Recent Advances in Estimating Term-Structure Models

David A. Chapman and Neil D. Pearson

In the past 10 years, increasingly sophisticated statistical techniques have been applied to the estimation of increasingly complex models of the term structure of interest rates. In reviewing this literature, we highlight the facts that have been established and the key unresolved issues. The data indicate that within a wide range of interest rates, mean reversion in rates is, at best, weak. Whether mean reversion is stronger for very high or very low levels of rates is an unresolved issue. The absolute volatility of rates increases as the level of rates increases, but the strength of this effect and the role and nature of either stochastic-volatility or regime-switching components in rates are still unclear. Unfortunately, these unresolved issues have important implications for fixed-income option pricing and risk measurement, including value-at-risk calculations.

Models of the term structure of interest rates are widely used in pricing interest rate derivatives and instruments with embedded options, such ascallable bonds and mortgage-backed securities. Many such models are based on the simplifying assumption that changes in interest rates of all maturities are driven by changes in a single underlying random factor, often taken to be the “short” or “instantaneous” rate of interest. These one-factor models include the well-known models of Vasicek (1977), Cox, Ingersoll, and Ross (1985; hereafter, CIR), Black, Derman, and Toy (1990), Black and Karasinski (1991), and Hull and White (1990); the models of Brennan and Schwartz (1979) and Courtadon (1982); the translated CIR model in Pearson and Sun (1994); and the empirical models estimated by Chan, Karolyi, Longstaff, and Sanders (1992; hereafter, CKLS). These and similar models differ in how they model expected changes in interest rates and the volatility of changes in rates; in particular, they differ in how they assume expected changes and volatilities of interest rates are related to their levels.

Certain decisions are crucial in modeling interest rate behavior. The first crucial decision is whether to use a one-factor model. In a one-factor model, the choices of the conditional mean and the conditional volatility functions are also important. The evolution of interest rates over time, and thus the prices of options and other derivatives, is determined entirely by these choices. Models of interest rate volatility (sometimes implicit) also play key roles in risk measurement—in value-at-risk (VAR) calculations, for example.

Unfortunately, theory provides little guidance about the modeling choices. As a result, the last decade has seen the development of a large and growing academic literature devoted to estimating how expected changes and volatilities of interest rates are related to their levels and, sometimes, other variables. This literature is scattered in different places and often emphasizes the statistical and econometric techniques used rather than the implications of the analysis for interest rate models. Therefore, it is not easily accessible to many practitioners. This article is the first step in remedying this problem. We summarize in one place the substantive implications of the recent academic literature for dynamic models of the evolution of interest rates.

Much of the recent academic literature we review focuses on one-factor models, but researchers have known since Litterman and Scheinkman (1991) that at least three factors are needed to fully capture the variability of interest rates. Why then consider one-factor models? As documented in both Litterman and Scheinkman and in the next section, roughly 90 percent of the variation in U.S. Treasury rates can be explained by the first factor, which can be interpreted as corresponding to changes in the general level of interest rates. Thus, any relationship between the level of interest rates and their expected

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changes and volatilities will be dominated by the influence of this first factor. In one-factor models, this factor is typically identified with the instantaneous or short rate of interest. The recent academic literature studying the behavior of the yields on short-term bonds or deposits can be interpreted as a detailed explanation of the first factor by using the yield on a particular instrument (e.g., one-month LIBOR) as a proxy for the short rate.

What can be learned from this large and growing literature? What model features appear to be essential in describing the fundamental properties of interest rates? First, this new literature does not provide conclusive evidence, based solely on the data, about whether interest rate levels tend to return to a constant long-run level and, if they do, whether this tendency is stronger for extreme levels of interest rates.

With respect to interest rate volatility, the level of the "absolute" volatility of the short rate, defined as the standard deviation of rate changes scaled by the square root of the time between changes, is clearly increasing. Inferences about the relationship between the level and volatility of the short rate are sensitive, however, to the treatment of the years between 1979 and 1982, the U.S. Federal Reserve's so-called experiment in targeting monetary aggregates rather than targeting interest rate levels. In particular, the data from the 1979–82 period suggest a strong relationship between volatility and the level of interest rates, whereas if this period is excluded or treated as a distinct regime with a lower probability of occurring, a much weaker relationship is suggested.

Finally, modeling the volatility of interest rates requires more than a simple "level effect"; that is, some sort of stochastic volatility apparently affects rates. But the additional volatility component can be described adequately (in a statistical sense) in a variety of competing ways.

In practice, term-structure models are often implemented by calibrating them to the prices of some subset of the traded instruments (e.g., U.S. T-bonds, interest rate swaps, or perhaps interest rate options). Although most of the literature we review does not discuss calibration, it nonetheless has implications for it. In calibration, the drift (under the risk-neutral probability) of the short-rate process is typically chosen to match the term structure of interest rates and the volatility function may be chosen to match the volatilities of long-term bonds or the prices of certain options. Choosing the volatility function always involves specifying the form (e.g., constant, proportional, or square root) of the function that relates interest rate volatility to the level of interest rates. Researchers specify this form either directly, in the Heath, Jarrow, and Morton (1992) framework, or through the choice of one of the other well-known "named" models, such as the CIR model. Similarly, the calibration of the drift often involves specifying the form of any mean reversion; for example, in the CIR and Hull–White models, the drift is linear in the level of the interest rate, whereas in the Black–Derman–Toy and Black–Karasinski models, the drift of the log of the short rate is linear in the log. The literature we review provides guidance about which functional forms are reasonable; thus, it has direct implications for the choice of model and model calibration.

A Look at Some Data

We begin our review with a brief examination of the most important facts about the time-series and cross-sectional properties of interest rate data. A number of data series have been used in the recent literature on estimating short-rate models, including the yields on short-term Eurodollar deposits, short-term T-bills, and the Federal funds rate. Panel A of Figure 1 is a plot of the daily level of the one-month Eurodollar rate, and Panel B shows the monthly time series of the rate's volatility, with volatility computed as the standard deviation of the daily rate changes within the month. These plots are qualitatively similar to those of all the commonly used short-rate proxies.

The following stylized facts emerge from a brief examination of Figure 1:

- **Short-rate fact #1.** The short-rate series is a "persistent" time series; that is, it spends long, consecutive periods above and below the (sample estimate of the) unconditional, or long-run, mean.
- **Short-rate fact #2.** In the 1979–82 period, the average level and volatility of the short rate was substantially higher than for other years in the 1971–2000 period.
- **Short-rate fact #3.** The volatility of the short-rate level appears to be both time varying and persistent.

For comparison, Figure 2 plots the level (Panel A) and volatility (Panel B) of another interest rate data series—the five-year constant-maturity Treasury (CMT) yield from 1962 through April 2000. Although this series has a lower mean and volatility than the short rate shown in Figure 1, the overall movements in levels and volatility are qualitatively similar. In particular, the five-year rate is also a persistent series, and the 1979–82 period was characterized by substantially higher yield levels and volatility than any other years in the sample. These observations are also true for CMT yields of maturities from 1 year to 30 years.
Figure 1. One-Month Eurodollar Yield and Volatility, 1971–April 2000

A. Daily Level of Rate

Level (annual %)

25
20
15
10
5
0

1/71 5/73 1/82 1/92 4/00

B. Monthly Volatility of Rate

Volatility (annual %)

2.0
1.5
1.0
0.5
0

4/71 7/79 10/87 1/96 4/00

The strong contemporaneous correlations between rates of different maturities suggest that the rate changes are determined by a limited number of common factors. Litterman and Scheinkman confirmed this intuition by showing that in the period of January 1984 to June 1988, virtually all the variability in weekly yield changes for bonds of maturities from six months to 18 years can be explained by three linear combinations of yield changes—the first three “principal components,” which we will name shortly.4

Table 1 shows the explanatory power of the first three principal components for five subperiods between February 1977 and May 2000. Consistent with the results in Litterman and Scheinkman, 99 percent of the variation in yield changes is explained by these three common factors, and (roughly) 88 percent of the variation is a result of the first factor alone. Figure 3 shows the sensitivity of yield changes at different maturities to each of these three factors in each of the five subperiods. The results for all the subperiods are consistent over time and consistent with the results in Litterman and Scheinkman. Consequently, we are justified in using their terminology of “level” (Factor 1), “slope” (Factor 2), and “curvature” (Factor 3) for these factors.5 Because the first factor explains approximately 90 percent of the variability in yield changes, the statistical properties of yield changes—in particular, any relationships between the level of interest rates and their expected changes and volatilities—will be dominated by this first factor. Thus, even though at least three factors are needed to explain yield changes fully, the first factor merits special attention. And because Figure 3 shows that this first factor is a shift in the general level of interest rates, this first factor is a good approximation of changes in the level of any interest rate. So, we can interpret the contribution of the recent academic literature on the behavior of the yields on short-term bonds or deposits as providing a detailed examination of the first factor.

The results so far have related to the properties of yield levels, but another important aspect of interest rate behavior is the “term structure of yield volatilities,” the function that relates yield volatility to time to maturity. Table 2 shows the average monthly volatility, defined as the sample average of the daily yield changes within each month, for Treasury yields of maturities from 1 month to 10 years.
Figure 2. Five-Year Constant Maturity Treasury Yield and Volatility, January 1962–April 2000

Table 1. Average Explanatory Power of First Three Principal Components by Subperiod, 1977–2000

<table>
<thead>
<tr>
<th>Period</th>
<th>Total Variability Explained</th>
<th>Explained Variability by Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Factor 1</td>
</tr>
<tr>
<td>2/25/77–2/19/82</td>
<td>99.2%</td>
<td>88.5%</td>
</tr>
<tr>
<td>2/26/82–2/13/87</td>
<td>99.2%</td>
<td>87.6</td>
</tr>
<tr>
<td>2/20/87–2/14/92</td>
<td>99.0%</td>
<td>86.6</td>
</tr>
<tr>
<td>2/21/92–2/14/97</td>
<td>99.0%</td>
<td>89.6</td>
</tr>
<tr>
<td>2/21/97–5/19/00</td>
<td>98.7%</td>
<td>83.1</td>
</tr>
</tbody>
</table>

Note: The principal components were extracted from the weekly changes of 3- and 12-month T-bill yields and 5-, 10-, and 10-year CMT yields.

Over the entire period, the volatility of yields was decreasing for increasing terms to maturity. This pattern continues in the four five-year subperiods ending in 1983. In the last three subperiods, however, the average term structure of volatility is either increasing or hump shaped. The two largest upward-sloping volatility yield curves occurred after 1983, whereas the two steepest declining volatility yield curves occurred in 1981 and 1982. The slope of the volatility curve appears to be negatively correlated with the levels of rates.

The comovements of the monthly volatilities of yield changes shown in Table 2 are similar to the comovements of yield changes. The contemporaneous correlations of monthly volatility are more than 0.90 for adjacent maturities and never less than 0.70, even for 1-month versus 10-year yields. As with changes in the yield levels, the first principal component of monthly volatility explains more than 80 percent of the changes in volatility, and the first two principal components together explain more than 90 percent of the volatility in yield changes.
Figure 3. Principal Components Factor Loadings in Different Subperiods

**A. February 25, 1977–February 19, 1982**

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Yield Change (annual %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-Month</td>
<td>1.0</td>
</tr>
<tr>
<td>1-Year</td>
<td>0.5</td>
</tr>
<tr>
<td>3-Year</td>
<td>0.0</td>
</tr>
<tr>
<td>5-Year</td>
<td>-0.5</td>
</tr>
<tr>
<td>10-Year</td>
<td>-1.0</td>
</tr>
</tbody>
</table>

**B. February 26, 1982–February 13, 1987**

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Yield Change (annual %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-Month</td>
<td>1.0</td>
</tr>
<tr>
<td>1-Year</td>
<td>0.5</td>
</tr>
<tr>
<td>3-Year</td>
<td>0.0</td>
</tr>
<tr>
<td>5-Year</td>
<td>-0.5</td>
</tr>
<tr>
<td>10-Year</td>
<td>-1.0</td>
</tr>
</tbody>
</table>

**C. February 20, 1987–February 14, 1992**

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Yield Change (annual %)</th>
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</thead>
<tbody>
<tr>
<td>3-Month</td>
<td>1.0</td>
</tr>
<tr>
<td>1-Year</td>
<td>0.5</td>
</tr>
<tr>
<td>3-Year</td>
<td>0.0</td>
</tr>
<tr>
<td>5-Year</td>
<td>-0.5</td>
</tr>
<tr>
<td>10-Year</td>
<td>-1.0</td>
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</tbody>
</table>

**D. February 21, 1992–February 14, 1997**

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Yield Change (annual %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-Month</td>
<td>1.0</td>
</tr>
<tr>
<td>1-Year</td>
<td>0.5</td>
</tr>
<tr>
<td>3-Year</td>
<td>0.0</td>
</tr>
<tr>
<td>5-Year</td>
<td>-0.5</td>
</tr>
<tr>
<td>10-Year</td>
<td>-1.0</td>
</tr>
</tbody>
</table>

**E. February 21, 1997–May 19, 2000**

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Yield Change (annual %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-Month</td>
<td>1.0</td>
</tr>
<tr>
<td>1-Year</td>
<td>0.5</td>
</tr>
<tr>
<td>3-Year</td>
<td>0.0</td>
</tr>
<tr>
<td>5-Year</td>
<td>-0.5</td>
</tr>
<tr>
<td>10-Year</td>
<td>-1.0</td>
</tr>
</tbody>
</table>

**Expected Changes in Short-Term Rates**

We consider the relationships between the level of interest rates and their expected changes and volatilities within the context of a simple continuous-time model of the form

\[ dr_t = \mu(r_t) dt + \sigma(r_t) dW_t, \]

where
- \( r_t \) = a proxy for the short rate at time \( t \),
- \( \mu \) = the drift function,
- \( \sigma \) = the diffusion function, and
- \( W \) = a Brownian motion

Drift \( \mu \) and diffusion \( \sigma \) are the volatility functions. Equation 1 states that changes in the short-rate over arbitrarily short time intervals are determined completely by functions of the current short-rate level and a normally distributed shock. Yan (2001) provides an introduction to theoretical models of the term structure, including those based on a single factor that evolves according to Equation 1.

The short rate in Equation 1 is said to follow a diffusion process, and the mean and variance of changes in the process defined in Equation 1 are, respectively,

\[ E(r_{t+\Delta t} - r_t | r_t) = \mu(r_t) \Delta t \]
Table 2. Term Structure of Yield Maturity: Monthly Volatility of Yield Changes by Period, January 1962–April 2000

<table>
<thead>
<tr>
<th>Period</th>
<th>1-Month Maturity</th>
<th>1-Year Maturity</th>
<th>3-Year Maturity</th>
<th>5-Year Maturity</th>
<th>10-Year Maturity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1962–2000 average</td>
<td>0.0749</td>
<td>0.0701</td>
<td>0.0655</td>
<td>0.0616</td>
<td>0.0551</td>
</tr>
<tr>
<td>Subperiod average</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1964–68</td>
<td>0.0332</td>
<td>0.0261</td>
<td>0.0301</td>
<td>0.0254</td>
<td>0.0215</td>
</tr>
<tr>
<td>1969–73</td>
<td>0.0765</td>
<td>0.0578</td>
<td>0.0530</td>
<td>0.0475</td>
<td>0.0351</td>
</tr>
<tr>
<td>1974–78</td>
<td>0.0908</td>
<td>0.0748</td>
<td>0.0598</td>
<td>0.0500</td>
<td>0.0367</td>
</tr>
<tr>
<td>1979–83</td>
<td>0.1977</td>
<td>0.1838</td>
<td>0.1429</td>
<td>0.1310</td>
<td>0.1190</td>
</tr>
<tr>
<td>1984–88</td>
<td>0.0715</td>
<td>0.0745</td>
<td>0.0775</td>
<td>0.0782</td>
<td>0.0792</td>
</tr>
<tr>
<td>1989–93</td>
<td>0.0445</td>
<td>0.0545</td>
<td>0.0589</td>
<td>0.0581</td>
<td>0.0525</td>
</tr>
<tr>
<td>1994–98</td>
<td>0.0418</td>
<td>0.0489</td>
<td>0.0585</td>
<td>0.0599</td>
<td>0.0569</td>
</tr>
<tr>
<td>Steepest up months</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>June 1986</td>
<td>0.0674</td>
<td>0.0770</td>
<td>0.1060</td>
<td>0.1120</td>
<td>0.1254</td>
</tr>
<tr>
<td>March 1996</td>
<td>0.0366</td>
<td>0.0728</td>
<td>0.1057</td>
<td>0.1027</td>
<td>0.0960</td>
</tr>
<tr>
<td>Steepest down months</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>February 1982</td>
<td>0.4631</td>
<td>0.2632</td>
<td>0.1765</td>
<td>0.1684</td>
<td>0.1562</td>
</tr>
<tr>
<td>January 1981</td>
<td>0.4319</td>
<td>0.3535</td>
<td>0.2641</td>
<td>0.1830</td>
<td>0.1420</td>
</tr>
</tbody>
</table>

Note: The monthly volatility of yield changes was computed as the standard deviation of the daily yield changes within the month. The 1-month rate is for T-bills; the 1- through 10-year rates are for CMT notes.

and

\[ E[(r_{t+\Delta t} - r_t)^2 | r_t] = \sigma^2(r_t) \Delta t. \] (3)

The standard deviation of the change in interest rate \( r_{t+\Delta t} - r_t \) is approximately \( \sigma(r_t) \sqrt{\Delta t} \).

A discrete-time version of Equation 1 describes how the interest rate evolves from time \( t \) to time \( t + \Delta t \):

\[ r_{t+\Delta t} - r_t = \mu(r_t) \Delta t + \sigma(r_t) \sqrt{\Delta t} \epsilon_{t+\Delta t}, \] (4)

where \( \Delta t \) is a discrete increment of time (e.g., one day) and \( \epsilon_{t+\Delta t} \) is a standard normal random variable. In the limit, as \( \Delta t \to 0 \), Equations 2 and 3 become exact, which is the sense in which the process in Equation 4 approximates the continuous-time process (Equation 1).

Our focus in this section is on the drift, \( \mu_t \), or equivalently, the expected change, \( \mu \Delta t \). Simple models, such as those of Vasicek and CIR, specify a linear relationship:

\[ \mu(r_t) = \alpha_0 + \alpha_1 r_t, \] (5)

where \( \alpha_0 \) is greater than 0 and \( \alpha_1 \) is less than 0.

Combining Equation 4 and Equation 5 implies

\[ r_{t+\Delta t} = (1 + \alpha_1 \Delta t) r_t + \alpha_0 \Delta t + \sigma(r_t) \sqrt{\Delta t} \epsilon_{t+\Delta t}. \] (6)

The interpretation of Equation 6 is that after the passage of \( \Delta t \) time units (i.e., at \( t + \Delta t \)), the short rate is equal to a fraction \((1 + \alpha_1 \Delta t)\) of its value at time \( t \) plus a fixed constant, \( \alpha_0 \Delta t \), plus a (mean zero) random shock with standard deviation \( \sigma(r_t) \sqrt{\Delta t} \).

The parameter \( \alpha_1 \) determines the rate at which \( r_t \) returns to its long-run average value, which turns out to be \( -\alpha_0 / \alpha_1 \).

It may seem obvious, both from the plots in Figure 1 and from introspection, that the short rate must tend to return to some long-run average value. The alternative is that the variance of the distribution of the short rate increases without bound as we look farther and farther into the future. This implausible behavior occurs when \( \alpha_1 \) is greater than or equal to zero.\(^7\) The simple question of whether \( \alpha_1 \) is negative has been remarkably difficult to determine, however, by using statistical methods alone. For example, CKLS, using monthly observations of the one-month Treasury yield and various volatility specifications, estimated values of \( \alpha_1 \Delta t \) that ranged from \(-0.18\) to \(-0.59\) and were not statistically significantly different from zero. Ait-Sahalia (1996), using daily observations of a one-week Eurodollar rate, found \( \alpha_1 \Delta t \) in the range of \(-0.014\) to \(-0.038\).

Should we assume a linear drift? A linear drift implies that the strength of mean reversion is the same for all levels of the short rate. Is this implication reasonable or demonstrably false? In other words, is mean reversion stronger for extremely low or high levels of \( r_t \)? Ait-Sahalia constructed a general specification test of a diffusion model of the form of Equation 1 and found that the test rejected a linear drift in favor of models that imply virtually no mean reversion for levels of the short rate between 4 percent and 22 percent and strong mean reversion for extreme levels of the short rate. Stanton (1997) obtained results that are roughly consistent with
those in Aït-Sahalia for high levels of the interest rate, despite differences in the data series and estimation approaches taken in the two papers. Conley, Hansen, Luttmer, and Scheinkman (1997), hereafter CHLS, also examined the linearity of the drift, although they were focusing on the fundamental connection between estimates of the drift and estimates of the diffusion function or volatility (which is the subject of the next section). CHLS assumed that the volatility can be written as a nonlinear function of the short rate:

$$\sigma(r) = \beta_0 r^{\beta_1}.$$  \hspace{1cm} (7)

This specification generalizes some well-known short-rate models of the form of Equation 1. For example, if $\beta_1$ equals 0, then Equation 7 is the model in Vasicek; if $\beta_1$ equals 1/2, Equation 7 is the specification in the "square-root" process used by CIR. CHLS found evidence of stronger mean reversion for both very high and very low levels of the short rate if $\beta_1$ is in the range of (roughly) 1/2 to 1. However, as $\beta_1$ became larger (roughly in the range of 3 to 4)—as volatility became more sensitive to the level of rates—their evidence was that mean reversion is stronger only for large values of the short rate.

Unfortunately, as with most things in life, the evidence in the data is not conclusive. Pritsker (1998) found that the specification test developed in Aït-Sahalia is very sensitive to the value of the drift coefficient, $\alpha$, and is not reliable for values of this parameter that are close to zero (as in Aït-Sahalia’s data). Thus, the evidence in Aït-Sahalia provides only limited support for a nonlinear drift function.

In the Chapman and Pearson (2000) study, we conducted a Monte Carlo simulation of realizations of interest rate processes with linear drift and then applied the estimators of Aït-Sahalia and Stanton to the simulated data. The results of this experiment showed that even when the drift of the (simulated) process is linear, these methods commonly suggest that mean reversion is stronger for high levels of the short rate. The source of this bias is the fact that when the interest rate is near the sample maximum (minimum), the interest rate changes must generally be negative (positive); otherwise, the maximum (minimum) would not be the maximum (minimum). Furthermore, the estimates of the strength of mean reversion in the extremes of the data are imprecise. This problem occurs, in part, because these estimation approaches use only extreme observations in trying to measure the strength of mean reversion in the tails, and (by definition) extreme observations are few.

Elerian, Chib, and Shepherd (forthcoming 2001) and Jones (2000) also found the evidence on mean reversion in the short rate to be inconclusive. They used a Bayesian approach to argue that whether mean reversion is stronger in the extremes than for more common levels of the short rate depends critically on whether mean reversion is assumed in the prior distribution. Intuitively, the role of the prior distribution is more important in the final interpretation of the data when the sample is (effectively) small than when it is large.

In as yet unpublished work, Durham (2001) and Li, Pearson, and Poteshman (2001) found evidence that is explicitly inconsistent with the presence of nonlinearity in the drift function of various short-rate proxies. Using a simulated maximum-likelihood estimator, Durham estimated a variety of models and conducted likelihood-based specification tests. His evidence suggests that simpler drift (and diffusion) specifications are preferable to more flexible forms and that the drift function appears to be constant, with stationarity of the short rate induced by the behavior of the volatility of the series, as in CHLS. Li et al. implemented a moment-based estimator that explicitly accounts for the bias described in Chapman and Pearson. After conditioning on the sample maximum, they found no evidence that the coefficients determining nonlinearity are statistically significantly different from zero, and their specification tests failed to reject a linear drift function.

In summary, we do not know the exact nature of the short-rate drift. Some limited evidence suggests that mean reversion is stronger in the tails than in the bulk of the distribution of the short rate, but we cannot interpret this evidence with any confidence. And the evidence that explicitly rejects nonlinearity is still preliminary in nature. The heart of the problem is that drawing inferences from the relatively few observations in the extremes of the data is an inherently difficult task.

Perhaps a more fundamental question than “do we know whether mean reversion is stronger in the extremes” is “do we care.” Unfortunately, the answer to this question is an emphatic yes. The existence and strength of nonlinear mean reversion have important implications for the likelihood of extreme interest rate changes and for the distribution of interest rate changes over long time horizons. As a result, questions of reversion have significant implications for VAR calculations over long horizons and for asset/liability management.

Moreover, nonlinear mean reversion may also have implications for pricing long-term bond and interest rate options. For this purpose, only the drift based on the risk-neutral probability matters. The discussion up to this point has been exclusively about the drift based on the original (or “physical”) probability because it can be estimated from the data. Nonlinearity in the drift based on the physical probability need not imply, however, nonlinearity in the risk-neutral drift. If the market price of risk (the premium for bearing interest rate
risk) is also a nonlinear function of the interest rate, this aspect of the determinant of interest rates can offset the nonlinearity of the drift based on the physical probability, resulting in a linear risk-neutral drift. At the moment, however, no convincing evidence exists that the market price of risk has the required nonlinearity.

Furthermore, we might care about the drift based on the physical probability if misspecifying it results in biased estimates of the diffusion function. Getting the volatility “wrong” would provide a mechanism through which the issues raised above could affect issues of pricing and hedging. This argument is similar to that made in Lo and Wang (1995) about the impact of return predictability on the pricing of equity options. Thus, we care deeply about whether nonlinear mean reversion exists and, if so, its strength. We simply don’t know the answers.

Volatility of the Short Rate
The diffusion function in Equation 1 defines the volatility of the short rate as a function of the level of the short rate. The most common choice of functional form in the empirical literature is the “constant volatility elasticity” parameterization shown in Equation 7. For a positive \( \beta_0 \) and \( \beta_1 \) equal to 0, Equation 7 gives the constant volatility of the Vasicek model; for \( \beta_1 = 1/2 \), Equation 7 gives the square-root volatility of the CIR model; and for \( \beta_1 = 1 \), it gives the linear proportional volatility used by Dothan (1978), Brennan and Schwartz, and Courtemad. A cursory glance at Figure 1 suggests that the Vasicek model will be hard to reconcile with the sample paths of any short-rate series because it implies constant volatility per unit of time. The question, then, is: What value of \( \beta_1 \) is consistent with the data?

CKLS estimated a variety of models of the general form of Equation 1 with a drift function given by Equation 5 or by a special case of it and a diffusion function of the form of Equation 7. They concluded that a value of \( \beta_1 \) equal to 1.5 provides the best fit to the historical relationship between the level of the short rate and its volatility. With \( \beta_1 \) equal to 1.5, CKLS concluded that no credible evidence exists of a single, permanent structural break in the short-rate process in October 1979, the beginning of the Federal Reserve experiment. CHLS, using daily data on the Federal funds rate from January 1970 to January 1997 and ignoring the possibility of a structural break, concluded that the value of \( \beta_1 \) is between 1.5 and 2.0.\(^{10}\)

Is a \( \beta_1 \) equal to 1.5 economically reasonable? This question is inherently subjective, but consider that a \( \beta_1 \) value of 1.5 implies that a 1 percent change in the level of the short rate results in a 1.5 percent increase in its volatility. Bliss and Smith (1998) argued that the conclusion that \( \beta_1 \) equals 1.5 in CKLS is not robust to a misspecification of the form of the structural break present in the short-rate data. Bliss and Smith could not reject the hypothesis that October 1979 to September 1982 was a temporary break in the CKLS data. When this break was specified explicitly, the point estimate of \( \beta_1 \) dropped from 1.5 to 0.948.\(^{11}\) Bliss and Smith also found that the CIR model (with \( \beta_1 \) set equal to 1/2) and the Brennan and Schwartz model (with \( \beta_1 \) equal to 1) were no longer inconsistent with the CKLS data.

This conclusion was robust to both the possibility of influential outliers (i.e., extreme observations that might bias the overall parameter estimates) and to a variety of alternative short-rate data sets. Bliss and Smith concluded that the evidence in the time series of short-term rates is more consistent with a temporary break and a lower estimate of \( \beta_1 \) than with no (permanent) break and a higher value of \( \beta_1 \). Because the tests in CHLS also ignored the possibility of a structural break of any kind, the analysis in Bliss and Smith is relevant to the conclusions of that study.

Aït-Sahalia and Stanton estimated the shape of the diffusion function, \( \sigma(r_t) \), by using, respectively, a flexible functional form and fully nonparametric estimators. These studies did not allow for the possibility of a structural break in the data, and as a result, the shapes of their estimated diffusion functions are roughly consistent with the findings of CKLS and CHLS. These results are not conclusive, however, because the statistical significance of the Aït-Sahalia estimate is difficult to assess and the estimator in Stanton does not provide a precise estimate of the exact shape of the diffusion function for high levels of the short rate.

The conclusion from these studies is that dealing with the 1979–82 subperiod is an unavoidable problem in estimating the diffusion or volatility function in Equation 1. The decision to allow or not allow for a structural break can have a statistically and economically significant impact on the estimates of short-rate volatility. This conclusion raises two questions.

First, how much past data should one use in estimating term-structure models? In particular, should one use data from 1979–1982? Provided that all data were generated by the same process, the use of more data is preferred to less because it will reduce the sampling variation in the estimates. This argument is especially weighty in the case of 1979–1982 because it is the only period in which high U.S. dollar interest rates have been observed; thus, it is the only source of information about interest rate volatility when interest rates are high.
Disregarding these data forces one to estimate the volatility for high rates by extrapolating the relationship between the level and volatility of rates estimated for lower levels of rates.

The argument against using these data is that the Federal Reserve experiment and the “stagflation” of the 1970s that led to it were anomalous and unlikely to be repeated. Thus, the 1979–82 data were not generated by the current interest rate process and contain little information about how volatile interest rates will be if they ever again become so high.

This issue is difficult, at best, to resolve. We can sidestep it, however, by obtaining the “right” answer to the second question: Is an alternative (unfortunately more complicated) model of volatility possible that simultaneously accounts for all the historical data? We turn to this issue next.

**Beyond Level Effects.** Models of the form of Equation 1 allow the volatility of the short rate to vary only with the level of the short rate. This assumption is very restrictive, and it can be tested explicitly within a more general framework. Brenner, Harjes, and Kroner (1996) suggested the model

\[ r_{t+1} - r_t = \alpha_0 + \alpha_1 r_t + \varepsilon_{t+1}, \tag{8} \]

where the shock to the short rate, \( \varepsilon_{t+1} \), satisfies the following conditions:

\[ E_t(\varepsilon_{t+1}) = 0, \tag{9} \]

\[ E_t(\varepsilon_{t+1}^2) = \sigma_t^2 = \psi_{t+1}^2 \gamma, \tag{10} \]

\[ \psi_{t+1}^2 = \theta_0 + \theta_1 \psi_t^2 + \theta_2 \xi_t + \beta \psi_t^2, \tag{11} \]

and \( \xi_t = \min(\varepsilon_t, 0) \). We refer to Equations 8 through 11 as the “BHK model.” Note that the parameter \( \gamma \) in Equation 10 corresponds to the parameter \( \beta_1 \) in Equation 7.

In the BHK model, the short rate changes over time in a manner similar to the way it changes in Equation 6. The important distinction between Equation 6 and Equation 8 is that the shock term has a more complicated volatility structure in Equation 8. The structure of the shocks in Equations 9–11 have several useful, if not immediately obvious, features.

First, Equation 11 does not introduce a second shock at time \( t + 1 \). This model retains a single source of uncertainty (i.e., it is still a single-factor model), but at the same time, the BHK model allows volatility to fluctuate with lagged shocks to the level of the short rate. The economic interpretation of this added flexibility is that it allows volatility to change with “news shocks.” Statistically, this aspect of the model makes the volatility function have a component that looks like the popular generalized autoregressive conditional heteroscedasticity (GARCH) models. In fact, if \( \gamma = 0 \), the BHK model reduces to a GARCH model. When \( \Theta_1 = \Theta_2 = \beta = 0 \), the volatility specification reduces to the diffusion function given in Equation 7.

Brenner et al. confirmed the results in Bliss and Smith that accounting for a structural break between 1979 and 1982 is important in estimating the sensitivity of short-rate volatility to level changes. They concluded that (1) news shocks play an important role (through the GARCH component) in explaining short-rate volatility, (2) even with the effect of news, level effects still play an important role in short-rate volatility, but the estimate of \( \gamma \) is close to the 1/2 assumed in the CIR model, and (3) the combination of news effects and level effects is important for pricing long-dated interest rate derivative instruments.

**Single-Factor Regime-Switching Models.** The volatility function can also be generalized beyond simple level effects by allowing for discrete shifts in regimes for the short-rate process. This choice might be motivated by shifting monetary policy in general or specifically by the 1979–82 Federal Reserve experiment. The simplest example of this approach is a two-regime model in which one regime corresponds to a high level and high volatility of the short rate and the second regime corresponds to “normal” short-rate behavior.

Gray (1996) added regime shifts to a model of the short rate that allows for both level and GARCH effects. In particular, he considered the following discrete-time specification of the evolution of the short rate:

\[ r_{t+1} - r_t = a_2 \beta_2 r_t + e_{t+1} \psi_{t+1}^2, \tag{12} \]

where \( i \) denotes the regime (in this case, 1 or 2), \( h_i \) for \( i = 1, 2 \) is a function (defined below) that describes the time-varying volatility of rate changes, and \( \varepsilon_{t+1} \) is an independent and identically distributed (over time) random shock with \( E(\varepsilon_{t+1}) = 0 \) and var(\( \varepsilon_{t+1} \)) = 1. The most general specification considered by Gray is

\[ h_{it+1} = \omega_i + a_i \varepsilon_t^2 + b_i h_t + \sigma_i^2 \tau_i, \tag{13} \]

where \( h_i \) is a probability-weighted average of \( h_t \) for \( i = 1, 2 \) and \( a_i \) is a constant that affects unconditional volatility. That is,

\[ h_t = p_{1t} \{ (\alpha_1 + \beta_1 \tau_t)^2 + h_{1t} \} + (1 - p_{1t}) \{ (\alpha_2 + \beta_2 \tau_t)^2 + h_{2t} \}
- p_{1t} \{ (\alpha_1 + \beta_1 \tau_t + (1 - p_{1t}) \{ (\alpha_2 + \beta_2 \tau_t \})^2, \tag{14} \]

where \( p_{1t} \), which depends on the level of the short rate, is the probability that the economy is in Regime 1 at date \( t + 1 \). The implication is, of course, that the probability of Regime 2 at \( t + 1 \) is \((1 - p_{1t+1})\). These probabilities may vary over time. For example, the probability of shifting out of the high-volatility regime may increase if the short rate has...
been in that regime a long time. For a given regime \( i \), Equation 13 is a combination of a simple GARCH model and the level-dependent square-root model of CIR. The primary innovation of this approach vis-à-vis earlier descriptions of short-rate dynamics is that the conditional mean and variance can differ from one regime to the other.

Using weekly data on one-month T-bill yields from January 1970 through April 1994, Gray found that using the generalized regime-switching model in Equations 12–14 and allowing for regime probabilities to depend on the level of the short rate provides the best explanation of the dynamics of the short rate. In particular, this specification produced smaller errors and errors that looked more like random draws from a constant distribution than did single-regime GARCH models, two-regime constant-volatility models, and even two-regime GARCH models. Allowing for state-dependent regime probabilities and the level effect within-regime volatility (the \( \sigma^2_{rt} \) term in Equation 13) is all important in capturing the observed features of time-varying volatility in the short rate.

Information on the economic, as opposed to statistical, significance of the generalized regime-switching model is limited. Gray showed that the model in Equations 12–14 not only provides the best explanation of the data over the 1970–94 period but also provides more accurate out-of-sample forecasts than a single-regime GARCH model. Gray did not address the pricing implications, however, of the generalized regime-switching model. Because the computational costs of estimating Equations 12–14 are considerable, understanding the pricing implications of this model is critical if short-rate regime-switching models are to be a viable alternative to the simpler diffusion specification of Equation 1.

Jump Components in Short-Rate Volatility.

Data for the short-rate yield contain numerous one-day spikes, as shown in Figure 4, which suggests that a general model of the volatility of the short rate given in Equation 1 should have a jump component. Johannes (2000) considered a continuous-time model of the following form:

\[
dr_t = \mu(r_t)dt + \sigma(r_t)dW_t + \gamma_t\exp(\xi) - 1)dJ_t, \tag{15}
\]

where the drift and diffusion functions are now defined with respect to the value of the short rate immediately before time \( t \), denoted \( r_{t-} \), the third term in Equation 1 defines the jump component of the short rate, \( \xi \) is the jump size (and is normally distributed with a constant mean and variance), and \( J_t \) is a pure jump process that provides for the random arrival times of the jumps.\(^{14}\)

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**Figure 4. Changes in the Daily Level of the Three-Month Treasury Yield, January 1962–April 2000**

![Graph showing changes in the daily level of the three-month Treasury yield from January 1962 to April 2000.](image-url)
Two primary motivations support the inclusion of a jump component in the specification of the short-rate process. Statistically, Johannes argued that pure diffusion models of the form of Equation 1 cannot simultaneously match the variance and the kurtosis of interest rate data.\textsuperscript{15} Perhaps more important, however, is that a diffusion model cannot account for the arrival of information surprises, which is accounted for in the third term of Equation 15.

Using daily observations of the three-month T-bill from January 1965 to February 1999, Johannes reached the following conclusions: (1) Even the single-factor model that best fits the data (and even stochastic-volatility models of the type introduced in the next section) cannot generate the observed frequency of "large" one-day moves in rates. (2) A jump component in the short rate accounts for a significant portion of the overall volatility of the short rate. (3) The share of volatility explained by—and the relative frequency of—jumps increases with the level of the short rate. Finally, (4) jumps in rates occur at both regularly scheduled announcements of macroeconomic news and at times of important but unpredictable political and economic events, both domestic and global.

Piazzesi (2001) examined the interaction between monetary policy and movements in the level and volatility of Treasury yields of different maturities. Specifically, she added jumps to interest rates corresponding to the meeting dates of the Federal Open Market Committee (FOMC) and the release dates of key macroeconomic data, because those dates represent the arrival of new information. In addition, she assumed that the Federal Reserve bases monetary policy on a set of data observable immediately prior to the FOMC meeting date.

Piazzesi used simulated maximum-likelihood methods and five years of daily swap spread data beginning in 1994 to estimate the model. She demonstrated that including a jump component in a multifactor model of the type described in the section “General Multifactor Structures” results in a substantial improvement in the fit of the model at the short end of the yield curve. The volatility structure of yields over horizons ranging from overnight to five years follows a “snake-shaped” pattern; that is, it drops at the very short end of the yield curve and then increases out to a maturity of two years. Piazzesi attributed the shape of the volatility curve to "policy inertia," although given the comparatively short sample period in her study, one should probably be careful about treating this effect as a robust fact. Piazzesi also identified interesting feedback effects from yields to monetary policy.

**Summary.** The behavior of the short rate is not easy to describe, and academic researchers have proposed a number of models of its behavior. The overall conclusions from the research reviewed in this section are as follows. (1) Although credible evidence suggests that volatility depends on the level of the short rate, the estimate of the sensitivity of short-rate volatility to the level of the short rate is not robust to alternative ways of treating the 1979–82 Federal Reserve experiment. (2) The level of the short rate alone (as in the simple diffusion model of Equations 1 and 7) is not sufficient to describe the volatility in the short-rate data from the 1960s through the 1990s. In particular, some sort of GARCH effect or other form of stochastic volatility seems to be involved. (3) Allowing the mean and variance of short rates, together with level and GARCH effects, to switch between two regimes (with the probability of switching depending on the level of the short rate) seems to provide an adequate description of short-rate dynamics. (4) Finally, some intriguing evidence suggests that a thorough characterization of yield volatility in a continuous-time model must allow for interest rates to make discrete jumps.

**Multifactor Models**

As we noted in the section “A Look at Some Data,” the movements in the term structure of interest rates appear to be determined, primarily, by three distinct factors—level, slope, and curvature. Our focus on models of the short rate was motivated by the evidence that approximately 90 percent of the variation in government bond yields can be attributed solely to the first factor, level. Additional factors, however, may affect the volatility of the short rate.

The tradition of considering multifactor models is long; early empirical work in this area includes Stambaugh (1988), Chen and Scott (1993), and Pearson and Sun. These papers were all based on two- and three-factor versions of the square-root model in CIR.

**Stochastic Volatility.** Stochastic-volatility models (hereafter, SV models) introduce an additional factor in attempts to explain short-rate dynamics. The SV models treat the volatility of the short rate as a distinct source of uncertainty about the rate’s evolution. Continuous-time stochastic-volatility models are two-factor short-rate models of the form

\[
\begin{align*}
    dr_t &= \kappa_1 (\mu - r_t) dt + \sigma_1 \sqrt{\sigma_2^2} dW_{1t}, \\
    d\log \sigma_t^2 &= \kappa_2 (\alpha - \log \sigma_t^2) dt + \xi dW_{2t},
\end{align*}
\]  

(16)  

(17)
where $q_1$ is the speed of reversion to the mean in the level of rates, $q_2$ is the speed of mean reversion in the stochastic-volatility function, $\gamma \geq 0$, and $W_1$ and $W_2$ are Brownian motions. 16 Equations 16 and 17 are a direct extension of the simple diffusion model given by Equations 1, 5, and 7. As in the simple model, the level effect works through $r_1^j$ in Equation 16, but in addition, a separate volatility effect is now working through the process for $\log\sigma^2$.

The results in Andersen and Lund (1997) indicate that the stochastic-volatility component of Equation 16 plays an important role in describing the dynamics of the short rate.17 In particular, using weekly yields on three-month U.S. T-bills from January 1954 to December 1995, the authors found in overall tests of model fit that the SV model describes the data better than the single-factor CKLS model. Andersen and Lund argued that the estimated parameters are economically (and statistically) plausible, although the SV model has a difficult time matching the fat tails of the short-rate distribution. Using their long time series and more precise estimation technique, Andersen and Lund obtained estimates for $\gamma$ of 0.544 in the SV model and 0.676 in their estimate of Equation 1, with drift and diffusion given by Equation 5 and Equation 7. 18 Neither estimate could be reliably distinguished from the 1/2 provided by the CIR model. Interestingly, the value of $\gamma$ seems to be unimportant in determining the ability of either the CKLS model or the SV model to explain the data.

Gallant and Tauchen (1998) generalized the SV model by allowing the level of the short rate to affect both the drift and the diffusion functions of the stochastic-volatility process, and they tested a variety of specifications, including the simple Vasicek and CIR models and the SV model. Using weekly observations on the three-month Treasury yield from January 1962 to August 1996, Gallant and Tauchen found evidence that is generally consistent with the results in Andersen and Lund; that is, a stochastic-volatility term is needed to account for short-rate dynamics. They also found that the level of rates affects both expected volatility and the diffusion function of the stochastic volatility.

Ball and Torous (1999) examined the discrete-time version of the SV model (hereafter, DSV model):

$$r_{t+\Delta t} - r_t = \kappa_1 (\mu - r_t) \Delta t + \sigma r_t^\gamma \sqrt{\Delta t} z_{1,t}^\gamma,$$  

(18)

$$\log \sigma^2_{t+\Delta t} - \log \sigma^2_t = \kappa_2 (\alpha - \log \sigma^2_t) \Delta t + \zeta \sqrt{\Delta t} z_{2,t}^\zeta,$$  

(19)

where $\Delta t$ is the measurement interval for a short-rate data series and $z_{i,t}$ for $i = 1, 2$ are standard normal random variables. They used a variety of short-rate data sets (including London Eurodollar, Euromark, Eurosterling, and Euroyen rates) and data chosen to replicate earlier studies (including CKLS). They concluded, consistently with Andersen and Lund and with Gallant and Tauchen, that stochastic volatility plays an important role in the dynamics of the various short rates. Stochastic volatility responds to economic shocks, but the effects of the shocks die out quickly. Ball and Torous also found that the greatest failure of the DSV model is in capturing the fat tails of the short-rate distribution. Finally, they found evidence of a level effect in short-rate volatility, even after controlling for stochastic volatility, but found that measuring $\gamma$ precisely is difficult.

In summary, Andersen and Lund, Gallant and Tauchen, and Ball and Torous—all concluded that stochastic volatility is a robust feature of short-rate data. Interpreting the results in Andersen and Lund and Gallant and Tauchen is complicated by two issues: (1) Although these papers demonstrate that SV models of the short rate can accommodate the anomalous 1979–82 period, neither paper estimated the model parameters while excluding data from that period. Therefore, it is difficult to determine the extent to which the observations of the period may have exerted an influence over the estimates and the evaluation of relative fit of alternative models. (2) None of the articles discussed in this section priced derivative securities using alternative short-rate models estimated over the full period (or any subperiods) of the data. The absence of evidence on the pricing performance of relative SV models makes the interpretation of the economic significance of stochastic volatility problematic.

The discussions so far of two-factor models and of regime-switching models raise a number of questions. What are the relative merits of stochastic-volatility versus regime-switching models? In a model that simultaneously allows for both, do stochastic-volatility effects dominate regime-switching effects, is the dominance the other way around, or are both features required to explain the properties of short rates? What are the relative pricing implications of a single-factor regime-switching model versus a stochastic-volatility model? The answers are likely to depend critically on whether or not the data include the volatile 1979–82 period—which raises other questions: How extensive is this dependence, and how should we analyze the trade-offs involved in using more or less of the time series data in constructing fixed-income pricing models? What are the relative computation costs between estimating these alternative structures and using them as complete pricing models? The answers to all of these questions remain unclear.
**General Multifactor Structures.** The stochastic-volatility model of the previous subsection is only one example of a multifactor term-structure model. The possible varieties of multifactor models are practically limitless. Of these possibilities, two generic classes, the affine and quadratic term-structure models, are of particular interest because of their convenience and tractability.

Dai and Singleton (2000) defined the short rate in a multifactor affine term-structure model as a linear combination of the factors or state variables

\[
 r_t = \delta_0 + \sum_{i=1}^{N} \delta_i Y_{it},
\]

(20)

where \(Y_{it}\) for \(i = 1, \ldots, N\) are the factors that determine the short rate (and, implicitly, all bond prices).

The factors evolve over time in a multidimensional affine diffusion process

\[
dy_t = \mathbf{K}(\mathbf{Y}_t - \mathbf{Y})dt + \mathbf{\Sigma} \sqrt{\mathbf{S}} dw_t,
\]

(21)

where \(\mathbf{K}\) and \(\mathbf{\Sigma}\) are \(N \times N\) matrixes, \(\mathbf{S}_i\) is a diagonal matrix whose \(i\)th diagonal element is

\[
(K_i)_{ii} = \alpha_i + \sum_{j=1}^{N} \beta_{ij} Y_{jt},
\]

(22)

and \(w_t\) is an \(N\)-dimensional vector of independent Brownian motions. In these models, yields can also be expressed as linear combinations of the state variables. Thus, a \(K\)-factor affine model is equivalent to picking \(K\) zero-coupon yields as factors. Alternatively, the factors could be expressed as a combination of yield levels and yield spreads. Examples of multifactor affine models include multifactor versions of the Vasicek and CIR models and models that mix Gaussian and square-root driving factors.

Ahn, Dittmar, and Gallant (forthcoming 2001) defined the short rate in a quadratic term-structure model as

\[
r_t = \alpha + \sum_{i=1}^{N} \beta_i Y_{it} + \sum_{i=1}^{N} \sum_{j=1}^{N} \psi_{ij} Y_{it} Y_{jt},
\]

(23)

where the \(Y\) state variables evolve according to

\[
dY_t = (\mathbf{\mu} + \mathbf{\xi} \mathbf{Y}_t) dt + \Sigma dW_t.
\]

(24)

Thus, \(Y\) is a multivariate Gaussian process in which \(\mathbf{\mu}\) is an \(N\)-vector of constants, \(\mathbf{\xi}\) and \(\Sigma\) are \(N \times N\) matrixes, and \(W_t\) is an \(N\)-vector of independent Brownian motions. Ahn et al. showed that the structure in Equations 23 and 24 includes the term-structure models of Longstaff (1989), as amended by Beaglehole and Tenney (1992), and Constantinides (1992). Using weekly observations of the yields on ordinary fixed-for-floating U.S. dollar interest rate swap contracts of 6-month, 2-year, and 10-year maturities between April 1987 and August 1996, Dai and Singleton (2000) estimated a variety of affine models. They concluded that it is important for a multifactor affine model to permit negative correlation among the factors. This feature is not found in common multifactor models, neither the square-root model of Chen (1996) nor the mixed Gaussian/square-root model of Balduzzi, Das, Foresi, and Sundaram (1996).

The preferred model in Dai and Singleton (2000) uses three factors, with only one factor affecting the conditional variances of yields. This model produced both the smallest pricing errors in their study and the closest match to the term structure of (unconditional) swap yield volatilities.

Ahn et al. estimated four quadratic models (including a version of the model in Constantinides) and two of the affine models estimated in Dai and Singleton (2000). Ahn et al. used monthly data on 3-month, 1-year, and 10-year yields from December 1946 through February 1991. They concluded that an unrestricted three-factor quadratic model provides a better fit to the yield data than any of the more commonly used models and a better fit than either of the affine models from Dai and Singleton (2000). Ahn et al. based this conclusion on an overall specification test, on tests of specific components of the means and variances of yields, and tests of the fitted model yields.

The best quadratic model produces fitted yields that match the various maturities more closely than the fit produced by the best affine model, but readers should recognize that the best quadratic model in these studies was still strongly rejected by the overall specification test (although it was rejected less convincingly than the alternative models) and it still failed to match components of the volatility of yields and some non-Gaussian features of the data. These failings are particularly important because they raise the issue of whether regime shifts are an essential part of any multifactor model that hopes to explain a data set that spans the volatile periods of 1973–1976 and 1979–1982.

Bansal and Zhou (2000) extended a two-factor affine model to explicitly incorporate the effects of regime shifts. In particular, they described short-rate dynamics as follows:

\[
x_{1t+1} - x_{1t} = \kappa_{1,t+1} (\theta_{1,t+1} - x_{1t}) + \sigma_{1,t+1} \sqrt{x_{1t}} u_{1t+1},
\]

(25)

\[
x_{2t+1} - x_{2t} = \kappa_{2,t+1} (\theta_{2,t+1} - x_{2t}) + \sigma_{2,t+1} \sqrt{x_{2t}} u_{2t+1},
\]
where $\mu_{1t+1}$ and $\mu_{2t+1}$ are independent, standard, normal random shocks to the two independent factors that determine short-rate dynamics and $s_{t+1}$ assumes a value of either 1 or 0 depending on the current regime. The state of $s_t$ determines the values of the drift and diffusion parameters for the processes $x_1$ and $x_2$, and $s_t$ evolves independently of the values of $x_1$ and $x_2$. The model in Equation 25 explicitly generalizes one- and two-factor CIR models.

In estimating the regime-switching model with monthly bond yield data on maturities from one month to five years for the 1964–95 period, Bansal and Zhou found that pure versions of the CIR model and a one-factor version of the regime-switching model are not consistent with the data. These models are not capable of matching the changes in either the volatilities of short- and long-term rates or the changing cross-correlations of short- and long-term rates nearly as well as the two-factor regime-switching model of Equation 25. According to Bansal and Zhou, these results differ from those in Dai and Singleton (2000) because by confining . . . [their analysis] . . . to the swap interest rate data beginning in 1987, much of the information regarding the most volatile periods of interest rates, e.g., 1979-82 and 1973-76, does not bear on the estimation. (pp. 19–20)

A potential method to resolve the issue of whether regime switching is an essential part of a description of government bond yields is to look at data from outside the United States. Evans (1998) examined monthly data on real and nominal yields on one-, three-, five-, and seven-year zero-coupon bonds from the United Kingdom from January 1983 to November 1995. He concluded that a single-factor CIR model is an adequate description of real yields but an adequate description of both the average level and the volatility of nominal yields requires a second regime. 24

Ang and Bekaert (2000) estimated a variety of regime-switching specifications by using monthly observations on three-month nominal yields and the spread between five-year and three-month yields in the United States, the United Kingdom, and Germany for 1972–1996. They found that U.S. short rates improve the estimates of regime dates and forecasts of U.K. and German rates but not the converse. Using the information in yield spreads generally resulted in models with smaller forecasting errors, and the dates of the two regimes corresponded closely to business cycle expansions and contractions. The means and variances of short rates implied by the multivariate switching model did not always, however, provide a close match to the data.

### Term Premiums and Model Selection

Up to now, our review has concentrated on the ability of various models to capture the key features of the time variation in either a proxy for the short rate or a small number of Treasury yields. An important tradition in term-structure modeling concentrates, however, on the properties of term (risk) premiums of bonds of different maturities and on the connection between forward rates and future spot rates. In particular, the expectations hypothesis of the term structure predicts that yields on long-term zero-coupon bonds are equal to the expected yields on a sequence of short-maturity bonds plus (possibly) a constant term premium. Alternatively, the expectations hypothesis implies that current forward rates are unbiased predictors of future spot rates.

These implications have been tested by estimating linear regressions involving either the yields on zero-coupon bonds or forward rates. Yield tests of the expectations hypothesis use the specification

$$
y_t(t) - y_{t-1}(t) = \alpha(t) + \beta(t) \gamma_t(t) + \epsilon_t,
$$

where

- $y_t(t)$ = yield to maturity at date $t$ on a zero-coupon bond maturing at date $t + \tau$
- $\alpha(t)$ = term premium depending on maturity $\tau$
- $\beta(t)$ = slope coefficient for the yield on a $\tau$-period bond
- $r_t$ = yield on a one-period zero-coupon bond
- $\epsilon_t$ = random disturbance term in the regression

The left-hand side of Equation 26 is the one-period holding-period return from a $\tau$-period bond purchased at time $t$ and sold at time $t + 1$. The normalization of the yield spread, $y_t(t) - r_t$, by $1/(t + 1)$ implies that the slope coefficients, $\beta(t)$'s, should all equal 1 if the expectations hypothesis is true.

Forward-rate tests of the expectations hypothesis are based on the specification

$$
r_{t+\tau} - r_t = \delta(t) + \gamma(t) f_{t}(t) - r_t + \nu_{t+\tau-1},
$$

where $\delta(t)$ and $\gamma(t)$ are regression coefficients, $\nu_{t+\tau-1}$ is the regression disturbance (observed at date $t + \tau - 1$), and $f_{t}(t)$ is the forward rate (observed at time $t$) for a one-period loan beginning at $t + \tau$. 25 If the expectations hypothesis is true, $\gamma(t) = 1$ for each maturity $\tau$. 

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Fama (1984a, 1984b), Fama and Bliss (1987), and Campbell and Shiller (1991), after examining predictions of the form of Equations 26 and 27, conclusively rejected the expectations hypothesis. In the yield specifications, the \( \beta \)'s were not only different from 1 but were often significantly negative and were more negative for the longer-maturity yields. In the forward-rate specifications, the \( \gamma \)'s were significantly less than 1.

These regression results are typically taken as evidence of time-varying term premiums, and the problem then becomes finding a term-structure model that can explain these patterns in the data.\(^\text{26}\) A reasonable expectation is that a term-structure model will reproduce the patterns in term premiums and forward rates that generate rejections of the expectations hypothesis, which would provide a criterion for distinguishing among the large set of dynamic models described in the “Multifactor Models” section. How do the models perform on this dimension?

The available evidence is very preliminary and, consequently, must be interpreted with caution. Duffee (forthcoming 2001) argued that affine models cannot simultaneously match yield-curve movements and bond return premiums without modification of the dependence of the market price of interest rate volatility. Duffee and Dai and Singleton (2001) demonstrated that by altering the dependence of risk premiums on factors, one can find parameter values for three-factor affine models that are consistent with the risk-premium regressions in Fama and Bliss and in Campbell and Shiller, although these models have a difficult time simultaneously explaining the yields on short-maturity bonds. Furthermore, all the models that provide the best fit to risk premiums assume constant volatility of yields, which guarantees that they are incapable of matching the observed volatility of interest rate data.

More important for the question at hand is the argument of Bansal and Zhou that their two-factor regime-switching model can reproduce the qualitative properties of regressions of holding-period returns and yield changes. Thus, preliminary evidence suggests that patterns in term premiums do not permit distinguishing among competing models. It may be necessary to examine data on the dynamics of interest rate options to narrow the field of acceptable dynamic models. For example, Collin-Dufresne and Goldstein (2001) demonstrated that the returns to at-the-money interest rate straddles are inconsistent with all two-factor and most three-factor affine models. They argued that the connection between interest rate volatility and straddle returns is more naturally accommodated in a Heath–Jarrow–Morton framework.

The ability of various model structures to resolve the facts about the dynamics of term premiums may be in doubt, but at least the basic facts have been established. The dynamics of the term structure of realized and expected volatility, in contrast, have received little attention in the empirical literature. Given the importance of the dynamics of conditional volatility, this omission is clearly a weakness in our current understanding.

**Conclusion**

After all this research, we know a few important facts about the evolution of interest rates. Unfortunately, the list of things we do not know is as long as the list of things we do know.

We know that within a wide range of interest rates, from about 4 percent to 15 percent, any mean reversion, or tendency of the short rate to revert to a long-run mean level, is weak at best. Within this range, the best estimate of next month’s, next quarter’s, or next year’s interest rate is today’s rate. We can conclude that the short rate is likely to be at least somewhat mean reverting; the alternative is to assume that the variance of its distribution increases without bound as we look farther into the future, and such a conclusion does not come from a statistical analysis of the data. For a wide range of interest rates, the mean reversion is so weak that it is difficult to discern in the data.

We do not know much about nonlinearity in the drift (i.e., the strength of mean reversion for very high and low levels of the short rate). Some studies have found that the drift in the short rate is a nonlinear function of the rate’s level, but this conclusion remains open to question. Also, we know little about the premium for bearing interest rate risk. Thus, on the one hand, we cannot assess the reasonableness of the assumption of the Vasicek, CIR, and Hull–White models that the drift of the short rate is linear in its level or the assumption of the Black–Derman–Toy and Black–Karasinski models that the drift of the log of the short rate is linear in its log. On the other hand, we have no compelling evidence that is inconsistent with these assumptions.

In the context of the simple diffusion model (Equation 1), we know that the “absolute” volatility of the short rate, defined as \( 1/\sqrt{\Delta t} \) times the standard deviation of changes \( r_{t+\Delta t} - r_t \), increases with the level of the short rate. Thus, models with absolute volatility that does not depend on the level of the rate, although tractable and convenient, are not consistent with the data. This level effect is also found in specifications that allow for stochastic volatility, which indicates that it is a robust feature of the data.
We do not know the strength of the relationship between interest rate volatility and its level. If the estimation includes data from the Federal Reserve experiment of 1979–1982 in the specification \( \sigma(r) = \beta_0 \beta_1 \), the data suggest that power \( \beta_1 \) is approximately 3/2. If the data from the 1979–82 period are not used, \( \beta_1 \) does not appear to be statistically significantly different from 1/2. One can plausibly argue that the 1979–82 data should not be used because the Federal Reserve experiment and stagnation of the 1970s that led to it are unlikely to be repeated. The counterargument is that this period is the only period in which very high U.S. dollar interest rates have been observed, so it offers a unique opportunity for study. Asking about the relationship between the level of the short rate and its volatility involves asking: If the short rate again becomes very high, what will be its volatility? If we disregard the 1979–82 data, we are reduced to extrapolating from the recent environment of low interest rates. In general, such extrapolation is not wise.

Given our lack of knowledge, proportional volatility specifications—such as those used in the Black–Derman–Toy and Black–Karasinski models and implementations of the Heath–Jarrow–Morton framework with proportional forward-rate volatilities—may be the best overall choices. But if we think the 1979–82 experience is no longer relevant, then a square-root specification may be the better choice. Strikingly, the square-root specifications, although popular among academics, are less popular among practitioners than models with constant and proportional absolute volatility that provide a poorer fit to the recent data.

Considering more general modeling frameworks, we know that the scalar diffusion (Equation 1) is not a complete description of the data. In particular, the short rate displays some sort of stochastic-volatility or GARCH effects. This finding is robust in a number of studies and is consistent with results for virtually all other financial time series. Furthermore, the level effect was found in conjunction with various forms of stochastic volatility.

We do not know, however, how the stochastic volatility should best be characterized. GARCH specifications, two-factor stochastic-volatility models, regime-switching models, and multifactor affine models have all been offered as possible descriptions of the data. So far, no consensus has developed about which approach is best.

These areas of our ignorance are important. The specification of interest rate volatility affects VAR calculations, other risk measures, and option pricing. The topic of option pricing is impossible to ignore, because embedded interest rate options are ubiquitous in the fixed-income markets. Nonlinearity in the drift has important effects on the distributions of interest rates over long horizons and thus can be important for VAR calculations over such horizons and for asset/liability management. Furthermore, when combined with a specification of the market price of interest rate risk, the drift affects the risk-neutral probability; therefore, it also has implications for option pricing.

Academics are inclined to rely on the conclusion that “more research is needed.” In dealing with our ignorance about which stochastic-volatility model is best, however, “more research” seems to be a reasonable prescription. Investigation of stochastic-volatility models has only just begun in earnest; so, more research may well clarify the situation.

We have less cause for optimism about clarification in the future in regard to the other unresolved questions about interest rate dynamics. Although work on the existence of nonlinear mean reversion is currently under way, whether the academic community will ever reach a consensus on this point is unclear. Despite the length of the available time series, the data speak only very softly. With respect to the volatility specification, the data speak more loudly—but no more clearly: The message we hear depends crucially on how we treat the 1979–82 period. Should models reflect the experience of those years, or was this period an aberration? The academic literature does not contain the answer to this question.

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Notes

1. Examples of various calibration exercises may be found in Black, Derman, and Toy, in Black and Karasinski, and in Hull and White (1994a, 1994b).
2. The data used in this section are available online from the Federal Reserve Board at www.federalreserve.gov/releases/H15/data.htm. This site also contains detailed descriptions of the construction of each data series.
3. All of these series have contemporaneous correlations well in excess of 0.90 at both weekly and monthly frequencies. However, they are not identical. For example, three-month Treasury yields are less volatile than the other data, and the Federal funds rate appears to have a number of days with large one-day movements.

5. Knez, Litterman, and Scheinkman (1994) found evidence for four factors in short-term interest rate series that include both government bond yields and high-grade commercial paper. Longstaff, Santa-Clar, and Schwartz (2000) argued that four factors are needed to reconcile the interest rate processes with the prices of interest rate caps and swaptions.

6. A Brownian motion is a continuous-time stochastic process with the properties that between any two dates \( s \) and \( t \) (with \( s > t \)), the increment \( W_t - W_s \) has a normal distribution with zero mean and variance of \( s - t \) and the increment is independent of the value of the process at all dates prior to \( t \).

7. In the class of continuous-time stochastic processes of the form of Equation 1, this statement is not strictly correct. Strict stationarity of the process depends on both the drift and the diffusion parameters, and the process can exhibit what is called “volatility-induced” stationarity for values of \( \alpha \geq 0 \). See Conley, Hansen, Luttmer, and Scheinkman (1997) for a discussion of this point.

8. Ait-Sahalia used 5,505 daily observations of the one-week Eurodollar rate from June 1973 to February 1995. Stanton used 7,975 daily observations of the three-month T-bill from January 1965 to July 1995.

9. The test rejected the null hypothesis too frequently when the null hypothesis was true. Pritsker found that when he used a significance level or size of 5 percent, for which the test statistic should reject a true null hypothesis 1 out of 20 times, it actually rejected the null 10 out of 20 times. Using this statistic, one could quite possibly reject a linear drift even when this specification is correct. Further results in Pritsker demonstrated that the test also had low power (relative to other feasible tests) in distinguishing among reasonable alternative hypotheses.

10. CKLS estimated \( \beta_3 \) with the use of the generalized method of moments introduced in Hansen (1982) applied to moment conditions from Equation 4, the discretized version of Equation 1. Ait-Sahalia argued that this estimator is not consistent (i.e., the estimate \( \hat{\beta}_3 \) does not converge to \( \beta_3 \) in arbitrarily large samples). CKLS extended moment-based estimators based on the infinite-time generator of the scalar diffusion in Equation 1 to derive consistent and asymptotically normal parameter estimates. CHLS concluded their analysis with the following comment: “The potential for fat-tailed distributions of interest rate differences may undermine the quality of statistical inference based on large-sample approximations. Assessing the finite-sample performance of estimation methods such as ours is an important topic for future research” (p. 558).

11. A precise interpretation of the statistical significance of a structural-break test of the form in Bliss and Smith is problematic. Intuitively, they were using information from outside the sample data. In essence, their test was based, in part, on data snooping, and the resulting test suffers from an implicit bias. Andrews (1993), Andrews and Ploberger (1994), Ploberger (1989), and Ploberger and Krämer (1990) examined the properties of a variety of test statistics that allow for the possibility of a permanent or temporary structural break over an arbitrary subset of the data.

12. Equation 11 has the form of an asymmetrical GARCH(1, 1) model. It is asymmetrical in the sense that if \( \theta_2 = 0 \), short-rate volatility responds differently to positive and negative shocks. See Bollerslev (1986) for an introduction to GARCH models and Bollerslev, Chou, and Kroner (1992) for a slightly dated review of the use of GARCH-style modeling in finance.

13. As Gray pointed out, defining the conditional regime-dependent volatility using \( h_t \) solves a difficult path-dependence problem present in the earlier regime-switching autoregressive conditional heteroscedasticity models of Cai (1994) and Hamilton and Susmel (1994).

14. The simplest example of a pure jump process, analogous to the Brownian motion that drives a continuous sample path process, is a Poisson process. See Merton (1990) or Protter (1990) for definition of a Poisson process and its generalizations.

15. The kurtosis of the distribution of a random variable is defined as the ratio of the fourth moment to the second moment. Intuitively, it measures the peaks and the tails of a distribution, typically relative to a normal distribution.

16. Longstaff and Schwartz (1992) described a two-factor equilibrium version of a stochastic-volatility model, constructed as in CIR. They provided some empirical evidence that can be interpreted as consistent with their model structure. Andersen and Lund (1997) and Gallant and Tauchen (1998) provided more detailed empirical studies of SV models.

17. Andersen and Lund also conducted a careful evaluation of a range of alternative models in the asymmetrical GARCH and exponential GARCH families that extended the analysis in Brenner et al. Based on both estimation from a long time series of interest rates and simulation, Andersen and Lund concluded that a low-order exponential GARCH model with a level effect on volatility (as in Brenner et al.) is the best-fitting GARCH model.

18. Andersen and Lund used the efficient method of moments estimator developed by Gallant and Tauchen, whereas CKLS used the generalized method of moments applied to the discrete-time version of Equation 1.

19. An affine model is one in which zero-coupon yields are linear with respect to underlying state variables. See Duffie and Kan (1996).

20. In Equation 21, \( W_t \) is a Brownian motion under the physical measure \( P \). Bond prices are calculated under the equivalent martingale measure \( Q \), and the coefficients determining the dynamics of \( Y_t \) under \( Q \) are related to the coefficients under \( P \) by the market price of risk process.

\[
\Delta_t = \sqrt{\lambda_t},
\]

where \( \lambda_t \) is an \( N \)-vector of constants.

21. In classical models of the term structure, such as CIR, these state variables are fundamental properties of the underlying equilibrium economy, such as the marginal productivity of capital and labor.

22. To derive an explicit formula for the price of a \( t \)-period zero-coupon bond, Ahn et al. also assumed that the stochastic discount factor (state prices) is a function of the short rate and the state variables. The price at time \( t \) of a default-free bond that pays one unit of account at time \( t + t \) is of the form \( V(t, t) = \exp [A(t) + B(t)Y_t + C(t)] \), where \( A(t), B(t), \) and \( C(t) \) are solutions of a system of ordinary differential equations. In contrast, the analogous affine bond-pricing equation is \( V(t, t) = \exp [D(t) - E(t)Y_t] \), where \( D(t) \) and \( E(t) \) also satisfy ordinary differential equations. See Duffie and Kan (1996), Dai and Singleton (2000), and Ahn et al. for details.


24. An interesting aspect of the analysis in Evans is that he found a role for regime changes other than the U.S. Federal Reserve experiment of 1979–1982. For the U.K. data, a regime switch was required to explain the period in October 1992 associated with the U.K.’s withdrawal from the European Monetary System.

25. In terms of the yields on zero-coupon bonds observed at time \( t \), \( f(t) \) is defined as \( f(t) = (r(t) - (t - T)Y(t)) \).

26. This exercise is difficult: Backus, Gregory, and Zin (1989), Den Haan (1995), and Chapman (1997)—all demonstrated that the time-varying term premiums documented in statistical evaluations of the predictions of Equations 26 and 27 are inconsistent with simple equilibrium models designed to explain, simultaneously, movements in the term structure and the macroeconomy.
References


