Appendix

A1. Capacity Thresholds

We define the following threshold capacity levels that depend on the number of firms n. We refer to these threshold capacity levels for describing the firms' optimal policies.

$$\bar{K}_1^{(n)} = \frac{1}{(n+1)} \frac{s_l(s_h(q_l - c_l) - s_l(q_h - c_h))}{q_l(s_h - s_l)} \tag{A-1}$$

$$\bar{K}_2^{(n)} = \frac{1}{(n+1)} \frac{s_h(s_h(q_l - c_l) - s_l(q_h - c_h))}{q_l s_h - q_h s_l} \tag{A-2}$$

$$\bar{K}_3^{(n)} = \frac{1}{(n+1)} \frac{q_l(q_l - q_h)s_h + c_h q_l(-s_l + s_h) + c_l(q_h s_l - q_l s_h)}{q_l(q_l - q_h)} \tag{A-3}$$

$$\bar{K}_4^{(n)} = \frac{1}{(n+1)} \frac{(q_h - c_h)s_h}{q_h} \tag{A-4}$$

$$\bar{K}_5^{(n)} = \frac{1}{(n+1)} \frac{(q_l - c_l)s_l}{q_l} \tag{A-5}$$

$$\bar{k}_1 = \frac{s_h q_l(s_h - s_l)}{s_l(q_l s_h - q_h s_l)} \tag{A-6}$$

$$\bar{k}_2 = \frac{s_h(-c_{yl}s_h + q_l s_h + (c_{yh} - q_h)s_l)}{3q_l s_h - 3q_h s_l} \tag{A-7}$$

$$\bar{k}_3 = (s_h(2q_h q_l s_h^2 - q_l^2 s_h^2 - c_{yh} q_l s_h s_l - 2q_h q_l s_h s_l + c_{yh} q_h s_l^2 + q_h^2 s_l^2 + c_{yl} s_h(q_l s_h - q_h s_l) - 2c_{zh}(q_l s_h(s_h - 2s_l) + q_h s_l^2)))/(-3q_l^2 s_h^2 + 6q_h q_l s_h(s_h - s_l) + 3q_h^2 s_l^2) \tag{A-8}$$

$$\bar{k}_4 = -\frac{s_h s_l(c_{yl}s_h - q_l s_h + (-c_{yh} + q_h)s_l)}{q_l s_h(2s_h - s_l) - q_h s_l^2} \tag{A-9}$$

$$\bar{c}_1 = \frac{(c_{yl} + 2q_h - q_l)q_l s_h^2 - (c_{yl}q_h + (c_{yh} + 2q_h)q_l)s_h s_l + q_h(c_{yh} + q_h)s_l^2}{2(q_l s_h(s_h - 2s_l) + q_h s_l^2)} \tag{A-10}$$

$$\bar{c}_2 = q_h + \frac{(c_{yl} - q_l)s_h(q_l s_h - q_h s_l)}{q_l s_h(2s_h - 3s_l) + q_h s_l^2} \tag{A-11}$$

(A-12)

A2. Proofs

Proof of Lemma 1. We need to show that the Hessian of Π_j is negative definite. First, we solve equations (1) and (2) to get the market clearing prices as a function of quantities:

$$p_i(w_j, w_{-j}) = q_i(1 - \sum_{a=i}^{m}(w_{ja} + w_{-j,a})) - \sum_{b=1}^{i-1} q_b(w_{jb} + w_{-j,b}).$$

Then, we construct the Hessian H for Π_j. For $a \leq b$, $\frac{\partial^2 \Pi_j}{\partial w_{ja} \partial w_{jb}} = -2q_a$. It is straightforward to show that the sign of the first leading principal minor is negative ($H_1 = -2q_1$), and the signs of the further leading principal minors alternate. This completes the proof.

$H_2 = Det[\begin{pmatrix} -2q_1 & -2q_1 \\ -2q_1 & -2q_2 \end{pmatrix}] = 4q_1(-q_1 + q_2) > 0,$

$H_3 = Det[\begin{pmatrix} -2q_1 & -2q_1 & -2q_1 \\ -2q_1 & -2q_2 & -2q_2 \\ -2q_1 & -2q_2 & -2q_3 \end{pmatrix}] = -8q_1(-q_1 + q_2)(-q_2 + q_3) < 0$

$H_i = (-1)^i 2^i q_1 \prod_{j=1}^{i-1}(-q_j + q_{j+1})$. ■

The following Lemma is essential to the proofs of Propositions 1 and 2.

Lemma A1 *Lagrangian of the oligopolist's problem given in (4) is as follows:*
$LG = w_{j,h}(p_h(w_j, w_{-j}) - c_h) + w_{j,l}(p_l(w_j, w_{-j}) - c_l) + \lambda(K - s_l w_{j,l} - s_h w_{j,h}) + \mu_l w_{j,l} + \mu_h w_{j,h}$.
Solutions that satisfy the optimality conditions to this problem are listed below:

i) $w_{j,h} = \frac{-c_h + c_l + q_h - q_l}{(1+n)(q_h - q_l)}$, $w_{j,l} = \frac{-c_l q_h + c_h q_l}{(1+n)(q_h - q_l) q_l}$, $\mu_h = 0$, $\mu_l = 0$, $\lambda = 0$.

ii) $w_{j,h} = \frac{K(1+n)q_l(s_h - s_l) + s_l(c_l s_h - q_l s_h - c_h s_l + q_h s_l)}{(1+n)(q_l s_h(s_h - 2s_l) + q_h s_l^2)}$, $w_{j,l} = \frac{-K(1+n)(q_l s_h - q_h s_l) + s_h(-c_l s_h + q_l s_h + c_h s_l - q_h s_l)}{(1+n)(q_l s_h(s_h - 2s_l) + q_h s_l^2)}$,
$\lambda = \frac{K(1+n)q_l(-q_h + q_l) + (c_l + q_h - q_l)q_l s_h - c_l q_h s_l + c_h q_l(-s_h + s_l)}{q_l s_h(s_h - 2s_l) + q_h s_l^2}$, $\mu_h = 0$, $\mu_l = 0$.

iii) $w_{j,h} = 0$, $w_{j,l} = \frac{q_l - c_l}{q_l + n q_l}$, $\mu_h = c_h - c_l - q_h + q_l$, $\mu_l = 0$, $\lambda = 0$.

iv) $w_{j,h} = 0$, $w_{j,l} = \frac{K}{s_l}$, $\mu_h = \frac{K(1+n)q_l(-s_h + s_l) + s_l(-c_l s_h + q_l s_h + c_h s_l - q_h s_l)}{s_l^2}$, $\lambda = \frac{-K(1+n)q_l + (-c_l + q_l)s_l}{s_l^2}$,
$\mu_l = 0$.

v) $w_{j,h} = 0$, $w_{j,l} = 0$, $\mu_l = c_l - q_l$, $\mu_h = c_h - q_h$, $\lambda = 0$.

vi) $w_{j,h} = \frac{q_h - c_h}{q_h + n q_h}$, $w_{j,l} = 0$, $\mu_l = c_l - \frac{c_h q_l}{q_h}$, $\mu_h = 0$, $\lambda = 0$.

vii) $w_{j,h} = \frac{K}{s_h}$, $w_{j,l} = 0$, $\mu_l = \frac{K(1+n)(q_l s_h - q_h s_l) + s_h(c_l s_h - q_l s_h - c_h s_l + q_h s_l)}{s_h^2}$, $\mu_h = 0$, $\lambda = 0$.

Proof of Lemma A1. Objective function of the oligopolist's problem is jointly concave on a convex set defined by linear constraints. Then, the optimal solution can be obtained by solving the first order conditions (Bazaraa et al., 2006). First, we solve equations (1) and (2) to get the market clearing prices as a function of quantities: $p_h(w_j, w_{-j}) = -q_l(w_{j,l} + w_{-j,l}) - q_h(-1 + w_{j,h} + w_{-j,h})$; $p_l(w_j, w_{-j}) = -q_l(-1 + w_{j,l} + w_{-j,l} + w_{j,h} + w_{-j,h})$. For brevity, we will use p_i for $p_i(w_j, w_{-j})$ in the remainder of the paper.

Then the first order conditions are: $\partial_{w_{j,h}} LG = 0$, $\partial_{w_{j,l}} LG = 0$, $\lambda(K - s_l w_{j,l} - s_h w_{j,h}) = 0$, $\mu_l w_{j,l} = 0$ and $\mu_h w_{j,h} = 0$. Simultaneous solution of these conditions yield the best response functions of the oligopolist firm given the actions of other firms (w_{-j}). Since we look for symmetric solutions, we plug in $(n-1)w_j$ for w_{-j}. We solve the resulting equation system for the equilibrium quantities $w_{j,l}, w_{j,h}$ and Lagrangian variables λ, μ_l, and μ_h. All possible solutions are as given in the Lemma. ■

Proof of Proposition 1. For the given set of parameters, we look for the solutions that are feasible within the solutions presented in Lemma A1. The feasibility conditions that need to be satisfied are as follows:

$w_{j,h} \geq 0$, $w_{j,l} \geq 0$, $\mu_h \geq 0$, $\mu_l \geq 0$, $\lambda \geq 0$, $w_{j,h} + w_{j,l} \leq 1/n$, and $s_h w_{j,h} + s_l w_{j,l} \leq K$.

When $c_h/c_l > q_h/q_l$, the optimal product line for an oligopolist is as follows. Note that there is only one feasible solution in all cases.

i) For $q_l - c_l \geq q_h - c_h$ and $K < \bar{K}_5^{(n)}$: $w^*_{j,h} = 0$ and $w^*_{j,l} = K/s_l$.

For $q_l - c_l \geq q_h - c_h$ and $K \geq \bar{K}_5^{(n)}$: $w^*_{j,h} = 0$ and $w^*_{j,l} = \frac{q_l - c_l}{(n+1)q_l}$.

Thus, only L is sold for all capacity levels.

ii.a) For $q_l - c_l < q_h - c_h$, $\frac{q_h - c_h}{s_h} < \frac{q_l - c_l}{s_l}$ and $K < \bar{K}_1^{(n)}$: $w^*_{j,h} = 0$ and $w^*_{j,l} = K/s_l$. Thus, only L is sold for all $K \leq \bar{K}_1^{(n)}$.

For $q_l - c_l < q_h - c_h$, $\frac{q_h - c_h}{s_h} < \frac{q_l - c_l}{s_l}$ and $\bar{K}_1^{(n)} < K < \bar{K}_3^{(n)}$:

$w^*_{j,h} = \frac{-K(1+n)q_l(s_l - s_h) + s_l(-c_h s_l + q_h s_l + c_l s_h - q_l s_h)}{(1+n)(q_h s_l^2 + q_l s_h(-2s_l + s_h))}$ and $w^*_{j,l} = \frac{K(1+n)(q_h s_l - q_l s_h) + s_h(c_h s_l - q_h s_l - c_l s_h + q_l s_h)}{(1+n)(q_h s_l^2 + q_l s_h(-2s_l + s_h))}$.

For $q_l - c_l < q_h - c_h$, $\frac{q_h - c_h}{s_h} < \frac{q_l - c_l}{s_l}$ and $K \geq \bar{K}_3^{(n)}$:

$w^*_{j,h} = \frac{-c_l + c_h + q_l - q_h}{(1+n)(q_l - q_h)}$ and $w^*_{j,l} = \frac{-c_h q_l + c_l q_h}{(1+n)q_l(q_l - q_h)}$.

Thus, both products L and H are sold for all $K > \bar{K}_1^{(n)}$.

ii.b) For $q_l - c_l < q_h - c_h$, $\frac{q_h - c_h}{s_h} \geq \frac{q_l - c_l}{s_l}$, and $K \leq \bar{K}_2^{(n)}$: $w^*_{j,h} = K/s_h$ and $w^*_{j,l} = 0$. Thus, only H is sold for all $K \leq \bar{K}_2^{(n)}$.

For $q_l - c_l < q_h - c_h$, $\frac{q_h - c_h}{s_h} \geq \frac{q_l - c_l}{s_l}$, and $\bar{K}_2^{(n)} < K < \bar{K}_3^{(n)}$:

$w^*_{j,h} = \frac{-K(1+n)q_l(s_l - s_h) + s_l(-c_h s_l + q_h s_l + c_l s_h - q_l s_h)}{(1+n)(q_h s_l^2 + q_l s_h(-2s_l + s_h))}$ and $w^*_{j,l} = \frac{K(1+n)(q_h s_l - q_l s_h) + s_h(c_h s_l - q_h s_l - c_l s_h + q_l s_h)}{(1+n)(q_h s_l^2 + q_l s_h(-2s_l + s_h))}$.

For $q_l - c_l < q_h - c_h$, $\frac{q_h - c_h}{s_h} \geq \frac{q_l - c_l}{s_l}$, and $K \geq \bar{K}_3^{(n)}$:

$w^*_{j,h} = \frac{-c_l + c_h + q_l - q_h}{(1+n)(q_l - q_h)}$ and $w^*_{j,l} = \frac{-c_h q_l + c_l q_h}{(1+n)q_l(q_l - q_h)}$.

Thus, both products L and H are sold for all $K > \bar{K}_2^{(n)}$.

■

Proof of Proposition 2. Similar to proof of Proposition 1, for the given set of parameters, we look for the solutions that are feasible within the solutions presented in Lemma A1.

When $c_h/c_l \leq q_h/q_l$, the optimal product line configuration for an oligopolist is as follows. Note that there is only one feasible solution in all cases.

i) For $\frac{q_h - c_h}{s_h} < \frac{q_l - c_l}{s_l}$, and $K \leq \bar{K}_1^{(n)}$: $w^*_{j,h} = 0$ and $w^*_{j,l} = K/s_l$. Thus, only L is offered for all $K \leq \bar{K}_1^{(n)}$ with positive quantity.

For $\frac{q_h - c_h}{s_h} < \frac{q_l - c_l}{s_l}$, and $\bar{K}_1^{(n)} < K \leq \bar{K}_2^{(n)}$:

$w^*_{j,h} = \frac{-K(1+n)q_l(s_l - s_h) + s_l(-c_h s_l + q_h s_l + c_l s_h - q_l s_h)}{(1+n)(q_h s_l^2 + q_l s_h(-2s_l + s_h))}$ and $w^*_{j,l} = \frac{K(1+n)(q_h s_l - q_l s_h) + s_h(c_h s_l - q_h s_l - c_l s_h + q_l s_h)}{(1+n)(q_h s_l^2 + q_l s_h(-2s_l + s_h))}$.

Thus, both products L and H is offered for the range $\bar{K}_1^{(n)} < K \leq \bar{K}_2^{(n)}$ with positive quantity.

For $\frac{q_h-c_h}{s_h} < \frac{q_l-c_l}{s_l}$, and $\bar{K}_2^{(n)} < K \leq \bar{K}_4^{(n)}$: $w^*_{j,h} = K/s_h$ and $w^*_{j,l} = 0$.
For $\frac{q_h-c_h}{s_h} < \frac{q_l-c_l}{s_l}$, and $K > \bar{K}_4^{(n)}$: $w^*_{j,h} = \frac{q_h-c_h}{(n+1)q_h}$ and $w^*_{j,l} = 0$.
Thus, only H is offered for all $K > \bar{K}_2^{(n)}$ with positive quantity.
ii) For $\frac{q_h-c_h}{s_h} \geq \frac{q_l-c_l}{s_l}$, and $K < \bar{K}_4^{(n)}$: $w_{j,h} = K/s_h$ and $w_{j,l} = 0$.
For $\frac{q_h-c_h}{s_h} \geq \frac{q_l-c_l}{s_l}$, and $K \geq \bar{K}_4^{(n)}$: $w_{j,h} = \frac{q_h-c_h}{(n+1)q_h}$ and $w_{j,l} = 0$.
Thus, only L is offered for all capacity levels with positive quantity.

■

Proof of Corollary 1. We define $\bar{k}_1 = \frac{s_h q_l(s_h-s_l)}{s_l(q_l s_h - q_h s_l)}$ which is necessary for the proof. Firm A chooses $w_{A,l}$ and $w_{A,h}$ to maximize

$$\begin{aligned}
\max_{w_{A,l},w_{A,h}} \quad & \Pi_A = w_{A,l}(q_l(1-[w_{A,l}+w_{B,l}+w_{A,h}+w_{B,h}])-c_l) \\
& +w_{A,h}(q_h(1-[w_{A,h}+w_{B,h}])-q_l[w_{A,l}+w_{B,l}]-c_h) \\
\text{st.} \quad & \qquad\qquad w_{A,l}s_l + w_{A,h}s_h \leq K_A.
\end{aligned}$$

Firm B solves the symmetric problem. We are looking for an equilibrium where $w_{A,l} > 0$,$w_{A,h} > 0$,$w_{B,l} = 0$,$w_{B,h} > 0$ and the capacity constraints bind. Thus, we can replace $w_{A,l}$ with $(K_A - w_{A,h}s_h)/s_l$ and similarly $w_{B,h}$ can be replaced with $(K_B - w_{B,l}s_l)/s_h$. Since $w_{B,l} = 0$, we have $w_{B,h} = K_B/s_h$. The optimal $w_{A,h}$ can then be solved using first order conditions ($\frac{d\Pi_A}{dw_{A,h}} = 0$), which yields

$$w_{A,h} = \frac{2K_A\frac{q_l}{s_l}(s_h-s_l) + K_B(\frac{q_l}{s_l} - \frac{q_h}{s_h})s_l - s_l s_h(\frac{q_l-c_l}{s_l} - \frac{q_h-c_h}{s_h})}{2(q_l s_h(\frac{s_h}{s_l}-2) + q_h s_l)} \tag{A-13}$$

For this to be an equilibrium it should satisfy, (i) $w_{A,h} > 0$, which requires $\frac{2\bar{k}_1 K_A + K_B}{3} \geq \bar{K}_2^{(2)}$. Note that this condition ensures that numerator in (A-13) is positive, denominator in (A-13) is always positive (this reduces to showing $\frac{q_h}{q_l} > \frac{s_h}{s_l}(2-\frac{s_h}{s_l})$ which is always true). Furthermore, we need (ii) $w_{A,l} > 0$ which is same as $K_A/s_h > w_{A,h}$ and this requires $\bar{K}_2^{(2)} \geq \frac{2K_A+K_B}{3}$. Finally for the above solution to be an equilibrium it should satisfy (iii) $\left.\frac{d\Pi_B}{dw_{B,h}}\right|_{w_{B,h}=K_B/s_h} \geq 0$ which leads to $K_B \geq \bar{K}_2^{(2)}$. Observe that conditions given in (i-iii) are the conditions of the corollary. The assumptions $\frac{q_l-c_l}{s_l} > \frac{q_h-c_h}{s_h}$ and $\frac{c_l}{q_l} > \frac{c_h}{q_h}$ ensures that $\bar{K}_2^{(2)} > 0$. Finally, it is straightforward to prove that $\bar{k}_1 > 1$, this reduces to showing $\frac{q_h}{q_l} > \frac{s_h}{s_l}(2 - \frac{s_h}{s_l})$ which is always true. Basically, the left hand side is always greater 1 and the right hand side is always smaller than 1. ■

Proof of Corollary 2. Firm Y chooses $w_{y,l}$ and $w_{y,h}$ to maximize

$$\begin{aligned}\max_{w_{y,l},w_{y,h}} \quad & \Pi_Y = w_{y,l}(q_l(1-[w_{y,l}+w_{z,l}+w_{y,h}+w_{z,h}])-c_{yl}) \\ & +w_{y,h}(q_h(1-[w_{y,h}+w_{z,h}])-q_l[w_{y,l}+w_{z,l}]-c_{yh}) \\ \text{st.} \quad & w_{y,l}s_l + w_{y,h}s_h \leq K.\end{aligned}$$

Firm Z solves the symmetric problem. We are looking for an equilibrium where $w_{y,l} > 0$,$w_{y,h} > 0$,$w_{z,l} = 0$,$w_{z,h} > 0$ and the capacity constraints bind. Thus, we can replace $w_{y,h}$ with $(K - w_{y,l}s_l)/s_h$ and similarly $w_{z,h}$ can be replaced with $w_{z,h} = K/s_h$. The optimal $w_{y,h}$ can then be solved using first order conditions ($\frac{d\Pi_Y}{dw_{y,h}} = 0$), which yields

$$w_{A,h} = (s_h(-c_{yl}s_h+q_ls_h+c_{yh}s_l-q_hs_l)+K(-3q_ls_h+3q_hs_l))/(2(q_ls_h(s_h-2s_l)+q_hs_l^2)) \quad \text{(A-14)}$$

For this to be an equilibrium it should satisfy, (i) $w_{y,h} > 0$, since the denominator is positive, for the numerator, we need $K < (s_h(-c_{yl}s_h + q_ls_h + (c_{yh} - q_h)s_l))/(3q_ls_h - 3q_hs_l) = \bar{k}_2$. Furthermore, we need (ii) $w_{y,l} > 0$ which requires $K > -((s_hs_l(c_{yl}s_h - q_ls_h + (-c_{yh} + q_h)s_l))/(q_ls_h(2s_h - s_l) - q_hs_l^2)) = \bar{k}_4$. Now we need to show there exist $\bar{k}_4 < K < \bar{k}_2$. Since $s_hq_l > q_hs_l$ and $s_h(q_l - c_{yl}) > s_l(q_h - c_{yh})$ in this case, $\bar{k}_4 < \bar{k}_2$. Finally for the above solution to be an equilibrium it should satisfy (iii) $\frac{d\Pi_Z}{dw_{z,h}}\Big|_{w_{z,h}=K/s_h} \geq 0$ which leads to $K < (s_h(2q_hq_ls_h^2 - q_l^2s_h^2 - c_{yh}q_ls_hs_l - 2q_hq_ls_hs_l + c_{yh}q_hs_l^2 + q_h^2s_l^2 + c_{yl}s_h(q_ls_h - q_hs_l) - 2c_{zh}(q_ls_h(s_h - 2s_l) + q_hs_l^2)))/(-3q_l^2s_h^2 + 6q_hq_ls_h(s_h - s_l) + 3q_h^2s_l^2) = \bar{k}_3$. Denominator is always positive. For numerator to be positive, $c_{zh} < ((c_{yl} + 2q_h - q_l)q_ls_h^2 - (c_{yl}q_h + (c_{yh} + 2q_h)q_l)s_hs_l + q_h(c_{yh} + q_h)s_l^2)/(2(q_ls_h(s_h - 2s_l) + q_hs_l^2)) = \bar{c}_1$. Since this case is $s_h/s_l > q_h/q_l > c_h/c_l, \bar{c}_1 > 0$. Also, $c_{zh} < c_{yh}$ is guaranteed when $c_{yh} > q_h + ((c_{yl} - q_l)s_h(q_ls_h - q_hs_l))/(q_ls_h(2s_h - 3s_l) + q_hs_l^2) = \bar{c}_2$.
■

Proof of Proposition 3.

i) Following Proposition 1, when $q_h - c_h > q_l - c_l$ and $\frac{q_h - c_h}{s_h} < \frac{q_l - c_l}{s_l}$, both in the monopoly case and the duopoly cases, the firm offers only L below a threshold capacity; $\bar{K}_1^{(n)}$ and $\bar{K}_1^{(n+1)}$ respectively where $\bar{K}_1^{(n+1)} = (n+1)/(n+2)\ \bar{K}_1^{(n)}$. The firms offer both product types above those thresholds. Thus, for all $\bar{K}_1^{(n+1)} < K < \bar{K}_1^{(n)}$, the solution in the duopoly market is to offer both H and L while the solution in the monopoly market remains to be to offering only L.

ii) Following Proposition 1, when $\frac{q_h - c_h}{s_h} > \frac{q_l - c_l}{s_l}$, both the monopoly case and the duopoly case solutions are to offer only H below a threshold capacity; $\bar{K}_2^{(n)}$ and $\bar{K}_2^{(n+1)}$ respectively where

$\bar{K}_2^{(n+1)} = (n+1)/(n+2)\ \bar{K}_2^{(n)}$. The firms offer both product types above those thresholds. Thus, for all $\bar{K}_2^{(n+1)} < K < \bar{K}_2^{(n)}$, in the duopoly case both H and L are offered while in the monopoly case solution remains to be to offering only H. ■

Proof of Proposition 4.

Following Proposition 2, when $(q_h - c_h)/s_h < (q_l - c_l)/s_l$, both the monopoly case and the duopoly case solutions are to offer only H above a threshold capacity; $\bar{K}_2^{(n)}$ and $\bar{K}_2^{(n+1)}$ respectively where $\bar{K}_2^{(n+1)} = (n+1)/(n+2)\ \bar{K}_2^{(n)}$. The firms offer both product types below those thresholds. Thus, for all $\bar{K}_2^{(n+1)} < K < \bar{K}_2^{(n)}$, in the duopoly case offering only H is the optimal solution while in the monopoly case solution remains to be offering both H and L. ■