^{M,a} \right) + c \left( I_B^{M,a} \right) \right],
\]
so the total impact of an increase in \( Z \) on outsiders’ utility under delegation is
\[
\frac{dU_0^{S,r}}{dZ} = \frac{1}{2} \left[ c \left( I_A^{M,a} \right) + c \left( I_B^{M,a} \right) \right] \frac{dI_A^{M,a}}{dZ} + \frac{\partial U_0^{S,d}}{\partial I_A} \frac{dI_A^{M,a}}{dZ} + \frac{\partial U_0^{S,d}}{\partial I_B} \frac{dI_B^{M,a}}{dZ}.
\]

\( \frac{\partial U_0^{S,d}}{\partial I_A} > 0 \) and \( \frac{\partial U_0^{S,d}}{\partial I_B} > 0 \) for \( Z \) large enough. The total impact of an increase in \( Z \) on outsider’s utility is
\[
\frac{dI_A^{M,a}}{dZ} = (1 - \alpha) \frac{1}{2Z} \left[ c \left( I_A^{S,a} \right) + c \left( I_B^{S,a} \right) \right].
\]
\( I_j^{k,a} \) is increasing in \( Z \) for all \( j \in \{A, B\} \) and \( k \in \{S, M\} \), yet as \( Z \) gets big, any party in control would invest \( I_j^{k,a} = I_{\min} \) where \( c' \left( I_{\min} \right) = e^{\hat{\theta}_0 - \hat{\theta}_1} \). Thus, for \( Z \) very large,
\[
\frac{dI_A^{S,a}}{dZ} \text{ equals the first term of } \frac{dI_A^{M,a}}{dZ},
\]
and the other two terms are strictly positive. Thus, there exists a \( \hat{Z} \) such that \( \frac{dU_0^{S,a}}{dZ} > \frac{dU_0^{S,r}}{dZ} \) for all \( Z > \hat{Z} \).

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27Note \( I_B^{M,a} \) satisfies \( \phi \left( I_B^{M,a} \right) = 0 \) where \( \phi \left( I, \theta, Z \right) = e^{\theta_0 - \theta^M} - c' \left( I_B^{M,a} \right) \). The result follows by totally differentiating \( \phi \) w.r.t. \( Z \).

28We do not need \( Z \) to be so big that \( I_B^{M,a} \geq e^{2(\hat{\theta}_1 - \theta^e)} V_A \). For intermediate values of \( Z \), \( I_B^{M,a} \geq I_A^{M,a} \), so \( \theta^S \geq \theta^e \). Depending on the value of \( Z \), \( \theta^M \in \left( \hat{\theta}_0, \hat{\theta}_1 \right) \), but \( \theta^M \) is strictly increasing in \( Z \). Thus, there is a \( \hat{Z} \) such that \( \frac{\partial U_0^{S,d}}{\partial I_B} > 0 \) for all \( Z > \hat{Z} \). Because \( \theta^S \) is inverted U-shaped in \( Z \), we cannot say that there is a unique \( Z \) such that \( \frac{\partial U_0^{S,d}}{\partial I_B} = 0 \).