

Small Worlds:

Modeling Attitudes towards Sources of Uncertainty*

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Abstract

In the literature on decision making under uncertainty, Ramsey and de Finetti originated the idea of distinguishing events according to whether they are ‘exchangeable’. In this paper, two events are exchangeable if the decision maker is indifferent to a permutation of their payoffs. Building on exchangeability and weak behavioral assumptions, we derive a comparability relation that delivers global probabilistic sophistication without requiring monotonicity or continuity. One advantage of our exchangeability based approach to global probabilistic sophistication is the intuitive manner in which it can be extended to accommodate non-global probabilistically sophisticated behavior. When it is not the case that every pair of events is comparable, we further introduce the concept of a *domain* - an analogue of Savage’s (1954) notion of a ‘small world’ - as a maximal collection of comparable events. Under similarly weak behavioral conditions we demonstrate probabilistic sophistication without monotonicity in any domain. Multiple domains (or small worlds) probabilistic sophistication offers a unifying perspective to modeling a decision maker’s attitudes towards different sources of uncertainty encompassing and extending the distinction between risk and ‘ambiguity’ often associated with Ellsberg-type behavior.

Keywords: Uncertainty, Risk, Ambiguity, Decision Theory, Non-Expected Utility, Utility Representation, Probabilistic Sophistication, Ellsberg Paradox.

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1. Introduction

1.1. Motivation

In their pioneering studies, Ramsey (1926) and de Finetti (1937) originated the idea of distinguishing events according to whether they are ‘exchangeable’ or ‘ethically neutral’, providing the basis for their construction of a decision maker’s subjective probability over events.¹ Savage’s (1954) subsequent formulation departs from this direction and nevertheless yields an overall subjective probability on the decision maker’s ‘big world’ consisting of all exhaustive and mutually exclusive contingencies. Building on Savage’s approach, Machina and Schmeidler (1992) and Grant (1995) provide more parsimonious characterizations of what is termed probabilistic sophistication, in which the choice behavior of a decision maker reflects her probabilistic belief in the sense that events are distinguished only by their subjective probabilities.

In the same work, Savage discusses how decision making tends to take place in a ‘small world’ - events relevant to a particular decision situation, that partition the state space. According to Savage, a small world embodies a balance between the meanings of two proverbs, “*Look before you leap*”, and “*You can cross that bridge when you get to it*”. He further interprets the latter proverb as meaning “*to attack relatively small problems of decision by artificially confining attention to so small a world that the former proverb applies*”. Savage also provides examples to illustrate the ubiquitous presence of ‘small-world thinking’ in decision making. His examples include betting on the temperature in Chicago, the sequence of heads and tails from repeatedly tossing a particular coin, and the decimal expansion of π . Notice that these examples of ‘small worlds’ correspond to different sources of risk or uncertainty. Under global probabilistic sophistication, however, the only attribute of an event that is relevant for decision making is its subjective probability, thus small worlds can not be distinct in any behavioral sense.

In what has come to be known as Ellsberg’s (1961) two-urn problem, originally proposed in Keynes (1921), one urn contains 50 red and 50 black balls while the second urn contains an unspecified combination of the two.² Say that two events are exchangeable if the decision maker is indifferent to permuting their payoffs. It is commonly observed that the event of drawing

¹de Finetti is associated with a number of results that have been described in terms of exchangeability. We refer specifically to his thinking on “Exchangeable Events” (Chapter 3) and his “Reflections on the Notion of Exchangeability” (Chapter 5) in Kyburg’s translation of de Finetti (1937). Ramsey focused on the partition of the state space into two ‘exchangeable’ events by what he called ‘ethically neutral’ events.

²Keynes suggested that “*If two probabilities are equal in degree, ought we, in choosing our course of action, to prefer that one which is based on a greater body of knowledge?*” Contemporaneously, Knight (1921) distinguished between ‘risk’ involving events with clearly defined underlying probabilities and ‘uncertainty’ involving vague (or ambiguous) events on which probabilities are not well defined. He suggested that people such as entrepreneurs would need to be incrementally compensated for facing uncertainty.

a red ball is exchangeable with the event of drawing a black ball from the same urn, but not across urns. In particular, people seem to prefer betting using the first ('known') rather than the second ('unknown'). By contrast, all the above events are deemed to have the same likelihood in the Savage setting, and are therefore exchangeable regardless of their sources. This suggests that each urn can be associated with a different small world, or source of uncertainty, and that the event – “drawing a ball of a particular color from the first urn” – may not be exchangeable with the event – “drawing a ball of the same color from the second urn”. That is, “equally likely” complementary events in one small world may not be exchangeable with “equally likely” complementary events in another small world.³

This observed pattern of behavior involving multiple sources of uncertainty and seemingly ‘equally likely’ events appears pervasive. Betting on the sequence of heads and tails from one’s own coin may be preferable to doing so with a coin from someone unfamiliar. One can come up with different ways to differentiate among urns, say your favorite uncle puts the unknown urn together. When betting on whether the 16th digit in the decimal expansion of $\sqrt{13}$ and $\sqrt{14}$ is even or odd, a Hong Kong resident may prefer to bet on $\sqrt{13}$ while a Bay area resident may prefer to bet on $\sqrt{14}$. (In Cantonese, 14 and 13 sounds respectively like “die for sure” and “live for sure”.) In playing Lotto, customers may prefer selecting their own numbers rather than having them picked by a computer.

In a recent experimental study, Fox and Tversky (1995) find that subjects tend to prefer betting on either of two complementary temperature intervals in San Francisco than betting on either of two complementary temperature intervals in Istanbul. Here, identification of subjects with the area where they live appears to be a factor in their choice behavior. In another experimental study, Denesraj and Epstein (1994) report that certain subjects prefer drawing a red bean from a bag with 7 red beans out of 100 beans than from a bag with 1 red bean out of 10 beans. In particular, subjects reported that “*although they knew, the probabilities were against them, they felt they have a better chance when there were more red beans*”. While some might hold the view that the latter behavior is sufficiently ‘irrational’ to merit ignoring, the finding serves to illustrate that even subtle differences between sources of risk or uncertainty can lead a decision maker to treat them differently.

The observed choice behavior over Ellsbergian urns has inspired a substantial literature in axiomatic models of decision making (see, e.g., Schmeidler, 1989; Gilboa and Schmeidler, 1989;

³There is a growing literature recognizing the importance of distinct sources of uncertainty in decision making. See, for example, Heath and Tversky (1991), Fox and Tversky (1995), Keppe and Weber (1995), as well as an early reference in Fellner (1961).

Nakamura, 1990). A number of these works posit the Knightian (1921) distinction between risk and uncertainty and classify events as being either subjectively ambiguous or subjectively unambiguous (see Epstein and Zhang, 2001; Ghirardato and Marinacci, 2001; Klibanoff, Marinacci and Mukerji, 2002; Nehring, 2001, 2002; Ghirardato, Marinacci and Maccheroni, 2002). The preceding discussion suggests the need to go beyond distinguishing between risk and uncertainty and develop a richer framework to classify events and model the decision maker’s attitudes towards risks arising concurrently from multiple distinct sources.

Building on exchangeability as the primitive, we develop a notion of comparability to capture the intuition behind a likelihood relation among events. Specifically, two disjoint events are comparable when one contains a subevent that is exchangeable with the other. Informally, one is motivated to view one event as ‘larger’ or ‘more likely’ than the other. When all disjoint events are comparable in this way, we show that very weak conditions - far weaker than Savage’s P3 and P4 - suffice to deliver probabilistic sophistication on the part of the decision maker. We also demonstrate how our approach can be adapted to encompass state dependent preferences.

To deal with the case in which not all events are comparable, we define a *domain* as a suitably maximal collection of comparable events. When conditioning on events within a domain, that domain can be viewed as an endogenously induced Savagian small world. Here too, very weak behavioral assumptions lead to non-global probabilistic sophistication on specific domains. To help link the decision maker’s probabilistic beliefs over distinct small worlds, we introduce a domain independence axiom requiring the conditional certainty equivalent for any act over a domain to not depend on payoffs associated with events outside that domain.

1.2. Comparability, Probabilistic Sophistication, and Domains

We illustrate the main ideas behind our contribution using an example. Consider the elicitation of equally likely events corresponding to the amount of rainfall in a particular city. The elicitation might proceed as follows: ask the decision maker to find some amount of rainfall that makes her indifferent to betting above versus below that amount, and denote this special point, if it exists, as r_{50} . We say that the event “rainfall $\geq r_{50}$ ” is exchangeable with the event “rainfall $\leq r_{50}$ ”, since the decision maker is indifferent to permuting the payoffs of any bet over the two events. We may also say that the two events are equally likely and r_{50} is the median of the subjective rainfall distribution. One can continue the elicitation by asking the decision maker to specify an amount of rainfall, say r_{25} , between 0 and r_{50} such that she is indifferent to exchanging the outcomes, x and y , in any bet of the form “win x if the rainfall is between 0 and r_{25} , y if it is

between r_{25} and r_{50} , and win $p(r)$ if the rainfall r is above r_{50} ,” where $p(r)$ is any given function. If the decision maker can specify r_{25} then we say that the event ‘rainfall is between 0 and r_{25} ’ is exchangeable with the event ‘rainfall is between r_{25} and r_{50} .’ One can likewise attempt to find the number r_{75} that makes ‘rainfall is between r_{50} and r_{75} ’ exchangeable with ‘rainfall is above r_{75} .’ It seems sensible to view r_{25} and r_{75} as the first and third quartiles of the subjective rainfall distribution if *all* four events constructed are exchangeable.

If exchangeability is viewed as a subjective ‘equal likelihood’ relation among events, then it can be used to derive an ‘at least as likely as’ relation as follows: E is ‘at least as likely as’ E' when $E \setminus E'$ contains a subset that is ‘as likely as’, or exchangeable with $E' \setminus E$.⁴ Whenever such a comparison between two events can be made we say that they are ‘comparable’. We caution the reader that ‘comparability’ among all events in a general setting does not give rise to probabilistic sophistication. On its own, comparability is a ‘pre-notion’ of likelihood that builds on exchangeability of subevents. One contribution of this paper is in establishing, under very weak additional conditions, that probabilistic sophistication follows from comparability of every pair of events. Besides being a more parsimonious characterization of global probabilistic sophistication in relation to the literature, this direction enables a simple approach to modeling departures from global probabilistic sophistication in terms of non-comparability among events. It is to this latter point that we address the remainder of the Introduction.

Suppose now, that the elicitation is repeated for another city, and denote the two cities and their respective medians as (C, r_{50}) and (C', r'_{50}) . If, as far as the decision maker is concerned, rain or its absence is a neutral event in either city (i.e., she does not care whether it rains nor how much), then according to Savage (1954) the decision maker is indifferent between betting on rainfall above the median in one city versus the other. It is not difficult to imagine how a decision maker might not be Savagian. Strict preference might exist if she is more familiar with one city than another. This can also happen for other reasons. For instance, if one city is the decision maker’s birthplace and the other is her current home, she might strictly prefer placing bets on her birthplace. Likewise, even if the decision maker is equally familiar with the distribution of rainfall in two cities, if one city is the home base of her favorite sports team and the other is the home city of her team’s traditional rival, she might feel better about betting on one city than the other. In these examples, it seems that the event $r \geq r_{50}$ is not exchangeable with $r' \geq r'_{50}$, even when each is exchangeable with its complement. Moreover, it seems reasonable to assume that $r' \geq r'_{50}$ is not exchangeable with *any* rainfall interval in the city C . Thus, one might say that while rainfall intervals are comparable within a city, this may not be so across cities.

⁴The set-difference, $E \setminus E'$, is the event E without any elements that are also in E' .

More generally, when it is not the case that any pair of events are comparable, it is natural to ask whether there are self-contained *domains* of events, such that the decision maker's preferences over acts restricted to any one domain exhibit probabilistic sophistication. It seems clear that rainfall in each city provides an example of such domains. In this spirit, we define a domain as a set of comparable events that is self-contained in the following key senses: (i) any pair of domain events are comparable; (ii) the union of all domain elements is in the domain; (iii) if E is the 'more likely' of two comparable events in the domain, then both the 'copy' of the other event within E , as well as the remainder, are also in the domain; and (iv) the collection resulting from adding any other collection of non-null event to the domain is not compatible with (i)-(iii). The first three conditions require the domain to be self contained, while the last requires it to be 'maximal'. Along with a technical condition,⁵ this description of a domain renders it a λ -system whose universal set may be a proper subset of the full state space.⁶ Domains of events arise endogenously according to the preferences of the decision maker and the manner in which she treats distinct sources of uncertainty. Another main contribution of this paper is in showing that, given weak assumptions, preferences restricted to a domain exhibit probabilistic sophistication.

1.3. Domain Independence and Utility Representation

Our discussion so far demonstrates how the presence of non-exchangeability may arise from distinct sources of uncertainty leading to deviation from global probabilistically sophisticated behavior. In situations involving multiple domains, choice behavior reduces to lottery preferences when taking bets purely within any single domain. For instance, restricting bets to any one of the cities in the example will lead to the impression that the decision maker is probabilistically sophisticated with well defined risk preferences over each domain. This suggests that the distinguishing features between domains are the risk attitudes of the decision maker within each domain. In particular, if the decision maker consistently prefers to play a lottery (with respect to the domain-specific subjective probability) in one city versus an 'identical' lottery in the other city, then it is reasonable to say that she is more risk averse in one domain. Alternatively, by pinning down the risk attitudes (or certainty equivalents) within each domain, it is possible to make comparisons between bets on single distinct domains.

In order to identify possible representations for acts involving payoffs across multiple do-

⁵The technical condition states that the domain is closed under countable unions.

⁶A λ -system, defined in Section 2, is a collection of events that possesses some key properties which allow it to form the basis of a probability measure.

mains, we need more structure. Consistent with the identification of domains with Savagian small worlds, we propose that ‘decisions across domains are separable’. Our *domain independence* axiom – similar to the sure thing principle – requires that a certainty equivalent for a conditional act that is adapted to a domain be *independent* of payoffs outside the domain.

To illustrate this, consider acts of the form $(p, H; q, T)$ where H and T correspond respectively to either ‘heads’ or ‘tails’ in a coin toss, and each of p and q corresponds to a payoff function that depends on the rainfall in one and only one of the two cities. If the coin toss is H , then the decision maker is paid according to p , otherwise the payoff corresponds to q . The payoff function, p , can depend on rainfall in one city or the other but not both, and the same is true of q (although both p and q can correspond to the same city). Since the decision maker may prefer to bet on a coin toss than a ‘subjective’ 50 – 50 bet on the rainfall distribution in either city, it is arguable that pure coin tossing events are not exchangeable with any rainfall intervals much in the same way that Ellsbergian decision makers prefer betting on a known versus an unknown urn. Thus in addition to the two city-domains, there is now another domain corresponding to pure coin toss events.

For expository ease, we assume that the risk preferences of the decision maker in each domain correspond to expected utility. This implies that the decision maker has von Neumann-Morgenstern utility functions $u_C, u_{C'}, u_{coin}$ for acts adapted to each of the three different domains. A preference for bets on C versus C' can be modeled by letting $u_{C'}$ be more concave than u_C , which is in turn more concave than u_{coin} . If p pays in city C , then its certainty equivalent c_p is given by:

$$u_C(c_p) = \int u_C(p(r))d\mu_C(r),$$

where μ_C is the subjective probability elicited on rainfall in city C . A similar expression holds for the certainty equivalent of p if it pays in city C' . The certainty equivalent for the pure coin toss bet, $(x, H; y, T)$, awarding payoffs of x and y is

$$u_{coin}(c) = \frac{1}{2}u_{coin}(x) + \frac{1}{2}u_{coin}(y).$$

Domain separability implies that the decision maker is indifferent between an act of the form $f = (p, H; q, T)$ and $(c_p, H; c_q, T)$, where c_p and c_q are the certainty equivalents of p and q , respectively. Thus the certainty equivalent of f can be determined from the above expression. Unless the risk attitudes in all three domains coincide, preferences implied by this representation are not probabilistically sophisticated. One may also explore mixed acts over joint intervals of

rainfall. For example, the act that pays \$100 if rainfall in *both* cities is above the subjective medians, and zero otherwise. Dealing with such acts requires additional assumptions over the domain structure of preferences. We investigate this further in the main text.

The domain recursive preference structure of the example is reminiscent of a growing literature that rationalizes Ellsbergian attitudes by means of a two stage valuation approach (see Segal (1990), Klibanoff, Marinacci and Mukerji (2002), Nau (2002) and Ergin and Gul (2002)). Whereas the recursive stages arise endogenously in our setting via domain independence, it is posited or exogenously imposed in the references. Segal (1990) illustrates that a two stage decision making model can rationalize some violations of probabilistic sophistication and the independence axiom. Klibanoff, Marinacci and Mukerji (2002) achieve this by supplementing the state space with ‘information states’ corresponding to possible probability distributions over the full state space. Nau (2002) and Ergin and Gul (2002) similarly enrich the state space with ‘credal states’ or ‘issues’, respectively.

1.4. Summary and Outline

The preceding discussion suggests a broadly applicable perspective on the modeling of probabilistically sophisticated behavior in the presence of multiple sources of uncertainty. Our approach does not require monotonicity or even Savage’s P4, while encompassing a range of observed behavior that go beyond the Knightian distinction between risk and uncertainty.

Section 2 contains preliminaries including formal definitions of the concepts introduced in the Introduction. Section 3 presents our main results concerning probabilistic sophistication in global as well as multiple domains settings. We also provide a discussion vis-a-vis the literature on probabilistic sophistication, including the case of state dependent preferences. In Section 4, we offer additional examples of preferences exhibiting multiple domains, and demonstrate how we can model the decision maker’s preference for acts whose payoffs are associated only with events in a single domain. To model attitudes towards mixed acts involving payoffs from distinct sources, we posit a domain independence axiom and other necessary conditions leading to a domain recursive representation. Concluding remarks are provided in Section 5.

2. Preliminaries

Let Ω be a space whose elements correspond to all states of the world. Let X be a set of payoffs and Σ a σ -algebra on Ω . Elements of Σ are events. The set of simple acts, \mathcal{F} , is comprised of all Σ -adapted and X -valued functions over Ω that have a finite range. As is customary, $x \in X$ is

identified with the constant act that pays x in every state. We will often devote attention and examples to subsets of Σ . An important role is played by λ -systems contained in Σ :

Definition 1. A λ -system in Σ is a collection of events, $\mathcal{A} \subseteq \Sigma$ that satisfies the following properties:

- i) The event $\widehat{\mathcal{A}} \equiv \bigcup_{E \in \mathcal{A}} E$ is in \mathcal{A} .
- ii) If $E \in \mathcal{A}$ then $\widehat{\mathcal{A}} \setminus E \in \mathcal{A}$.
- iii) If $E, E' \in \mathcal{A}$ are disjoint, then $E \cup E' \in \mathcal{A}$.

Observe that the event $\widehat{\mathcal{A}} \equiv \bigcup_{E \in \mathcal{A}} E$, referred to as the *envelope* of \mathcal{A} , may be *strictly* contained in Ω . We say that $f \in \mathcal{F}$ is *adapted to a collection of events*, $\mathcal{A} \subseteq \Sigma$, whenever $f^{-1}(x) \cap \widehat{\mathcal{A}} \in \mathcal{A}$ for every $x \in X$. For any collection of disjoint events, $E_1, E_2, \dots, E_n \subset \Omega$, and $f_1, f_2, \dots, f_n, g \in \mathcal{F}$, let $f_1 E_1 f_2 E_2 \dots f_n E_n g$ denote the act that pays $f_i(\omega)$ if the true state, $\omega \in \Omega$, is in E_i , and pays $g(\omega)$ otherwise. We say that $E \in \Sigma$ is *null* if $f E h \sim g E h \forall f, g, h \in \mathcal{F}$.⁷

The decision maker has a non-degenerate binary preference relation on \mathcal{F} :

Axiom 1 (P1). \succsim is a weak order on \mathcal{F} .

Axiom 2 (P5). There exists $f, g \in \mathcal{F}$ such that $f \succ g$.

To capture the sense in which events are similar, we introduce a binary relation over events via \succeq :

Definition 2 (Event Exchangeability). For any pair of disjoint events $E, E' \in \Sigma$, $E \approx E'$ if for any $x, x' \in X$ and $f \in \mathcal{F}$, $x E x' E' f \sim x' E x E' f$.

Whenever $E \approx E'$ we will say that E and E' are *exchangeable*. Note that given P1, all null events are exchangeable. Exchangeability may be viewed as a pre-notion of ‘equally likely’: two events are ‘equally likely’ if the decision maker is indifferent to a permutation of their payoffs. Without further structure this interpretation is not formally justified since, as the next example demonstrates, \approx is not necessarily transitive, and therefore not an equivalence relation.

Example 1. Consider the partition $\{A, B_1, B_2, C\}$ of Ω . Let $X \equiv [0, 1]$ and the utility representation over acts $x A y_1 B_1 y_2 B_2 z$ be given by

$$V(x, y_1, y_2, z) = x + z + \frac{y_1 + y_2}{2} + \frac{y_1 - y_2}{4} x$$

⁷This definition may not be robust relative to the presence of ‘extreme’ forms of state dependence. For instance, in Aumann’s example of a man who will lose his taste for life should his sick wife die, the event ‘sick wife dies’ will also be classified by us as null. For further discussion of this issue see Karni (2003) and references therein.

It is straight forward to check that the representation is monotonic. It should also be clear that $A \approx B_1 \cup B_2$ and $C \approx B_1 \cup B_2$. On the other hand, it is certainly not the case that $A \approx C$ due to the asymmetry between x and z arising in the last term of the utility function. ■

Intuitively, an event is ‘at-least-as-likely’ as any of its subevents. Exchangeability supplies the motivation underlying a similar comparison across disjoint events, $E, E' \in \Sigma$: if a subevent of E is exchangeable with E' , then it is also intuitive to view E as ‘at-least-as-likely’ as E' . Building on this intuition, we define the following exchangeability based relation between any two events.

Definition 3 (Event Comparability). *For any events, $E, E' \in \Sigma$, $E \succeq^C E'$ whenever $E \setminus E'$ contains a subevent that is exchangeable with $E' \setminus E$.*

Just as \approx gives a pre-notion of ‘equal likelihood’ among events, \succeq^C provides a pre-notion of an ‘at-least-as-likely’ relation. The event E is ‘at least as likely’ as E' if outside their intersection the ‘more likely’ event (i.e., $E \setminus E'$) contains a ‘copy’ of the ‘less likely’ event (i.e., $E' \setminus E$). Since \emptyset is a subevent of any event and \emptyset is exchangeable with itself, $e \subseteq E$ implies $E \succeq^C e$.⁸

A subevent of $E \setminus E'$ that is exchangeable with $E' \setminus E$ is termed a *comparison event*. For any $E, E' \in \Sigma$, we say that E and E' are *comparable* whenever $E \succeq^C E'$ or $E' \succeq^C E$. Moreover, define $E \succ^C E'$ whenever $E \succeq^C E'$ and it is not the case that $E' \succeq^C E$. Likewise, define \sim^C as the symmetric part of \succeq^C .

Suppose that the decision maker behaves as if she assigns a unique probability measure to each event and the measure of an event is its only relevant characteristic for the purpose of her decision making. Clearly, if two events are equally likely then their set differences are also equally likely and thus exchangeable. If Σ is sufficiently ‘fine’ any event will contain a subevent with arbitrary yet smaller likelihood, and therefore any two events in the decision maker’s world are comparable. Thus an important attribute characterizing this decision maker’s global probabilistic sophistication is the fact that *all events are comparable*, or alternatively, \succeq^C is complete.

When preferences are not probabilistically sophisticated, it may not be the case that every event is comparable to every other event. However, as argued in the Introduction, it is possible

⁸One might also wish to assert that E is ‘at-least-as-likely’ as E' whenever $E \setminus E'$ is exchangeable with an event that contains $E' \setminus E$. While the latter is also compelling, adding it to Definition 3 turns out to be redundant if the state space is atomless or composed of equal sized atoms. While it may be useful in exploring cases of unequal atoms, changing Definition 3 comes at a non-trivial cost in additional structure if one is to obtain probabilistic sophistication. This issue is not unique to our work – the majority of papers in this literature tend to focus on non-atomic state spaces. The additional conditions that are necessary for the general atomic case are discussed in Chateauneuf, 1985.

that probabilistic sophistication may still be exhibited on certain collections of events. We are led to identify such collections according to the following intuition: first, as in the case of global probabilistic sophistication discussed above, every event in the collection should be comparable with every other event; second, likelihood is generally defined relative to some benchmark event - in the case of global probabilistic sophistication the benchmark event is Ω . Thus the collection, say \mathcal{A} , should contain a ‘universal’ event which we take to be its envelope, $\widehat{\mathcal{A}}$. Finally, consider two events, E and E' , in the collection, \mathcal{A} , that can potentially be described via a probability measure relative to $\widehat{\mathcal{A}}$. If $E \succeq^C E'$ then $E \setminus E'$ contains a subset, say e , that is ‘as likely as’ $E' \setminus E$. Thus if the likelihood (relative to $\widehat{\mathcal{A}}$) of $E' = (E \cap E') \cup (E' \setminus E)$ is known, it should be equal to that of $\xi \equiv (E \cap E') \cup e$, and it seems sensible to require $\xi \in \mathcal{A}$. If, in addition, the likelihood (relative to $\widehat{\mathcal{A}}$) of E is known, then one can readily calculate the likelihood of $E \setminus \xi = E \setminus (E' \cup e)$, which should therefore also be in \mathcal{A} . Given these considerations, we characterize collections over which the decision maker may be probabilistically sophisticated as follows:

Definition 4. *A collection of events, $\mathcal{A} \subseteq \Sigma$ is **homogeneous** if it satisfies the following:*

1. *If $E, E' \in \mathcal{A}$, then E and E' are comparable.*
2. *$\widehat{\mathcal{A}} \in \mathcal{A}$.*
3. *For any $E, E' \in \mathcal{A}$ such that $E \succeq^C E'$, if $e \subseteq E \setminus E'$ is a comparison event, then $(E \cap E') \cup e \in \mathcal{A}$ and $E \setminus (E' \cup e) \in \mathcal{A}$.*

It is important to observe that if every event is comparable to every other event, then Σ itself is homogeneous. For proper homogeneous subsets of Σ , the definition implies that if E and $E' \subseteq E$ are in \mathcal{A} , then $E \setminus E'$ is in \mathcal{A} (\emptyset plays the role of the comparison event). The logic behind the definition of a homogeneous collection resembles that which leads to the construction of λ -systems of events. Indeed, we have the following result:

Lemma 1. *If $\mathcal{A} \subseteq \Sigma$ is homogeneous, then \mathcal{A} is a λ -system.*

Proof: Since $\widehat{\mathcal{A}} \in \mathcal{A}$, and any $E \in \mathcal{A}$ is a subset of $\widehat{\mathcal{A}}$, the event $\widehat{\mathcal{A}} \setminus E$ is also in \mathcal{A} . Properties (i) and (ii) in Definition 1 are therefore satisfied. Suppose $E, E' \in \mathcal{A}$ are disjoint. Since $\widehat{\mathcal{A}} \setminus E \in \mathcal{A}$ and $E' \subseteq \widehat{\mathcal{A}}$, it must be that $\widehat{\mathcal{A}} \setminus (E \cup E') \in \mathcal{A}$. The latter, in turn, implies that $E \cup E'$, the relative complement of $\widehat{\mathcal{A}} \setminus (E \cup E')$, is also in \mathcal{A} . Thus property (iii) in Definition 1 is satisfied and \mathcal{A} is a λ -system. ■

Example 2. Consider an urn containing 100 balls of four different colors: red (R), green (G), blue (B), and white (W) – see Zhang (1999). The decision maker has limited knowledge about the numbers r , g , b , and w of colored balls. Specifically, $r + g + b + w = 100$ and $r + b = g + w = 50$. Denote composite events by the letters of their colors (e.g., BGW corresponds to the event ‘Blue or Green or White ball drawn’). It seems reasonable to assume that the events BG, BR, GW, RW are comparable. Moreover, since $r = g$ and $b = w$, it seems reasonable that B and W are considered exchangeable, as are R and G . Finally, the intuition underlying Ellsbergian behavior suggests that a bet of \$100 on BR (a 50 – 50 proposition) is strictly preferred to a bet of \$100 on BW (where the odds are unknown); thus neither of B and W is comparable with either R or G . Based on this analysis, we can identify several homogeneous collections:

- i) $\{\emptyset, B, W, BW\}$
- ii) $\{\emptyset, G, R, GR\}$
- iii) $\{\emptyset, BW, GR, BGRW\}$ – similar to the Ellsberg urn with unknown mixture of two colors.
- iv) $\{\emptyset, BG, RW, BGRW\}$ – similar to the Ellsberg urn with known mixture of two colors.
- v) $\{\emptyset, GW, BR, BGRW\}$ – similar to the Ellsberg urn with known mixture of two colors.
- vi) $\{\emptyset, GW, BR, BG, RW, BGRW\}$

Notice that the last collection is the λ -system example in Zhang (1999) of subjectively unambiguous events (see also Epstein and Zhang, 2001). It is also the union of the two homogeneous collections preceding it. In particular, this serves to demonstrate that a homogeneous collection is not generally exhaustive and can be strictly contained within another homogeneous collection. In Section 4 we restrict attention to ‘exhaustive’ or maximal homogeneous collections, which we identify with Savageian ‘Small Worlds.’ ■

We also need the following definitions:

Definition 5. \succeq^C is a *likelihood relation* over a λ -system of events, $\mathcal{A} \subseteq \Sigma$ if the following conditions hold:

- i) \succeq^C is a weak order over \mathcal{A}
- ii) $\widehat{\mathcal{A}} \succ^C \emptyset$ and for every $A \in \mathcal{A}$, $A \succeq^C \emptyset$ and $\widehat{\mathcal{A}} \succeq^C A$

iii) for every $A, B, C \in \mathcal{A}$ such that $C \cap (A \cup B) = \emptyset$, $A \succeq^C B \Leftrightarrow A \cup C \succeq^C B \cup C$

Note that the last requirement is satisfied by the definition of \succeq^C .

Definition 6. μ is an **agreeing probability measure** for \succeq^C over a λ -system, $\mathcal{A} \subseteq \Sigma$, if it is a probability measure over \mathcal{A} and for every $A, B \in \mathcal{A}$, $A \succeq^C (\succ^C)B \Leftrightarrow \mu(A) \geq (>)\mu(B)$.

Finally, we require \succeq to be ‘bounded’:

Axiom 3 (Weak Event Archimedean Property). If $\{e_i\}_{i=0}^\infty \subseteq \Sigma$ is a sequence of disjoint events with $e_i \approx e_{i+1}$ for every $i = 0, \dots$, then e_0 is null.

Axiom 3 prevents pathological situations in which Σ contains an infinite number of disjoint and ‘equally likely’ non-null events.

3. Probabilistic Sophistication Without Monotonicity

While homogeneity may be appealing, on its own it is not sufficient for the existence of a likelihood relation, let alone a unique agreeing probability measure. As we discuss later, Savage’s P3 in conjunction with P1, P5, and the Event Archimedean Property is sufficient to ensure the existence of a unique agreeing probability measure that coincides with \succeq^C when it is complete over Σ .⁹ Grant (1995), however, convincingly argues that an insistence on Savage’s P3 may descriptively exclude some decision makers that rely on subjective probabilities for decision making. In particular, he cites the example of a mother that strictly prefers tossing a coin to determine how an indivisible good is to be distributed among her two children. Another example noted by Grant (1995) is that of induced preferences, which are quasi-convex.

In addition, mean-variance preferences commonly used in financial economics and management science, are probabilistically sophisticated by definition, yet violate both P3 and P4 (as well as both of Grant’s (1995) weakening of P3 and Machina and Schmeidler’s (1992) P4*).¹⁰ In order that \approx can be interpreted as an ‘equal likelihood’ relation between events in a homogeneous collection, we need to impose additional structure such that \approx is transitive, at least when restricted to the collection. It is desirable that any such structure be as parsimonious as possible if one is to develop a better understanding of deviations from global probabilistic sophistication. The following example provides some insight into both the limitations of an

⁹ Savage’s P3 states that for any non-null event, $E \subseteq \Omega$, act $f \in \mathcal{F}$ and any $x, y \in X$, $x \succeq y \Leftrightarrow xEf \succeq yEf$.

¹⁰ Savage’s P4 states that for any events $E, E' \in \Sigma$ and $x^*, x_*, y^*, y_* \in X$ with $x^* \succ x_*$, $y^* \succ y_*$, $x^*Ex_* \succeq x'E'x_*$ implies $y^*Ey_* \succeq y'E'y_*$. Machina and Schmeidler’s (1992) more restrictive P4* requires that for any $f, g \in \mathcal{F}$ and whenever $E \cap E' = \emptyset$, $x^*Ex_*E'f \succeq x'E'x_*Ef$ implies $y^*Ey_*E'g \succeq y'E'y_*Eg$.

exchangeability based approach to probabilistic sophistication and the nature of the additional assumptions that are called for to ensure probabilistic sophistication.

Example 3. Consider the ‘mother’ example supplied by Grant (1995) and mentioned above. If there are only two outcomes in the world of the decision maker - namely, receipt of the indivisible good by Child 1 or by Child 2 - then a plausible representation for the mother’s preferences is the utility function $U(p) = p(1 - p)$, where p is the probability that Child 1 receives the indivisible good and is subjectively generated by some device deemed by the mother to be uniform. According to the definition of exchangeable events, any event with probability $p \in [0, 0.5]$ is exchangeable with its complement. ■

In the example, \approx fails to deliver a notion of likelihood because given three disjoint events, E, E' and A such that $\mu(E) = \mu(E') = 0.4$ and $\mu(A) = 0.2$, the mother’s preference behavior leads to the conclusion that $E \approx E'$ while $E \cup A \approx E'$. To ensure an ‘equal likelihood’ interpretation for \approx , a natural condition is to assume that whenever two events are exchangeable, adding a disjoint non-null event to one of them makes the combined event more likely:

Axiom 4 (Weak Event Non-satiation). *For any disjoint $E, A, E' \in \Sigma$, if $E \approx E'$ and A is non-null, then $E \cup A \succ^C E'$.*

Notice that Axiom 4 rules out the behavior in Example 3. To further motivate and understand Axiom 4 consider three equivalent forms:

Proposition 1. *Given P1, the following are equivalent:*

- i) *Axiom 4*
- ii) *Let $E, A, E' \in \Sigma$ with A both non-null and disjoint from E . Then $E \succeq^C E'$ implies $E \cup A \succ^C E'$.*
- iii) *For any $E, E' \in \Sigma$, $E \succ^C E'$ whenever $E \succeq^C E'$ and there is some comparison event, $e \subseteq E \setminus E'$, such that $E \setminus (e \cup E')$ is not null.*
- iv) *Let $E, A, E' \in \Sigma$ be disjoint, E and E' be exchangeable, and A be non-null. Then for every $e' \subseteq E'$ there are outcomes, $x, x' \in X$ and act $f \in \mathcal{F}$ such that $x(A \cup E)x'e'f \not\prec x'(A \cup E)x'e'f$.*

According to Part (ii) of the proposition, Axiom 4 can be equivalently stated using a \succeq^C relation between E and E' . Part (iii) of the proposition can be trivially asserted for disjoint events as $E \succ^C E'$ if and only if whenever $e \subseteq E$ is a comparison event, then $E \setminus e$ is not null – an

intuitive property of any likelihood relation that can be represented by an agreeing probability measure. The fourth part of Proposition 1 is an equivalent way of asserting Axiom 4 only in terms of the preferences of the decision maker. It can be understood as a requirement of richness on the outcome space: X must contain payoffs that can ‘discriminate’ between events that are strictly ordered in terms of likelihood, so that such events are not deemed exchangeable. Note that this is not the case in the coin tossing mother example. On the other hand, if the mother can conceive of access to an additional *divisible* good, say chocolate that she can distribute among her two children, it will likely no longer be the case that any event is exchangeable with its complement; for instance, if E is a probability 0.6 event, then it is reasonable to suppose that the mother is not indifferent between giving *each* child a piece of chocolate if E is realized and nothing otherwise, versus giving each child a piece of chocolate if the complement of E is realized and nothing otherwise.

The Axiom is also sufficient (though not necessary) for another intuitively sensible implication:

$$E' \sim^C E \Leftrightarrow E \approx E'.$$

The next result establishes that Axiom 4 is necessary for any exchangeability based likelihood relation in which non-null sets are strictly more likely than the empty set. In particular, Axiom 4 is a minimal requirement for any sensible theory of probabilistic sophistication in which exchangeable events are equally likely.

Lemma 2. *Assume P1 and that \succeq° is a likelihood relation over Σ with (i) a symmetric part that agrees with \approx on disjoint sets, and (ii) $A \succ^\circ \emptyset$ for all non-null $A \in \Sigma$. Then for any disjoint $E, E', A \in \Sigma$ such that A is not null, $E \approx E'$ implies that $E \cup A \succ^\circ E'$.*

Proof: Assume that $E, E', A \in \Sigma$ are disjoint, A is not null, and $E \approx E'$ (meaning that $E \sim^\circ E'$). Note that $A \succeq^\circ \emptyset \Leftrightarrow E \cup A \succeq^\circ E$. Transitivity of \succeq° implies that $E \cup A \succeq^\circ E'$. If $E \cup A \sim^\circ E'$ then $E \cup A \sim^\circ E$. In particular, the cancellation property (iii) of a likelihood relation means that $A \sim^\circ \emptyset$ – a contradiction. Thus $E \cup A \succ^\circ E'$. ■

Our first major result delivers global probabilistic sophistication as a result of P1, P5, Axiom 3, Axiom 4 and the assumption that every event is comparable to every other event (i.e., \succeq^C is complete).

Theorem 1. *Assume P1. Then \succeq^C is complete and Axioms P5, 3 & 4 are satisfied if and only if there exists a unique probability measure, μ , representing \succeq^C such that any two events with*

the same measure are exchangeable, and the decision maker is indifferent between any two acts that induce the same lottery with respect to μ . Moreover, if Σ is atomless then the measure is convex-valued, otherwise Σ is atomic with each atom having the same mass.

Theorem 1 requires very weak conditions on preference. In particular, it does not require continuity of preferences. As an example, consider $\Omega = [0, 1]$ with Σ the usual Borel σ -algebra. Let $X = [0, 1]$ and assume that the decision maker ranks any simple act according to the expected value of the lottery it induces (with respect to the uniform measure on $[0, 1]$), and if two lotteries have the same mean the decision maker prefers the one with smaller variance. It should be clear that the decision maker's preferences are lexicographic and therefore not continuous. Since acts are evaluated based on the probability measure they induce, the decision maker is probabilistically sophisticated and her preferences satisfy the hypothesis of Theorem 1.

Theorem 1 also requires little if any monotonicity of preferences: assuming a sufficiently rich outcome space, both the coin-tossing mother example and induced preferences can be accommodated simultaneously, as can mean-variance, trimmed mean, Winsorized mean, and certain lexicographic preferences.¹¹ We will shortly return to discussing this result in the context of existing literature.

It is useful to investigate conditions that ensure the existence of a continuous utility representation over the act-induced lottery space. Other approaches (e.g., Savage, 1954; Machina and Schmeidler, 1992; Grant, 1995; Epstein and Zhang, 2001) impose variants of Savage's P6,¹² and rely on some form of monotonicity to establish the continuity properties of a utility representation. The relative absence of any notion of monotonicity in our approach prevents us from being able to follow this path; P6 is not enough to establish order denseness – $f \succ g$ implies the existence of $h \in \mathcal{F}$ such that $f \succ h \succ g$ – necessary for the existence of a utility representation.¹³ Instead, we adopt a notion of continuity that is more appropriate for our setting. Beforehand, define $\{a_i\}_{i=1}^m$ to be a *uniform m -partition of $E \in \Sigma$* whenever the a_i 's are disjoint, $\bigcup_{i=1}^m a_i = E$, and $a_i \approx a_j$ for any $i, j = 1, \dots, m$. Furthermore, for any partition of Ω , $\{e_i\}_{i=1}^k \subset \Sigma$ and $f_1, \dots, f_k \in \mathcal{F}$, let $(f_i, e_i)_k$ denote the act that pays f_i in event e_i (for $i = 1, \dots, k$). Note that $g E (f_i, e_i)_k$ is therefore the act that pays $g(\omega)$ if $\omega \in E$ and $f_i(\omega)$ if $\omega \in e_i \setminus E$ (for $i = 1, \dots, k$).

Assuming X is a compact metric space, we write $\{f_n\} \rightarrow f$, for $f = (x_i, e_i)_k$, whenever $f_n = \hat{f}_n E_n (f_{i,n}, e_i)_k$ for each n , with $\max_{x \in f_{i,n}(e_i)} \|x - x_i\| \rightarrow 0$ for every $i = 1, \dots, k$, and for

¹¹For the α -Winsorized mean, instead of trimming, outliers are concentrated at the α and $1 - \alpha$ percentiles.

¹²Savage's P6 requires that whenever $f \succ g$, then for any $x \in X$ there is a sufficiently fine finite partition of Ω , say $\{E_i\}_{i=1}^n \subset \Sigma$, such that $x E_i f \succ g$ and $f \succ x E_i g$ for every $i = 1 \dots n$.

¹³This condition, also known as 'dense ordering', guarantees a functional representation. The latter need not be continuous. See Kreps (1988) for details.

arbitrary integer $m > 0$, there exists a uniform m -partition of some non-null E , say $\{a_i\}_{i=1}^m$, and $N > 0$ such that for every $n > N$ $E_n \subseteq a_1$. The continuity axiom can now be stated in terms of ‘convergence in acts’:

Axiom 5 (Continuity). *X is a compact metric space, and whenever $\{f_n\} \rightarrow f$, $\{g_n\} \rightarrow g$, and $f_n \succeq g_n$ for each n , it must be that $f \succeq g$.*

As might be expected, adding this axiom leads to a continuous utility representation over induced lotteries.

Theorem 2. *Axioms P1, P5, 3-5 are satisfied and \succeq^C is complete if and only if for every $f, g \in \mathcal{F}$*

$$f \succeq g \Leftrightarrow U(F_f) \geq U(F_g)$$

where for any $h \in \mathcal{F}$, F_h is the lottery induced by h from μ , characterized in Theorem 1, and $U : \{F_h | h \in \mathcal{F}\} \mapsto \mathbb{R}$ is non-constant and continuous in the topology of weak convergence.

An analogous theorem can also be derived when X is only a separable metric space, but the definition of ‘act convergence’ needs to be appropriately modified to reflect the possibility of outcomes becoming unbounded as their supports approach a null event.

Homogeneous Collections

The examples discussed in the Introduction provide vivid instances of deviation from global probabilistic sophistication. Given our much weaker setup, departures from probabilistic thinking can be traced to violations of P1, P5, Axiom 3, Axiom 4, or the assumption that \succeq^C is complete (i.e., any two events are comparable). Despite the example of the coin-tossing mother (in a highly restricted outcome space), it seems most fruitful to explore deviations from the last assumption. As motivated in the Introduction, there are decision making situations where completeness of \succeq^C over Σ is not a compelling assumption. In such cases \succeq^C is not guaranteed to be transitive even if Axioms 3 and 4 hold (see Example 1). One may hope, however, that when restricted to a homogeneous collection \succeq^C can be shown to be a likelihood relation if \succ satisfies P1, P5 and Axioms 3 and 4. Examination of the proof of Theorem 1 confirms that this is generally true whenever the homogeneous collection is a σ -algebra. However, without imposing an algebraic structure on a homogeneous collection we are unable to show that Axioms 3 and 4 are sufficient for transitivity of \succeq^C . Consider, instead, the following strengthenings of these conditions:

Axiom 3' (Event Archimedean Property). Let $\{e_n, d_n\}_1^\infty$ be a collection of disjoint events in Σ and set $d_0 = \emptyset$. If $d_{n-1} \cup e_n \approx e_{n+1} \cup d_{n+1}$ for every $n \geq 1$, then $e_1 \cup d_1$ is null.

To motivate Axiom 3', consider that if $d_{n-1} \cup e_n \approx e_{n+1} \cup d_{n+1}$ for every $n \geq 1$, then by adding d_n to each side of the relation one intuitively obtains that $e_1 \cup d_1$ is 'as likely as' $d_1 \cup e_2 \cup d_2$, which is in turn 'as likely as' $d_2 \cup e_3 \cup d_3$, which is in turn as likely as $d_3 \cup e_4 \cup d_4$, and so on. If the relation 'as likely as' is transitive, the construction leads to an infinite number of disjoint and equally likely events: $e_1 \cup d_1$, $d_2 \cup e_3 \cup d_3$, $d_4 \cup e_5 \cup d_5$, etc. Thus to prevent pathological situations in which Σ contains an infinite number of disjoint and 'equally likely' non-null events, we require that such a construction is possible only if $e_1 \cup d_1$ is null. Note that the axiom asserts this property even if the relation 'as likely as' is not transitive; moreover, setting $d_n = \emptyset$ for every $n > 0$ implies Axiom 3.

Consider also the following strengthening of Axiom 4,

Axiom 4' (Event Non-Satiation). For any disjoint $E, E', A \in \Sigma$, if $x(E \cup A)x'E'f \sim xEx'(E' \cup A)f$ for every $x, x' \in X$ and $f \in \mathcal{F}$ then A is null.

In particular:

Lemma 3. Given P1, Axiom 4' implies Axiom 4.

Notice that while the converse of Lemma 3 is not true, there is some similarity between part (iv) of Proposition 1 and event non-satiation.

In this slightly richer setting, we can extend our results to homogeneous collections (λ -systems) that are closed under countable unions.

Theorem 3. Assume P1, Axiom 3', Axiom 4', and that $\mathcal{A} \subseteq \Sigma$ is closed under countable disjoint unions and contains at least one non-null event. Then \mathcal{A} is homogeneous if and only if \succeq^C is a likelihood relation over \mathcal{A} with a unique agreeing probability measure such that any two events with the same measure are exchangeable. If \mathcal{A} is atomless then the measure is convex-valued, otherwise \mathcal{A} is atomic with each atom having the same mass.

Note that the representation of \succeq^C by a unique probability measure applies to any homogeneous collection, including homogeneous 'sub-collections' of homogeneous collections. Clearly, investigating the latter is redundant. Instead, we are led to the following:

Definition 7 (Domains). $\mathcal{D} \subseteq \Sigma$ is a *small world event domain* if it is a homogeneous subset of Σ with the following properties:

- i) \mathcal{D} contains at least one non-null event.
- ii) If $\mathcal{A} \subseteq \Sigma$ is a non-empty collection of events and $\mathcal{A} \cup \mathcal{D}$ is homogeneous, then $\mathcal{A} \subseteq \mathcal{D}$.
- iii) \mathcal{D} is closed under countable disjoint unions.

For brevity, we will henceforth refer to any small world event domain simply as a ‘domain’. The first condition is akin to ‘non-triviality’. The second property, a maximality condition, guarantees that the domain is a ‘maximal’ homogeneous structure and justifies the association of \mathcal{D} with the notion of a Savagian ‘small world’. The third condition is technical and ensures that Theorem 3 delivers probabilistic sophistication on domains. Note the following properties: (i) If Σ is a domain then it is the only domain and under the hypothesis of Theorem 1 the decision maker is probabilistically sophisticated, (ii) if Σ does not contain domains, then no two non-null events are exchangeable.

Denote by $\mu_{\mathcal{D}}$ the unique measure representing \succeq^C on a domain, \mathcal{D} , satisfying the hypothesis of Theorem 3. One would like to conclude from Theorem 3 that the decision maker is indifferent between any two acts, say f and f' , that are identical outside a domain, while inducing the same lottery on the domain. Unfortunately, barring other conditions, this is not generally true unless \mathcal{D} is a σ -algebra, or the identical lotteries in question are binary. This situation is not unique to our approach; in both Epstein and Zhang (2001) and Kopylov (2002) monotonicity is crucial to demonstrate indifference between such a pair, f and f' . In place of a monotonicity condition, consider the following criterion:

Axiom 6. Let $\{e_i\}_{i=1}^m, \{e'_i\}_{i=1}^m \subset \Sigma$ be uniform m -partitions of $E \in \Sigma$ such that $e_i \sim^C e'_j$ for any $i, j \in \{1, \dots, m\}$. Then for any $f \in \mathcal{F}$, $\{x_i\}_{i=1}^m$ and permutation $\sigma : \{1, \dots, m\} \mapsto \{1, \dots, m\}$, $x_1 e_1 \dots x_m e_m f \sim x_{\sigma(1)} e'_1 \dots x_{\sigma(m)} e'_m f$.

Consider that exchangeability of two events amounts to indifference between permutations of payoffs among the events. Axiom 6 can be interpreted to say that if two different partitions of the same event are ‘exchangeable’ (in the sense of \sim^C) event by event, then the two partitions are ‘exchangeable’.

Let \mathcal{D} be a domain and denote by $\mathcal{F}_{\mathcal{D}}$ the set of acts adapted to \mathcal{D} . Given the unique measure, $\mu_{\mathcal{D}}$ over \mathcal{D} from Theorem 3, the portion of each act in $\mathcal{F}_{\mathcal{D}}$ that is adapted to \mathcal{D} induces a distribution. Let $\Delta_{\mu}(\mathcal{F}_{\mathcal{D}})$ be the set of all such distributions induced by $\mu_{\mathcal{D}}$ and $\mathcal{F}_{\mathcal{D}}$. If $f \in \mathcal{F}_{\mathcal{D}}$ then the *conditional probability distribution* associated with it through μ is denoted as $F_{f|\mathcal{D}}$. Finally, we identify the set of deterministic lotteries, $\{\delta_x \in \Delta_{\mu}(\mathcal{F}_{\mathcal{D}}) \mid x \in X\}$, with X .

Theorem 4. *Suppose P1 and Axioms 3', 4', 5 and 6 hold, and $\mathcal{D} \subseteq \Sigma$ is a domain. Then there exists $U_{\mathcal{D}} : \Delta_{\mu}(\mathcal{F}_{\mathcal{D}}) \times \mathcal{F} \mapsto \mathbb{R}$, continuous in its first argument (in the topology of weak convergence), such that $\mu_{\mathcal{D}}$ is characterized in Theorem 3, and for every $f, g \in \mathcal{F}$ adapted to \mathcal{D} and $h \in \mathcal{F}$:*

$$f \widehat{\mathcal{D}} h \succeq g \widehat{\mathcal{D}} h \Leftrightarrow U_{\mathcal{D}}(F_{f|\mathcal{D}}, h) > U_{\mathcal{D}}(F_{g|\mathcal{D}}, h).$$

The ‘representation’ in Theorem 4 is not fully satisfactory since it does not generally allow one to pin down risk preferences over a domain unless the domain envelope is the entire state space. Section 4 addresses this by specializing to the case where $U_{\mathcal{D}}(\cdot, h)$ is independent of h for any \mathcal{D} . In the remainder of this section we continue to relate our approach to the literature on probabilistic sophistication. In one subsection we compare our characterization of probabilistic sophistication (via Theorem 1) to that of Machina and Schmeidler (1992), and Grant (1995). In the next, we discuss how exchangeability based probabilistic sophistication is related to the literature on state dependent preferences. The last subsection discusses Theorem 3 in the context of the literature on non-global probabilistic sophistication.

3.1. Homogeneity, Monotonicity, and Probabilistic Sophistication

Machina and Schmeidler (1992) show that P1, P3, P4*, P5 and P6 are necessary and sufficient for the existence of a continuous probabilistically sophisticated utility representation of \succeq that agrees with first degree stochastic dominance (see Footnotes 9, 10 and 12 for definitions). Grant (1995) weakens P3 to either one of two variants: conditional upper (or lower) eventwise monotonicity (P3^{CU} or P3^{CL}). These state that for any $x, y \in X$, $h \in \mathcal{F}$ and disjoint non-null $E, E' \in \Sigma$,

$$\text{P3}^{CU}: \quad x(E \cup E')f \succ y(E \cup E')f \Rightarrow xEyE'f \succ y(E \cup E')f$$

$$\text{P3}^{CL}: \quad x(E \cup E')f \succ y(E \cup E')f \Rightarrow x(E \cup E')f \succ xEyE'f$$

Grant also uses a modified form of P4 stating that for all $w, x, y, z \in X$, $f, g \in \mathcal{F}$, and disjoint $E, E' \in \Sigma$:

$$\text{P4}^{CE}: \quad x(E \cup E')f \succ xEyE'f \sim yExE'f \succ y(E \cup E')f \Rightarrow wEzE'g \sim zEwE'g$$

In the language of this paper, Grant’s P4^{CE} supplies a sufficient condition for E and E' to be exchangeable. Assuming P1, P4^{CE}, P5, a slightly strengthened form of P6, and one of P3^{CU} or P3^{CL}, Grant (1995) shows the existence of a probabilistically sophisticated representation with

either quasi-convex or quasi-concave upper contour sets in the space of probability distributions.

Our purpose in this section is to compare the existing approaches to probabilistic sophistication with ours. We begin with the Machina and Schmeidler (1992) axiomatization. Note first that P4* ensures that their axioms lead to a unique convex valued probability measure where the measures of two events coincide if and only if the events are exchangeable. Thus their axioms imply that Σ is homogeneous (i.e., all events are mutually comparable). It is also easy to show that monotonicity (i.e., P3) implies weak event non-satiation. One can therefore interpret that, in establishing global probabilistic sophistication in Theorem 1, we weaken the Machina and Schmeidler axioms as follows: P3 \rightarrow Axiom 4, P4 \rightarrow homogeneity of Σ , and P6 \rightarrow Axiom 3. Thus aside from monotonicity considerations, our assumptions substantively weaken those of Savage (1954) or Machina and Schmeidler (1992). Moreover, consider the following result:

Proposition 2. *Assume P1, Savage’s P3, and that Σ is homogeneous. Then for any $x^*, x_* \in X$ with $x^* \succ x_*$, disjoint $E, E' \in \Sigma$, and $f \in \mathcal{F}$, $x^*Ex_*E'f \succeq x^*E'x_*Ef \Leftrightarrow E \succeq^C E'$, and the latter relation is strict if and only if $E \succ^C E'$.*

The proposition establishes two things: given a weak ordering satisfying P3, Machina and Schmeidler’s P4* is an implication of the homogeneity of Σ ; moreover, \succeq^C is, in this case, the comparative likelihood relation represented in their probabilistically sophisticated setting. In other words, to arrive at their representation theorem one need only add P3 to the list of conditions in our Theorem 2.

As stated, our axioms do not encompass those of Grant (1995) whose approach, in particular, can accommodate Example 3. On the other hand, Grant’s axioms are more demanding in the sense that taken as a whole they imply homogeneity of Σ , require a form of continuity that is not needed in Theorem 1, and rule out many probabilistically sophisticated functional forms that are behaviorally reasonable and admissible under our axioms. Consider the following result:

Proposition 3. *Assume P1 and that for every non-null $A \in \Sigma$, $f \in \mathcal{F}$, there exist $x, x' \in X$ such that $xAf \succ f \succ x'Af$. Then either one of $P3^{CU}$ or $P3^{CL}$ implies Axiom 4.*

The condition “for every non-null $A \in \Sigma$, $f \in \mathcal{F}$ there exist $x, x' \in X$ such that $xAf \succ f \succ x'Af$ ” is a form of non-satiation in outcomes: there is always something sufficiently good (resp. bad) that the decision maker is happy (resp. reluctant) to substitute for the payoff scheme determined by f on A . It can therefore be viewed as a ‘richness’ assumption on both \succeq and the outcome set, X . Indeed, it is a challenge to find an intuitively behavioral example in a state independent setting where the state space cannot be so ‘enriched’. To further emphasize

the importance of ‘enriching’ the outcome space, we note that whenever X contains only two outcomes, Σ is non-atomic, and \succeq can be represented via a continuous and probabilistically sophisticated utility function, Axiom 4 is satisfied if and only if the representation is monotonic in the sense of P3.¹⁴

Proposition 3 establishes that in the presence of a ‘rich’ outcome space, either one of Grant’s P3^{CU} and P3^{CL} implies Axiom 4. Moreover, since in this case Grant’s unique measure representing probabilistic sophistication agrees with \succeq^C , his axioms (taken together) imply both homogeneity of Σ and Axiom 3. In other words, probabilistically sophisticated preferences that satisfy Grant’s axioms also satisfy ours *provided that the outcome space is sufficiently rich to ensure that Axiom 4 is also satisfied*. We note once more that the mother example in which there are only two equally ranked outcomes (i.e., Example 3) does not satisfy Axiom 4 but does satisfy Grant’s conditions. However, if there are at least two outcomes such that the mother strictly prefers placing more probability on one versus the other in almost every binary lottery, then Axiom 4’, and thus Axiom 4, is satisfied.

Theorem 1 applies to many instances in which P3^{CU} and P3^{CL} are violated, including but not limited to cases where upper contour sets of preferences over distributions are not always quasi-concave or quasi-convex. Moreover, in addition to violating P3 and its variants, some common choice models, such as mean-variance preferences, also violate P4 and its variants while leaving the homogeneity of Σ intact. However, there is one other advantage to an exchangeability based approach to probabilistic sophistication beyond the fact that it encompasses other known models (given a ‘richness’ assumption). As demonstrated with Theorem 4, our exchangeability based approach lends itself particularly well to studying deviations from global probabilistic sophistication, and does not confound such deviations with other behavioral considerations such as monotonicity or continuity.

In summary, there is an analogy between the standard axioms required for probabilistic sophistication and ours. All approaches share P1 and some form of P5. We weaken the standard continuity axiom (e.g., P6) to the event Archimedean property, and weaken the P4 variants to the assumption that all events are comparable. Monotonicity is replaced by Axiom 4. Grant’s P3^{CU} and P3^{CL}, however, are not strictly contained within our set of axioms, unless the outcome space is enriched so as to satisfy the hypothesis of Proposition 3.

¹⁴We thank I. Gilboa for pointing this out.

3.2. State Dependence

One often interprets Savage's P3 and P4 to jointly assert a separation between tastes and beliefs, or 'state independence' (see, for example, Sarin and Wakker, 2000). Theorem 1 and Theorem 2 are state independent in the sense that the decision maker to which these theorems apply only cares about the likelihood of an event and not its identity. The axioms leading to these two results are weaker than P3 and P4, thus prompting the obvious question: which of our assumptions embodies the sense of 'state independence'? A possible answer is the same one we provided for ruling out Ellsbergian or source dependent behavior; i.e., \succeq^C is complete, and thus all events are comparable. For instance, in the Aumann-Savage correspondence documented in Dreze (1987), two equally likely events – 'rain in Chicago' and 'no-rain in Chicago' – are not exchangeable with respect to the outcomes 'trip to Chicago with umbrella' and 'trip to Chicago without an umbrella'. While they may well be exchangeable with respect to other outcomes (e.g., trip to San Francisco with, versus without, an umbrella), the two events are not exchangeable with respect to the 'Chicago plus/minus umbrella' payoffs even though they are associated with unique probabilities. In particular, the two events are not comparable via \succeq^C .

Informally, one is able to distinguish between incompleteness of \succeq^C due to Ellsbergian or similar behavior, versus the type of state dependence outlined above. In this subsection we explore the extent to which one can formally distinguish the two types of incompleteness. In so doing we sketch a basis for understanding state dependence and 'small world' attitudes vis-a-vis exchangeability. Rather than provide a comprehensive treatment of the subject, our goal is to outline how an exchangeability based approach may contribute to the literature on state dependence whose primary focus concerns global probabilistic sophistication based on the expected utility specification (see Karni, Schmeidler and Vind, 1983; Dreze, 1987; Karni, 1993; and Karni, 2003).

In the Aumann example, one can conceivably discern the probabilistic sophistication of the decision maker by restricting attention to acts whose payoffs are not related to the identity of events. For example, in assessing probabilistic sophistication, one could focus on the behavior of the decision maker with respect to pure monetary outcomes. Note that in order to accommodate the coin-tossing mother from Example 3 we require the outcome space to be enriched beyond two outcomes, while to accommodate probabilistic sophistication in the presence of state dependence we require precisely the opposite. To proceed formally, let \succeq_Y correspond to the preference relation obtained by restricting the set of outcomes to $Y \subseteq X$. Define \approx_Y and \succeq_Y^C similarly, then

Definition 8 (State Independent Preferences). \succeq is state independent when $\succeq_Y^C = \succeq_{Y'}$, for every pair of outcome sets, $Y, Y' \subseteq X$, such that \succeq_Y and $\succeq_{Y'}$ satisfy P5 and Axiom 4'.

A preference relation is state independent when essentially every outcome set can be used to elicit subjective probabilities within a domain. We require P5 and Axiom 4' so as to exclude scenarios such as Example 3 which seems intuitively state independent. We caution the reader that our definition does not fully capture the intuition behind ‘state independence’. For instance, in the Aumann-Savage example one might consider a very simple outcome space, spanning ‘trip to Chicago with x ’, where $x \in X = \{\text{expensive umbrella, average umbrella, cheap umbrella}\}$, in which intuitively there appears to be state dependence. However, since the ‘connection’ between the outcomes and the states is uniform across outcomes, \succeq is ‘state independent’ according to our definition. Notice that this mis-labeling is shared by other definitions of state dependence, such as the requirement that \succeq satisfies P3 and P4. In applications, however, our notion of state independence seems to accord with intuition. Moreover, we take as uncontentious the claim that if \succsim is *not* ‘state independent’ according to our definition, then it is *state dependent*. In any case, if \succeq is state independent, then any structure characterizing \succeq^C – in particular, deviations from global probabilistic sophistication – can only be due to ‘small worlds’ behavior. Thus, at least in this case, we can identify the source of incompleteness of \succeq^C .

If \succeq is state dependent, the natural question is how, if possible, can one deduce a state independent structure for \succeq^C and the consequent ‘small worlds’ beliefs (i.e., Theorem 3)? To formally answer this, let \mathcal{R} denote the set of all restricted outcome relations such that $\succeq_Y \in \mathcal{R}$ implies \succeq_Y satisfies P5 and Axiom 4', and is itself a state independent preference relation in the sense of Definition 8. For instance, in the Aumann example, while \succsim may not be state independent, its restriction to monetary outcomes might be.

Definition 9 (State Independent ‘Beliefs’). $\succeq_{\mathcal{R}}^C$ is termed a **state independent event comparability relation** on Σ if \mathcal{R} is not empty and for any $\succeq_Y \in \mathcal{R}$, $\succeq_Y^C = \succeq_{\mathcal{R}}^C$.

Thus as long as every state independent restriction of \succeq induces the same comparability relation, this relation, denoted as $\succeq_{\mathcal{R}}^C$, can be said to be itself state independent. Note, first of all, that for any $E, E' \in \Sigma$ and $Y \subseteq X$, $E \succeq^C E' \Rightarrow E \succeq_Y^C E'$. Thus, since \succeq^C is generally a partial order, $\succeq_{\mathcal{R}}^C$, if it exists, can be seen as a ‘more complete’ extension (or superset) of \succeq^C . Thus existence of $\succeq_{\mathcal{R}}^C$ allows one to identify the component of incompleteness of \succeq^C due to state dependence as the set difference between $\succeq_{\mathcal{R}}^C$ and \succeq^C . Moreover, if $\succeq_{\mathcal{R}}^C$ exists and is complete then Theorem 3 implies global probabilistic sophistication *in the presence of state dependence*.

In contrast with existing literature on state dependent preferences, this characterization does not hinge on an expected utility framework.

Some forms of state dependence, for instance situations involving moral hazard (see Karni, 2003), may not allow for the existence of the relation, $\succeq_{\mathcal{R}}^C$. Thus, while it complements the literature on probabilistic sophistication with state dependence, the approach sketched above is not comprehensive. We leave a more thorough investigation to future research.

3.3. Relation to Subjectively Unambiguous Events

One focus of the current literature on decision making under uncertainty is to distinguish events on which a decision maker is globally probabilistically sophisticated, from those on which the decision maker's likelihood relation does not admit a probabilistic representation. The former are often called 'unambiguous' while the latter are termed 'ambiguous' events. In this subsection we relate this literature to our approach based on domains. It is worth emphasizing that examples abound in which the noncomparability of events *does not* stem from the presence of ambiguity in any particular way. Thus *prima facie* a focus on domains differs in scope and purpose from the literature on ambiguity.

Epstein and Zhang (2001) define unambiguous events (see below) using Savage-like primitives; Ghirardato and Marinacci (2002) rely on degrees of what they term ambiguity aversion to identify unambiguous events; Nehring (2001) and Ghirardato, Marinacci and Maccheroni (2003) define an event to be unambiguous if the event's probability is the same under every measure in an endogenously determined set of measures representing an incomplete comparative likelihood relation;¹⁵ Klibanoff, Marinacci and Mukerji (2002) define ambiguous events by relying on benchmark lotteries with respect to an 'objective' randomization device (or other appropriate enrichment of the state space). These references do not generally agree on the ambiguity of specific events as inferred from the decision maker's choice behavior.

All the definitions discussed above have their compelling aspects. The definition given by Epstein and Zhang (2001), which applies to preferences satisfying Savage's P3, appears to be the least dependent on imposing additional structure on \succeq . The definitions offered by Nehring (2001) and Ghirardato, Marinacci and Maccheroni (2003) are also quite general but, besides a version of Savage's P4, require additional global structure over preferences to ensure the existence of a Bewley-like partial qualitative likelihood relation. Ghirardato and Marinacci's (2002) can

¹⁵Siniscalchi (2003) notes that multiple prior models can also be characterized by a unique set of measures, each of which provides a subjective expected utility representation when \succeq is restricted to a particular convex set of acts that contains all the constant acts. While one can view this as 'probabilistic sophistication' over domains of acts, this approach does not share our aim of characterizing probabilistic attitudes towards collections of events.

be interpreted as specializing to a particular class of preferences which reduce to expected utility in the absence of ambiguity. Klibanoff, Marinacci and Mukerji (2002) arguably make the most imposing requirements over preferences, but their definition is remarkably clean: by *a priori* enriching their state space with an ‘objective’ randomization device, they can compare risk preferences over objective lotteries and preferences over Savage acts. In such a comparison between a binary bet on an event versus an objective lottery, deviation from P4 implies that the event is ambiguous.

A natural question is what overlap, if any, exists between event classification through domains and classification into ‘ambiguous’ versus ‘unambiguous’ events. We focus on the definition supplied by Epstein and Zhang (2001). Examples to differentiate domains from the other theories are straightforward to construct.

Epstein and Zhang (2001) define subjectively unambiguous events as follows: $E \in \Sigma$ is **EZ-unambiguous** if the following two conditions are satisfied (i) for any $x, x', z, z' \in X$, $f \in \mathcal{F}$ and $A, B \in \Sigma$ such that $A, B \subseteq E^c$ and $A \cap B = \emptyset$,

$$zExAx'Bf \succeq zEx'AxBf \Rightarrow z'ExAx'Bf \succeq z'Ex'AxBf.$$

and, (ii) condition (i) holds if E is everywhere replaced by E^c . It is straight forward to check that in all the urn examples considered so far EZ-unambiguous events constitute domains or unions of domains. In particular, each unambiguous event is comparable to every other. An obvious question is whether ‘EZ-unambiguity’ implies comparability more generally. The answer turns out to be no, as the following example illustrates.

Example 4. Consider the four-color urn with the only information about the color combinations being that $g > r > w = b$. Assume a monotonic utility representation for \succeq given by

$$V(x_B, x_W, x_R, x_G) = x_B + x_W + x_R + x_G + (x_G + 1)(x_R + 2)$$

with $x_B, x_W, x_R, x_G \in \mathbb{R}_+$. Moreover, the event GB is unambiguous. To see this, consider that fixing the payoff on GB , the decision maker’s attitudes towards permuting payoffs on R and W are summarized by

$$V(x_{GB}, x_W, x_R, x_{GB}) \geq V(x_{GB}, x_R, x_W, x_{GB}) \Leftrightarrow (x_{GB} + 1)(x_R - x_W) \geq 0 \Leftrightarrow (x_R - x_W) \geq 0$$

and are therefore independent of the fixed payoff on GB . Thus the first requirement of ‘EZ-

unambiguity' is satisfied. It is easy to check that the complementary condition holds as well. Analogously, $\{\emptyset, GB, RW, GW, RB, GBRW\}$ is the λ -system of EZ-unambiguous events. However, not every EZ-unambiguous event is comparable to every other. In particular, GB is not comparable to RW (since G and R are not exchangeable). ■

In the above example, note that the decision maker's preferences satisfy Savage's P3 and even P4 restricted to EZ-unambiguous events.

Another natural question is whether the domain language can be used to characterize the distinction between 'ambiguous' and 'unambiguous' events. We observe instead that a preference-based definition of 'unambiguous' events may not always accord with intuition. For instance, consider the two-urn Ellsberg example discussed in the introduction and two decision makers. The first exhibits the conventional preference pattern of preferring to bet on the known urn while the other exhibits the opposite pattern of preferring to bet on the unknown urn. How is an observer who does not know which urn is 'ambiguous' or which decision maker is 'ambiguity averse' to tell both the difference between the urns and the difference between the two decision makers? From the perspective of the observer there is sufficient symmetry to prevent such inference – doing so requires information external to the primitives at hand. In particular, any definition that identifies a domain of events as ambiguous in the case of one decision maker will simultaneously identify it as unambiguous in the case of the other decision maker. This difficulty also seems inherent to the definitions in Epstein and Zhang (2001), Nehring (2001), Ghirardato and Marinacci (2002) and Ghirardato, Marinacci and Maccheroni (2003). In other words, the 'knowledge' that in one urn the distribution of balls is unambiguous while it is ambiguous in the other is *exogenous* and cannot be reliably inferred through observation of the decision maker's preferences alone. Klibanoff, Marinacci and Mukerji (2002) avoid this by explicitly incorporating the exogenous knowledge about the known urn into the decision maker's preferences (i.e., the known urn is a randomization device which, by definition, receives differential treatment in the decision maker's preferences).

While we refrain from proposing a definition of unambiguous events, we can say (relative to state independent outcomes) that 'ambiguity' corresponds to the presence of multiple domains, or more generally, having at least two non-null events that are not comparable. Lack of comparability suggests that there are different sources, interpretations, or 'issues', in the language of Ergin and Gul (2002), associated with the noncomparable events, and thus the presence of ambiguity. One may hope to say something more concrete than this, but such hope may be in vain. For instance, suppose that one were to expand the outcome space to lotteries, as in

the Anscomb-Aumann approach or the enriched state space approach advocated by Klibanoff, Marinacci and Mukerji (2002). One can then define a domain to be unambiguous if risk preferences within that domain are identical to risk preferences over objective lotteries associated with the same distribution functions. This intuition relies on the assumption that the lotteries in the expanded outcome space are somehow unambiguous *a priori*. In other words, one relies on the ‘neutral-ness’ of objective lotteries. However, as will be discussed in Subsection 4.1, examples of distinct domains over different lottery spaces are not uncommon. For example, consider the two-urn example where the first urn contains 50 red balls and 50 black balls, while the second urn has one red ball and one black ball. Although the proportion of balls is the same in each urn, a decision maker may prefer to draw from the urn containing more balls, reasoning that there will be less ‘regret’ if he is disappointed by the realized outcome (e.g., ‘Darn! I had the winning ball in my hand!’).

4. Domains, Ellsbergian Behavior and Representations

We begin the section by discussing examples of multiple domains risks. This motivates additional axioms relating to the structure of domains, which are necessary for characterizing domain recursive utility for mixed acts across distinct domains.

4.1. Examples of State Independent Multiple Domains Risks

In addition to the discussion in the Introduction on choice behavior in a multiple domain context, we illustrate here how multiple domains can arise in additional settings. We assume throughout this subsection that preferences are state independent.

Examples involving Urns

Consider an urn containing 50 red balls (R), plus a combination of green (G) and blue (B) balls that sum to 50. It seems reasonable that people would find B exchangeable with G and BG exchangeable with R . In other words, there are two domains: $\{\emptyset, G, B, BG\}$, and $\{\emptyset, R, BG, \Omega\}$.

Based on the possible homogeneous collections identified in Example 2, we can readily identify the following domains in the four-color urn: $\mathcal{D}_{BW} = \{\emptyset, B, W, BW\}$, $\mathcal{D}_{GR} = \{\emptyset, G, R, GR\}$, $\mathcal{D}_{BGRW} = \{\emptyset, BW, GR, \Omega\}$ and $\mathcal{D}_\lambda = \{\emptyset, GW, BR, BG, RW, \Omega\}$.

Returning to the two-urn experiment, let $\{R_k, B_k\}$ and $\{R_u, B_u\}$ denote partitions of Ω associated with drawing a single ball from each of the two Ellsberg urns: an urn containing 50 red (R_k) and 50 black (B_k) balls (the ‘ k ’ or *known* urn) and an urn containing 100 red

(R_u) and black (B_u) balls of unknown ratio (the ‘ u ’ or *unknown* urn). Commonly reported behavior suggests that there is exchangeability between R_k and G_k and between R_u and G_u but not between either R_k and G_k and either of R_u and G_u . In other words, $\{\emptyset, R_k, G_k, \Omega\}$ and $\{\emptyset, R_u, G_u, \Omega\}$ are potentially domains. By examining the decision maker’s preference for more general acts of the form, $x_1 R_k \wedge R_u$ $x_2 R_k \wedge B_u$ $x_3 B_k \wedge R_u$ x_4 , one may identify additional domains involving the *conjunction* of events in the two urns. We shall defer the discussion of this question until after we have developed our results for a domain recursive utility representation for mixed acts.

Additional Examples

The Introduction describes several multiple-domain variations of the single small world examples due to Savage – decimal expansion of different numbers, tossing different coins, and temperature/rainfall in different cities. Introspection suggests that people may prefer to bet on the decimal expansion of a nice number than a strange one, on one’s own coin than a coin from someone peculiar, and on uncertainty arising from a familiar source than a foreign one. Examples of multiple domains risks can be found in daily life. For instance, giving consumers the choice of picking their own numbers appears to be a demand-enhancing feature of state lottery design. The Denesraj-Epstein (1994) study suggests that even when two sets of events are clearly compatible with having objective probabilities, they may be differentiated by virtue of event noncomparability in relation to certain acts. At the same time, when bets are restricted to a single source of uncertainty the same individuals may exhibit preferences that are compatible with probabilistic sophistication.

Since attitudes towards uncertainty within a domain are completely described by attitudes towards risk, the presence of multiple domains enables the modeling of distinct risk preferences within distinct domains. For instance, a decision maker may be probabilistically sophisticated when betting only on temperature intervals in any given city. That decision maker, however, may have different certainty equivalents for the same ‘subjective lottery’ in different cities. If he prefers to bet on a subjective lottery in a more ‘familiar’ city, one may say that the decision maker exhibits ‘uncertainty aversion’ towards risks arising from the less familiar city-domain. This can be rationalized by requiring the decision maker to be more ‘risk averse’ when betting on the unfamiliar city-domain. In the insurance context, even when hazard rates are available, individuals appear to exercise ‘mental accounting’ when considering unusual and particularly disastrous risks (see, Eisner and Strotz, 1961; Kunreuther et al, 1978). Our perspective may also

shed light on other decision situations involving uncertainty from distinct sources, e.g., domestic (versus foreign) and local (versus national) biases of equity investors.¹⁶ In the next subsections, we will address the question of representing the decision maker's preference over acts that are not adapted within a single domain.

4.2. Domain Independence and Separation

Theorem 4 concerns the representation of the decision maker's preference among acts that are adapted within individual domains. To further pin down possible representations over mixed acts, a natural way to proceed is to assume, as Savage did, that decisions across small worlds are separable. The identification of domains with Savagian small worlds leads to the following hypothesis:

Axiom 7 (Domain Independence). *X is connected, and whenever $f \in \mathcal{F}$ is adapted to events in a domain \mathcal{D} , there is some $x \in X$ such that for all $g, h \in \mathcal{F}$, $f \widehat{\mathcal{D}} g \sim x \widehat{\mathcal{D}} g \Leftrightarrow f \widehat{\mathcal{D}} h \sim x \widehat{\mathcal{D}} h$.*

Domain independence is similar to Savage's sure thing principle in that for each act adapted to the domain, there exists a certainty equivalent over that domain that is *independent* of payoffs outside the domain. While the separation among small world 'beliefs' is accomplished endogenously via Theorem 3, Axiom 7 asserts an analogous separation in terms of 'valuation'. This assumption is not particularly normative and we do not have empirical evidence to defend it on descriptive ground. We view it as a falsifiable hypothesis to investigate empirically. The axiom is also instrumental in delivering a global representation of preferences exhibiting multiple domains. Domain independence, combined with previous assumptions, delivers a well defined (up to monotonic transformation) utility function representing the decision maker's preferences over each domain.

Proposition 4. *If \succeq satisfies P1, P5 and Axioms 3-7, then for every domain, \mathcal{D} , and acts $g, f \in \mathcal{F}$ adapted to \mathcal{D} , there exists a unique probability measure on, μ on \mathcal{D} , and a continuous function, $W_{\mathcal{D}} : \Delta_{\mu}(\mathcal{F}_{\mathcal{D}}) \mapsto \mathbb{R}$ such that $W_{\mathcal{D}}(\Delta_{\mu}(\mathcal{F}_{\mathcal{D}})) = W_{\mathcal{D}}(X)$, and for every $h \in \mathcal{F}$,*

$$f \widehat{\mathcal{D}} h \succeq g \widehat{\mathcal{D}} h \Leftrightarrow W_{\mathcal{D}}(F_{f|\mathcal{D}}) \geq W_{\mathcal{D}}(F_{g|\mathcal{D}}).$$

Proof: Suppose for some $x, y \in X$, $x \widehat{\mathcal{D}} h \succ y \widehat{\mathcal{D}} h$. Note that Axiom 7 rules out the possibility that $y \widehat{\mathcal{D}} h' \sim x \widehat{\mathcal{D}} h'$ for any $h' \in \mathcal{F}$. Assume, therefore, that $y \widehat{\mathcal{D}} h' \succ x \widehat{\mathcal{D}} h'$ for some $h' \in \mathcal{F}$.

¹⁶There is a literature (see, e.g., French and Poterba, 1991) on the 'home market bias' phenomenon that portfolios of equity investors are disproportionately weighted in favor of domestic stocks, relative to foreign stocks. There are further evidence (see Huberman, 2001; Coval and Moskowitz, 2001) that, within the same country, there may also be a similar bias in favor of shares in companies that are geographically proximate. Recent theoretical work that attempt to rationalize this behavior in terms of uncertainty aversion includes Uppal and Wang (2002).

Let $\{e_i\} \subset \Sigma$ be the coarsest partition of Ω to which both $x \widehat{\mathcal{D}} h$ and $x \widehat{\mathcal{D}} h'$ are adapted. Note that by Axiom 5 the restriction of \succeq to $\{(x_i, e_i) \mid x_i \in X, i = 1, \dots\}$ is continuous in the sense that upper and lower contour sets are closed. Since X is connected, one can find some $h'' \in \mathcal{F}$ such that $y \widehat{\mathcal{D}} h'' \sim x \widehat{\mathcal{D}} h''$ – a contradiction of Axiom 7. Thus it must be that $x \widehat{\mathcal{D}} h' \succ y \widehat{\mathcal{D}} h'$ for any $h' \in \mathcal{F}$, or alternatively, the ranking is independent of h' and $W_{\mathcal{D}}(\cdot, h)$ is monotonically equivalent to $W_{\mathcal{D}}(\cdot, h')$. ■

If, according to the decision maker's preferences, the state space is a sequence of nested domains - much like a filtration - then the decision maker's preference representation will emulate that structure and appear as a recursive nesting of the domain utility functions from Proposition 4. Before proving a general result along these lines, we illustrate nested domains and the representation implied by domain independence.

Examples

For the three-color urn containing 50 red balls plus a distribution of blue and green balls that sum to 50, we have defined two domains: $\mathcal{D}_0 = \{\emptyset, R, GB, \Omega\}$ and $\mathcal{D}_1 = \{\emptyset, G, B, GB\}$. Let W_0 and W_1 represent risk preferences in the two domains, respectively. Note that by Proposition 4, $W_1(\delta_{c^*}) > W_1(\delta_{c_*})$ if and only if $W_0(\frac{1}{2}\delta_{c^*} + \frac{1}{2}\delta_c) > W_0(\frac{1}{2}\delta_{c_*} + \frac{1}{2}\delta_c)$ for any $c \in X$. This 'monotonicity' implies that one can always find some continuous $\hat{v}_0 : X \times \mathbb{R} \mapsto \mathbb{R}$ that is strictly increasing in its second argument, such that $W_0(\frac{1}{2}\delta_{x_R} + \frac{1}{2}\delta_{c^*}) > W_0(\frac{1}{2}\delta_{x'_R} + \frac{1}{2}\delta_{c_*})$ if and only if $\hat{v}_0(x_R, W_1(\delta_{c^*})) > \hat{v}_0(x'_R, W_1(\delta_{c_*}))$. In turn, identifying x_R with δ_{x_R} , the act (x_R, x_G, x_B) has a recursive utility representation

$$V(x_R, x_G, x_B) = \hat{v}_0(\delta_{x_R}, W_1(\frac{1}{2}\delta_{x_G} + \frac{1}{2}\delta_{x_B})),$$

where $\hat{v}_0(\delta_{x_R}, W_1(\delta_{c_{GB}})) = W_0(\frac{1}{2}\delta_{x_R} + \frac{1}{2}\delta_{c_{GB}})$.

In the case of the four-color urn, one can represent a general act as $(x_B, x_W, x_R, x_G) \sim (c_{BW}, c_{BW}, c_{RG}, c_{RG})$, where the indifference relation results from the domain independence of the domains identified in Subsection 4.1. Let W_1 represent \succeq over the domain generated by $\{B, W\}$ (and, by symmetry, $\{R, G\}$) in accordance with Proposition 4. Likewise, let W_0 represent the risk preferences in the domain spanned by BW and RG . Then the following recursive utility function represents \succeq :

$$V(x_B, x_W, x_R, x_G) = \hat{v}_0(W_1(\frac{1}{2}\delta_{c_B} + \frac{1}{2}\delta_{c_W}), W_1(\frac{1}{2}\delta_{c_R} + \frac{1}{2}\delta_{c_G})),$$

where $\hat{v}_0(W_1(\delta_{c_{BW}}), W_1(\delta_{c_{RG}})) = W_0(\frac{1}{2}\delta_{c_{BW}} + \frac{1}{2}\delta_{c_{RG}})$ is continuous, symmetric, and strictly increasing in both its arguments. Note that $\{GW, BR, BG, RW, \}$ are mutually comparable, as are $\{BW, RG\}$. Moreover, events in the latter collection are not generally comparable with those in the former unless W_1 and \hat{v}_0 exhibit the same risk attitudes over their respective domains (in which case the representation is globally probabilistically sophisticated).

Domain Separation

An important question to address when considering how decisions are made across domains is how and whether domains ‘overlap’. All of the examples considered thus far have the following structure: domain envelopes are either disjoint or ‘nested’ within a larger domain. To what extent can this be justified more generally?

To attempt an answer to this, suppose \mathcal{D}_1 and \mathcal{D}_2 are domains, and consider any $E_1 \in \mathcal{D}_1$, such that $E_1 \not\sim^C \widehat{\mathcal{D}}_1$ and $E_1 \cap \widehat{\mathcal{D}}_2 \in \mathcal{D}_2$; then in all the urn examples, $\widehat{\mathcal{D}}_2 \subseteq \widehat{\mathcal{D}}_1$. For instance, in the four color urn example \mathcal{D}_λ contains an event, GW , that contains the event G from \mathcal{D}_{GR} – note that GR is not in \mathcal{D}_λ – and $\widehat{\mathcal{D}}_{GR} \subseteq \widehat{\mathcal{D}}_\lambda$. It is easy to verify that the same is true for all the other examples. To rationalize this structure more generally, set $E_1 \cap \widehat{\mathcal{D}}_2 = E_2 \in \mathcal{D}_2$ and let $E'_1 = (E_1 \setminus E_2) \cup E'_2$ where E'_2 is any element of \mathcal{D}_2 such that $E_2 \sim^C E'_2$. It should be clear that $E_1 = (E_1 \setminus E_2) \cup E_2 \sim^C E'_1$ for every E'_1 so constructed. Thus at the level of intuition, if one can assign a unique probability to E_1 then one can assign the same probability to every event in $\mathcal{A} \equiv \{E'_1 = (E_1 \setminus E_2) \cup E'_2 \mid E'_2 \sim^C E_2, E'_2 \in \mathcal{D}_2\}$. It therefore seems reasonable to require that $\mathcal{A} \subseteq \mathcal{D}_1$. Now, if \mathcal{D}_2 is non-atomic or generated from exchangeable atoms, then the union of all events in $\{E'_2 \mid E'_2 \sim^C E_2, E'_2 \in \mathcal{D}_2\}$ is $\widehat{\mathcal{D}}_2$. Thus the envelope of \mathcal{D}_1 should contain $\widehat{\mathcal{D}}_2$.

The preceding discussion makes a compelling case for the following assumption:

Axiom 8 (Domain Separation). *If \mathcal{D}_1 and \mathcal{D}_2 are domains, and for some $E_1 \in \mathcal{D}_1$, where $E_1 \not\sim^C \widehat{\mathcal{D}}_1$ and $E_1 \cap \widehat{\mathcal{D}}_2 \in \mathcal{D}_2$, then $\widehat{\mathcal{D}}_2 \subseteq \widehat{\mathcal{D}}_1$.*

4.3. Characterizing Domain Recursive Utility

The preceding subsection discusses examples of ‘nested domains’ with the feature that each act can be reduced to a certainty equivalent in multiple steps akin to backward induction. In each step an act is adapted to a set of domains whose envelopes are disjoint, so that one can use domain independence to simplify the act further. To generalize this idea we introduce the following:

Definition 10. \succeq is said to be **finite domain recursive** whenever it induces a sequence of domains $\{\mathcal{D}_{t,i}\}$, where for each $t = 0, \dots, T < \infty$, $i = 1, \dots, n_t$ with $n_0 = 1$, $n_t \leq n_{t+1} < \infty$, and such that the following must be satisfied:

1. $\widehat{\mathcal{D}}_{0,1} = \Omega$.
2. For each $0 \leq t < T - 1$, $1 \leq i \leq n_t$ and $1 \leq j \leq n_{t+1}$, either $\widehat{\mathcal{D}}_{t+1,j} \in \mathcal{D}_{t,i}$ where i is unique in $1, \dots, n_t$, or $\widehat{\mathcal{D}}_{t,i} \cap \widehat{\mathcal{D}}_{t+1,j} = \emptyset$.
3. Letting $E_{t,i} \equiv \widehat{\mathcal{D}}_{t,i} \setminus \bigcup_{j=1}^{n_{t+1}} \widehat{\mathcal{D}}_{t+1,j}$, if $e \subseteq E_{t,i}$ and $e \in \Sigma$, then $e \in \widehat{\mathcal{D}}_{t,i}$.

Diagrammatically, this structure is analogous to a $T + 1$ -stage tree with a single base node, and whose nodes at stage t correspond to distinct domains. Any two domains within the same stage have disjoint envelopes, and the envelope of a domain at stage t is an event in at most one domain at stage $t - 1$. The envelope of the base domain at stage 0 is the full state space. To gain a better understanding of the last part of the definition, consider first that the event $E_{T,i}$, for $i = 1, \dots, n_T$ is identically equal to $\widehat{\mathcal{D}}_{T,i}$, and thus *every* subevent of $\widehat{\mathcal{D}}_{T,i}$ is adapted to $\mathcal{D}_{T,i}$; in other words, every act is perforce adapted to each of the $\mathcal{D}_{T,i}$'s. Now, by part (ii) of the definition, a domain at stage $t < T$ may contain the disjoint envelopes of some domains from stage $t + 1$, but the union of the latter may not exhaust the envelope of the former – the residual part is the event $E_{t,i}$.

For the three color urn, with 50 red balls and 50 balls that can each be blue or green, there is only one domain in each stage. The stage 1 domain is $\{\emptyset, G, B, GB\}$, while the base domain is $\{\emptyset, R, GB, \Omega\}$. In this case, setting $E_{1,1} = \emptyset$ and $E_{0,1} = R$ satisfies Definition 10.

If \succeq is finite domain recursive, then the reduction of an act via domain independence is achieved by ‘backward induction’ through the tree of domains, resulting in the following *finite domain recursive representation*:

$$\forall f, g \in \mathcal{F} \quad f \succeq g \Leftrightarrow \hat{v}_{0,1}(f) \geq \hat{v}_{0,1}(g)$$

where $\hat{v}_{0,1}$ is continuous and defined recursively as follows: let $\alpha_1, \dots, \alpha_{n_{t,i}}$ index the $n_{t,i}$ domains whose envelopes are contained in $\mathcal{D}_{t,i}$, and let

$$\hat{v}_{t,i}(f) = v_{t,i} \left(\hat{F}_{f|E_{t,i}}, \hat{v}_{t+1,\alpha_1}(f), \dots, \hat{v}_{t+1,\alpha_{n_{t,i}}}(f) \right),$$

where $v_{t,i}$ is a real-valued function that is strictly increasing in all arguments from the second up to the last, $\hat{F}_{f|E_{t,i}}$ is the distribution induced by f via $\mu_{\mathcal{D}_{t,i}}$ conditional on the event $E_{t,i}$,

and whenever f is adapted to $\widehat{\mathcal{D}}_{t,i}$, $\hat{v}_{t,i}(f) = W_{\mathcal{D}_{t,i}}(F_{f|\mathcal{D}_{t,i}})$.

The following is necessary for a domain recursive representation:

Axiom 9 (Nesting). *Let $f \in \mathcal{F}$ be a non-constant act. Then f is adapted to some domain, \mathcal{D} , such that f is not constant on \mathcal{D} .*

Together with the other axioms posited, the above is also sufficient for a domain recursive representation.

Theorem 5. *Suppose \succeq induces a finite number of domains in Σ . Then \succeq also satisfies P1, P5, and Axioms 3-9 if and only if it has a continuous and non-constant finite domain recursive representation.*

Proof: We first prove that there is at least one domain, say $\mathcal{D}_{T,1}$, to which *all* acts are adapted. To see this, consider any pair of acts, $f, g \in \mathcal{F}$. Let P be the partition of Ω generated by the join of $f^{-1}(\Omega)$ and $g^{-1}(\Omega)$. Note that by Axiom 9 P is adapted to some domain, \mathcal{D} , containing more than one element of P . Since both f and g are coarser than P , they must both be adapted to \mathcal{D} . Now, consider another act, h , not adapted to P . The same argument demonstrates that f, g , and h are adapted to *some* domain. This can not be continued indefinitely, by the assumption of a finite number of domains. Eventually, there must be at least one domain, say $\mathcal{D}_{T,1}$, to which *all* acts are adapted. In particular, the latter implies that $\mathcal{D}_{T,1}$ must contain all elements of Σ that are in $\Omega \cap \widehat{\mathcal{D}}_{T,1}$.

Note that by domain independence, the decision maker is indifferent between every act and some act of the form $x \widehat{\mathcal{D}}_{T,1} f$. If $\widehat{\mathcal{D}}_{T,1} = \Omega$ we are done. Consider, therefore, the case $\widehat{\mathcal{D}}_{T,1} \neq \Omega$ and restrict attention all acts of the form $x \widehat{\mathcal{D}}_{T,1} f$. Axiom 9 implies that every such act must be adapted to some domain other than $\mathcal{D}_{T,1}$. The same argument from the previous paragraph can be used to show that there is a domain, $\mathcal{D}_\alpha \neq \mathcal{D}_{T,1}$ to which *all* acts of the form $x \widehat{\mathcal{D}}_{T,1} f$ are adapted. By Axiom 8 there are two possibilities: $\widehat{\mathcal{D}}_{T,1}$ is or is not a subset of $\widehat{\mathcal{D}}_\alpha$. In the latter case, write $\mathcal{D}_{T,2} \equiv \mathcal{D}_\alpha$, while in the former case note that arbitrariness of $x \widehat{\mathcal{D}}_{T,1} f$ requires that $\widehat{\mathcal{D}}_{T,1}$ be an event in \mathcal{D}_α , so it is sensible to write $\mathcal{D}_{T-1,1} \equiv \mathcal{D}_\alpha$. This pattern may be continued, but by the assumption of a finite number of domains, must eventually end at some $\mathcal{D}_{0,0}$ whose envelope is Ω , and where T is ‘normalized’ to appropriately reflect the number of stages in the construction. The domain recursive representation follows from Proposition 4 - we omit the details. Necessity is likewise easy to show. ■

Up to this point, all the representations in the examples are domain recursive. If one restricts \succeq to a finite set of simple acts, Theorem 5 can be easily extended to the case where the number

of domains is not specified.

Corollary to Theorem 5: *If \succeq satisfies P1, P5, Axioms 3-9 then the restriction of \succeq to acts adapted to some finite partition of Ω has a continuous and non-constant finite domain recursive representation.*

Proof: The argument closely follows that of the proof of Theorem 5. ■

With the above results, we are ready to re-visit the two-urn experiment and address the question of whether there are additional domains and the corresponding question of representing the preference for general acts $x_1 R_k \wedge R_u$ $x_2 R_k \wedge B_u$ $x_3 B_k \wedge R_u$ x_4 . For brevity, we refer to such an act as (x_1, x_2, x_3, x_4) . As observed earlier, the commonly observed Ellsbergian choice pattern $-(x, x, y, y) \sim (y, y, x, x) \succ (x, y, x, y) \sim (y, x, y, x)$ – potentially yields the domains $\mathcal{D}_k = \{\emptyset, R_k, B_k, \Omega\}$ and $\mathcal{D}_u = \{\emptyset, R_u, B_u, \Omega\}$. Consider $R_k \wedge R_u$ and $B_k \wedge B_u$, corresponding to the payoffs x_1 and x_4 , respectively. If these are exchangeable events, then the decision maker is indifferent between (x, y, x, y) and (y, y, x, x) , contradicting $(y, y, x, x) \succ (x, y, x, y)$. Thus to be compatible with Ellsbergian behavior, $R_k \wedge R_u$ and $B_k \wedge B_u$ must not be exchangeable; similarly, $B_k \wedge R_u$ and $R_k \wedge B_u$ cannot be exchangeable.

To identify all domains, suppose that the decision maker reasons as follows: there are r red and b black balls in the unknown urn, where $r + b = 100$. The number of combinations of drawn balls corresponding to each event is given by: $R_k \wedge R_u \Rightarrow 50r$, $R_k \wedge B_u \Rightarrow 50b$, $B_k \wedge R_u \Rightarrow 50r$, and $B_k \wedge B_u \Rightarrow 50b$, so that $R_k \wedge R_u \approx B_k \wedge R_u$ and $B_k \wedge B_u \approx R_k \wedge B_u$ (i.e., the decision maker is indifferent to permuting x_1 and x_3 , or x_2 and x_4). Note that it is not possible for this decision maker to concurrently exhibit Ellsbergian behavior *and* deem $R_k \wedge R_u$ exchangeable with both $R_k \wedge B_u$ and $B_k \wedge R_u$. That is, the decision maker is *not* indifferent to permuting x_1 and x_2 , or x_3 and x_4 . In addition to the previously identified domain $\mathcal{D}_u = \{\emptyset, R_u, B_u, \Omega\}$, this exchangeability structure suggests that \mathcal{D}_k should be expanded to $\{\emptyset, R_k, B_k, S, S^c, \Omega\}$, where $S = R_k \wedge R_u \cup B_k \wedge B_u$ and S^c is the complement of S (\mathcal{D}_k mimics the domain structure of the four color urn). Two further domains that can be identified are: $\mathcal{D}_{R_u} = \{\emptyset, \{R_k \wedge R_u\}, \{B_k \wedge R_u\}, R_u\}$ and $\mathcal{D}_{B_u} = \{\emptyset, \{R_k \wedge B_u\}, \{B_k \wedge B_u\}, B_u\}$. Notice that the envelopes of \mathcal{D}_{R_u} and \mathcal{D}_{B_u} are nested within \mathcal{D}_u , and one can therefore represent \succeq with a finite domain representation.

Although one can appeal directly to Theorem 5, it is instructive to see explicitly how the representation arises. According to domain independence, any act can be decomposed as follows: $(x_1, x_2, x_3, x_4) \sim (c, c', c, c')$, where c is a certainty equivalent payoff associated with receipt of

x_1 and x_3 in \mathcal{D}_{R_u} , and c' is defined similarly with respect to \mathcal{D}_{B_u} . Equivalently, one can write $W_{\mathcal{D}_{R_u}}(\delta_c) = W_{\mathcal{D}_{R_u}}(\frac{1}{2}\delta_{x_1} + \frac{1}{2}\delta_{x_3})$ where $W_{\mathcal{D}_{R_u}}$ is the utility function from Proposition 4. From the symmetry between the two domains we infer that the decision maker has the same risk preferences over \mathcal{D}_{B_u} as over \mathcal{D}_{R_u} . Moreover, assuming the draws from each urn are independent, it seems reasonable that these risk preferences coincide with those over \mathcal{D}_k . Thus, $W_{\mathcal{D}_{B_2}} = W_{\mathcal{D}_{R_2}} = W_{\mathcal{D}_k}$, and $W_{\mathcal{D}_k}(\delta_{c'}) = W_{\mathcal{D}_k}(\frac{1}{2}\delta_{x_2} + \frac{1}{2}\delta_{x_4})$. The decomposed act, (c, c', c, c') , is evaluated via $W_{\mathcal{D}_u}(\frac{1}{2}\delta_c + \frac{1}{2}\delta_{c'})$. If $W_{\mathcal{D}_u}$ is ‘more risk averse’ than $W_{\mathcal{D}_k}$, then the decision maker will prefer bets over the known urn.¹⁷ Consequently, one has the recursive representation:

$$(x_1, x_2, x_3, x_4) \succeq (x'_1, x'_2, x'_3, x'_4) \Leftrightarrow \\ \hat{v}_{\mathcal{D}_u}(W_{\mathcal{D}_k}(\frac{1}{2}\delta_{x_1} + \frac{1}{2}\delta_{x_3}), W_{\mathcal{D}_k}(\frac{1}{2}\delta_{x_2} + \frac{1}{2}\delta_{x_4})) \geq \hat{v}_{\mathcal{D}_u}(W_{\mathcal{D}_k}(\frac{1}{2}\delta_{x'_1} + \frac{1}{2}\delta_{x'_3}), W_{\mathcal{D}_k}(\frac{1}{2}\delta_{x'_2} + \frac{1}{2}\delta_{x'_4}))$$

where $\hat{v}_{\mathcal{D}_u}$ is increasing and symmetric in its arguments, and $\hat{v}_{\mathcal{D}_u}(\delta_x, \delta_y) = W_{\mathcal{D}_u}(\frac{1}{2}\delta_x + \frac{1}{2}\delta_y)$.

The recursive structure arises due to the manner in which the domains are nested. If the domain structure is changed, so will the representation. In particular, if $R_k \wedge R_u$ is exchangeable with $R_k \wedge B_u$, while $B_k \wedge R_u$ is exchangeable with $B_k \wedge B_u$, then domain independence implies that the roles of $W_{\mathcal{D}_k}$ and $W_{\mathcal{D}_u}$ are reversed. In this case too, if $W_{\mathcal{D}_u}$ is ‘more risk averse’ than $W_{\mathcal{D}_k}$, the decision maker prefers bets over the known urn.

It is also worth mentioning that the Corollary to Theorem 5 is useful in deriving a recursive representation even when \succeq is not *finite* domain recursive. Consider, for instance, the rainfall example from the Introduction. Note that acts of the form $(p, H; q, T)$ reside in one of four partitions corresponding to p and q (i) both conditioned on rainfall in city C , or (ii) both conditioned on rainfall in city C' , or (iii) one conditioned on C while the other is conditioned on C' . In each case, the Corollary to Theorem 5 gives a representation. In turn, if the representation for each of the four partitions is subjective expected utility, then each must be consistent with the one proposed in the Introduction for *all* four partitions.

The nested domain structures discussed in this section lead to a partitioning of events into those that are conditioned upon (e.g., second stage) versus those that are conditionally aggregated (e.g., first stage). The accompanying nested-domain representations are reminiscent of the two-stage approach suggested by Segal (1987, 1990) and extended by Klibanoff, Marinacci and Mukerji (2002), Nau (2002), and Ergin and Gul (2002). Segal showed that both Allais and Ellsberg type behavior can be rationalized if the decision maker approaches a choice problem,

¹⁷If the payoffs are monetary and the decision maker has monotonic preferences, a ‘more risk averse’ utility function always assigns a weakly lower certainty equivalent to the same lottery.

like the two-urn experiment, in two-stages. The other papers vary in the formality of their approach and underlying motivation, but in spirit share the objective of rationalizing a two-stage approach to the two-urn problem. In Klibanoff, Marinacci and Mukerji (2002), the unknown urn functions as a source for multiple distributions; the decision maker first assigns a certainty equivalent to each unknown scenario and then aggregates. If the aggregation function is concave, the decision maker prefers bets on the known urn. Nau (2002) also assumes a particular pattern to the stages (payoffs in unambiguous states are first aggregated conditional on ambiguous states). The functional similarity with our domain recursive representation is clear. In our case, however, the representation is silent on the cognitive stages in decision making, or on the ability of the decision maker to conceptualize ‘ambiguity’ in terms of a multitude of distributions. Indeed, our formulation does not commit the decision maker to a first stage utility that aggregates over ambiguous states. Ergin and Gul (2002) take a more neutral stand on the timing of the stages, axiomatizing a representation where the state space is exogenously resolved into a partition of ‘issues’.

Our approach is differentiated by two aspects. First, the state-space partitions determining the stages are endogenously determined. Second, the stages are not limited to two as is the case with Klibanoff, Marinacci and Mukerji (2002) and Nau (2002). Fundamentally, the reason the latter are constrained to two stages is that the papers are concerned primarily with only two types of events: ambiguous and unambiguous. While Ergin and Gul (2002) also consider only two issues, it is clear that their approach can be extended to more than two stages. In this regard, their approach is closest to ours.

The nesting assumption has its limitations. Situations naturally arise in which a recursive domain structure is far from obvious even when acts are restricted to a particular partition. For instance, consider an act that pays \$100 if rainfall in *both* cities is above the subjective medians, and zero otherwise. One approach to this problem is to view the state space spanned by the two cities as resembling a ‘continuous’ version of the two urn problem. If such an identification is possible, one can impose the kind of domain structure suggested above for the two urn problem. While experiments may reveal a clear pattern, there does not appear to be clear intuition for why decision makers would feel compelled to make this analogy. However, if the C and C' domains are seen to correspond to the ‘known’ and ‘unknown’ urns, respectively, then repeated use of the Corollary to Theorem 5 along with a domain specific expected utility assumption

leads to the following representation for an act $f(r, r')$ on the product of C and C' :

$$V(f) = \int u_{C'} \left(u_C^{-1} \left(\int u_C(f(r, r')) d\mu_C(r) \right) \right) d\mu_{C'}(r').$$

5. Anyone Can (Sometimes) Be Probabilistically Sophisticated

The literature on probabilistic sophistication concerns contingencies that Savage (1954) envisaged as states of the ‘big’ world - a description of the world leaving no relevant aspect undescribed. At the same time, Savage observed that decisions are generally made in smaller worlds, which contain events summarizing the relevant aspects of the contingencies pertaining to specific decision situations. This perspective on small worlds leaves with us the question of consistency in decision making from one small world to another. Under Savage’s formulation or more generally probabilistic sophistication, this question has a ready answer, namely, that events in all small worlds are comparable (in our language) to events in any other small world. In the aggregate, small worlds are all derived from and remain similar to a single big world. In other words, a Savagian small world is one in which events are all comparable, and all small worlds containing mutually comparable events are ‘similar’ to each other.

Motivated in part by the pervasiveness of small world decision making, there has been a number of finite-state axiomatizations of subjective expected utility as well as Choquet expected utility (see, e.g., Stigum, 1972; Gul, 1992; Nakamura, 1990; Chew and Karni, 1994). These works do not explicitly address the question of consistency in decision making from one finite-state small world to another finite-state small world. Global probabilistic sophistication provides one avenue to address the question. This issue remains open with the finite-state axiomatizations of Choquet expected utility (Nakamura, 1990 and Chew and Karni, 1994) which does not admit a natural approach to scale upwards from small world decision making to decision making in larger worlds. Put simply, subjective expected utility has an abundance of structure, making the small-world-to-large-world transition straight forward, while Choquet expected utility offers very little structure with which to work.

In this paper, we attempt to go to the heart of the question by offering a precise meaning to small world decision making without requiring comparability of events across different small worlds. Drawing on the de Finetti-Ramsey idea of exchangeability, we develop a concept of comparability to capture the intuition of similarity among events, differing from each other only by a sense of likelihood, and further define a *domain* as a maximal collection of comparable events. When conditioning on events within a domain, the domain can be viewed as an en-

dogenously induced Savagian small world. Our approach goes beyond the Knightian distinction between risk and uncertainty. We characterize small-world probabilistic sophistication on acts adapted to individual domains via a condition – event non-satiation – which considerably weakens Savage’s P3. At the same time our exchangeability based approach delivers multiple domain probabilistic sophistication in the presence of state dependence encompassing the case of pure state independence.

We offer an efficient and weak axiomatization of probabilistic sophistication on (possibly) multiple domains. This enables the modeling of the decision maker’s preference for single-domain acts arising from different domains. To model the decision maker’s preference on mixed acts over distinct domains, we introduce additional preference assumptions and arrive at a recursive utility representation. Overall, our approach enables decision theoretic modeling to generally take place at the small world level, deferring the question of consistent extension of decision making across distinct worlds as and when the relevant domain arises. As Savage puts it, cross the bridge when you come to it.

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A. Appendix

Proof of Proposition 1:

(i) \Rightarrow (ii): Without loss of generality, assume $E \cap E' = \emptyset$. If this is not so, then one can rewrite all that follows replacing E with $E \setminus E'$ and E' with $E' \setminus E$. Suppose $E' \succeq^C A \cup E$. If $A \setminus E'$ is null, then $E \approx e'$ for some $e' \subseteq E' \setminus A$. Since $E' \setminus e'$ contains $A \cap E'$, it is not null. Axiom 4 then implies that $E' = (E' \setminus e') \cup e' \succ^C E$, which contradicts $E \succeq^C E'$. Suppose, on the other hand, $A \setminus E'$ is not null. Since $E \succeq^C E'$, there is some $e \subseteq E$ such that $e \approx E'$. Since $(A \setminus E') \cup (E \setminus e)$ is not null, it follows from Axiom 4 that $A \cup E \succ^C E'$.

(ii) \Rightarrow (iii): Let $e \subset E \setminus E'$ such that $e \approx E' \setminus E$ and assume that $E \setminus (e \cup E')$ is not null. Then since the latter is disjoint from $e \cup (E \cap E')$ and $e \cup (E \cap E') \succeq^C E'$, part (ii) implies $E = \left(E \setminus (e \cup E')\right) \cup \left(e \cup (E \cap E')\right) \succ^C E'$.

(iii) \Rightarrow (iv): The statement of (iv) is equivalent to requiring that for every event $e' \subseteq E'$, $A \cup E$ is not exchangeable with e' . Note that if $e' \approx A \cup E$, then $E' \succeq^C A \cup E$, violating part (iii). Thus (iii) implies (iv).

(iv) \Rightarrow (i): Since $E \cup A \succeq^C E'$, if $E' \succeq^C E \cup A$ then there must be some $e' \subseteq E'$ such that $e' \approx E \cup A$, which in turn contradicts (iv). Thus $E \cup A \succ^C E'$. ■

Proof of Theorem 1: We first prove \succeq^C is a likelihood relation with a unique agreeing measure for any homogeneous σ -algebra. The result for Σ follows from that. Note that if E, E' are members of a homogeneous algebra, then any comparison between E and E' is also in the algebra. The latter property is used extensively in the proof. First consider the following proposition:

Proposition A.1. *Assume P1, Axiom 4 and let $\mathcal{A} \subseteq \Sigma$ be a homogeneous algebra of events. Then for any $E, E', E'' \in \mathcal{A}$, $E \succeq^C E'$ and $E' \succeq^C E''$ imply $E \succeq^C E''$.*

Proof: We first prove related results.

Lemma A.1. *Assume P1 and Axiom 4, and consider any disjoint $a, b, c, d \in \Sigma$. Then $a \cup b \approx c \cup d$, b and d are comparable, and $a \approx c$ implies $b \approx d$*

Proof: Assume that it is not the case that $b \approx d$. Then without loss of generality, let $b \succ^C d$, which by Proposition 1 implies that there is some $b' \subset b$ such that $b' \approx d$ and $b \setminus b'$ is not null. By the definition of exchangeability, $a \cup b' \approx c \cup d$ which violates Axiom 4 since $a \cup b \setminus a \cup b'$ is not null. ■

Lemma A.2. *Assume P1, Axiom 3 and Axiom 4. Then if E, E' and E'' are disjoint members of a homogeneous collection, \mathcal{A} , and $E \approx E'$ and $E' \approx E''$ then $E \approx E''$.*

Proof: This is trivial if any of the events are null, so assume otherwise. Suppose that $E \succ^C E''$. Then by Axiom 4 there is some non-null $e_1 \in E$ such that $E \setminus e_1 \approx E''$, and since \mathcal{A} is a homogeneous algebra, $e_1 \in \mathcal{A}$. By homogeneity of \mathcal{A} and Axiom 4, $E' \succ^C E \setminus e_1$; thus there is some $e_2 \in E'$ such that $e_2 \in \mathcal{A}$ and $E' \setminus e_2 \approx E \setminus e_1$. The events e_1 and e_2 are disjoint, so Lemma A.1 implies that $e_1 \approx e_2$. The fact that $E'' \approx E'$ can be similarly used to establish the existence of a set $e_3 \in E''$ disjoint from e_1 and e_2 such that $e_3 \approx e_2$. Similarly, $E \setminus e_1 \approx E''$ leads to $e_4 \in E \setminus e_1$ such that $e_4 \approx e_3$, etc. Clearly this can be continued to construct an infinite sequence of non-null events that are disjoint such that $e_{i+1} \approx e_i$, in violation of Axiom 3. ■

Lemma A.3. *Assume P1 and Axiom 4. Let $a_1, a_2, a_3, b_1, b_2, b_3$ be disjoint events in a homogeneous algebra \mathcal{A} , and consider the configuration of events as illustrated in Figure 2. Suppose $a_1 \cup b_3 \approx a_2 \cup b_2$, $a_2 \cup b_1 \approx a_3 \cup b_3$, then $a_1 \cup b_1 \approx a_3 \cup b_2$.*

Proof: The proof is accomplished in 3 steps. Step 1 consists of establishing that there exist events a'_1, a'_3 and b'_3 such that $a'_1 \approx a'_3$ and $a'_1 \cup b'_3 \approx a_2 \approx a'_3 \cup b'_3$ (all events are in \mathcal{A} unless otherwise indicated). Step 2 consists of proving that there are events $b'_1 \subseteq b_1, b'_2 \subseteq b_2$ such that $b'_1 \approx a_3 \setminus a'_3, b'_2 \approx a_1 \setminus a'_1$, and $b_1 \setminus b'_1 \approx b_2 \setminus b'_2$. The final step concludes from this that $a_1 \cup b_1 \approx a_3 \cup b_2$.

Step 1: Since $a_1 \cup b_3 \succeq^C a_2$ and $a_3 \cup b_3 \succeq^C a_2$ there is some $\hat{a}_1 \cup \hat{b}_3 \approx a_2$ and $\check{a}_3 \cup \check{b}_3 \approx a_2$, with $\hat{a}_1 \subseteq a_1, \check{a}_3 \subseteq a_3$, and $\hat{b}_3, \check{b}_3 \subseteq b_3$. Let $\hat{a}_2 \approx \hat{b}_3$ and $\check{a}_2 \approx \check{b}_3$, where $\hat{a}_2, \check{a}_2 \subseteq a_2$. Set $a'_2 \equiv a_2 \setminus (\hat{a}_2 \cup \check{a}_2)$. By Axiom 4 and Lemma A.1 there must be $a'_1 \subseteq a_1$ and $a'_3 \subseteq a_3$ such that $a'_1 \approx a'_2 \approx a'_3$. Lemma A.2 proves that $a'_1 \approx a'_3$. Defining $b'_3 \equiv \hat{b}_3 \cup \check{b}_3$ gives $a'_1 \cup b'_3 \approx a_2 \approx a'_3 \cup b'_3$.

Step 2: From $a_1 \cup b_3 \approx a_2 \cup b_2$, $a_3 \cup b_3 \approx a_2 \cup b_1$, and the last step, Lemma A.1 implies that $(a_1 \setminus a'_1) \cup (b_3 \setminus b'_3) \approx b_2$ and $(a_3 \setminus a'_3) \cup (b_3 \setminus b'_3) \approx b_1$. Thus there are $b'_1 \subseteq b_1$ and $b'_2 \subseteq b_2$ such that $b'_1 \approx a_3 \setminus a'_3$ and $b'_2 \approx a_1 \setminus a'_1$. By Lemma A.1, $b_1 \setminus b'_1 \approx b_3 \setminus b'_3$ and $b_3 \setminus b'_3 \approx b_2 \setminus b'_2$, thus Lemma A.2 implies that $b_1 \setminus b'_1 \approx b_2 \setminus b'_2$.

Step 3: Write $a_1 \cup b_1 = a'_1 \cup (a_1 \setminus a'_1) \cup b'_1 \cup (b_1 \setminus b'_1) \approx a'_3 \cup b'_2 \cup (a_3 \setminus a'_3) \cup (b_2 \setminus b'_2) = a_3 \cup b_2$. ■

Now, given $E, E', E'' \in \mathcal{A}$, suppose that $E \succeq^C E'$ and $E' \succeq^C E''$. Let $e' \in \mathcal{A}$ be a comparison subset between E' and E'' . Note that $e' \subseteq E' \setminus E''$; moreover $e' \approx E'' \setminus E'$. Proposition 1 and the assumption that \mathcal{A} is an algebra gives $E \succeq^C e' \cup (E' \cap E'')$. Thus there is some $\hat{e} \subseteq E \setminus (e' \cup (E' \cap E''))$ such that $\hat{e} \approx (e' \cup (E' \cap E'')) \setminus E$. We can now apply Lemma A.3 as follows. Let the lower circle in Figure 2 correspond to E'' . This can be broken up into two pieces: $E'' \setminus E' \equiv a_3 \cup b_3$ and $E'' \cap E' \equiv b_2 \cup c$. Likewise, let e' correspond to $a_2 \cup b_1$, so that $a_2 \cup b_1 \approx a_3 \cup b_3$. Finally, let $a_1 \cup b_3 \equiv \hat{e}$ and set:

$$\xi = (e' \cup (E' \cap E'')) \cap E$$

Diagrammatically, ξ corresponds to $b_1 \cup c$. Note that we identify the left and right circles with

subsets of E and E' , respectively. It follows that:

$$\begin{aligned}
b_1 &= \xi \cap e' \\
a_2 &= e' \setminus b_1 \\
b_3 &= \hat{e} \cap E'' \\
a_1 &= \hat{e} \setminus b_3 \\
b_2 &= \left((e' \cup (E' \cap E'')) \setminus E \right) \cap E'' \\
a_3 &= E'' \setminus (\hat{e} \cup E')
\end{aligned}$$

$\hat{e} \approx (e' \cup (E' \cap E'')) \setminus E$ means that $a_1 \cup b_3 \approx a_2 \cup b_2$. Since $a_2 \cup b_1 \approx a_3 \cup b_3$, Lemma A.3 implies $a_1 \cup b_1 \approx a_3 \cup b_2$. Moreover, since $E'' \setminus E = a_3 \cup b_2$ and $a_1 \cup b_1 \subseteq E \setminus E''$, by definition $E \succeq^C E''$. ■

Proposition A.1 establishes that \succeq^C is a weak order (transitive and complete) over \mathcal{A} . Condition (ii) in the definition of a likelihood relation is satisfied by \succeq^C due to the presence of non-null events (P5) and Axiom 4. The last condition is automatically satisfied by the definition of comparability. The following result helps to complete the proof.

Lemma A.4. \succeq^C is either atomless and tight or \mathcal{A} consists of a finite number of exchangeable atoms.

Proof: Assume first that \mathcal{A} contains an atom, a , and let a^c be its relative complement in $\widehat{\mathcal{A}}$. Note that for any $e \in \mathcal{A}$ it cannot be that $a \succ^C e$ since a cannot be partitioned into two or more non-null events. Thus $a^c \succeq^C a$. If $a \succeq^C a^c$ then Lemma 3 implies that $a \approx a^c$ and \mathcal{A} therefore consists of two atoms. Suppose instead that $a^c \succ^C a$. Then there is some $a_1 \subseteq a^c$ with $a_1 \approx a$; note that a_1 must be an atom in \mathcal{A} . Moreover, since $a^c \setminus a_1 \in \mathcal{A}$ it must be that $a^c \setminus a_1 \succeq^C a$. In turn this implies the presence of another atom $a_2 \approx a$ in $a^c \setminus a_1$ with a, a_1 and a_2 disjoint. According to Axiom 3, this can be continued at most a finite number of times, proving that the set of non-null events in \mathcal{A} is finite. Transitivity of \approx (Lemma 3) implies that each atom is exchangeable.

Assume now that \mathcal{A} is atomless. To demonstrate tightness, we must show that whenever $E \succ^C E'$, there are $A, B \in \mathcal{A}$ such that $A \cap E' = \emptyset$ and $B \subset E$ such that $E \succ^C A \cup E'$ and $E \setminus B \succ^C E'$. By definition, $E \succ^C E'$ implies that there is some $e \subset E \setminus E'$ such that $e \approx E' \setminus E$ and $E \setminus (e \cup E')$ is not null. Since \mathcal{A} is atomless, $E \setminus (e \cup E')$ can be split into two disjoint non-null events, ξ_1 and ξ_2 , both in \mathcal{A} . Proposition 1 implies that $E \succ^C E' \cup \xi_1$ where $\xi_1 \cap E' = \emptyset$, as well as $E \setminus \xi_1 \succ^C E'$. Thus \succeq^C is tight. ■

Lemma A.4 establishes that \succeq^C is a likelihood relation that is either atomic with equal likelihood over each atom, or atomless and tight. In the latter case, we can show that \succeq^C is fine. To do this, for any $E \in \Sigma$ we construct a finite partition of $\widehat{\mathcal{A}}$ at least as fine as $E, \{e_i\}$, starting with $e_1 \equiv E$. Next, homogeneity implies that either $E \succeq^C \widehat{\mathcal{A}} \setminus E$ or $\widehat{\mathcal{A}} \setminus E \succeq^C E$. In the former

case, let $e_2 \equiv \widehat{\mathcal{A}} \setminus E$ and $\{e_1, e_2\}$ forms a partition containing events at least as fine as E . In the latter case, define e_2 as the comparison subset of $\widehat{\mathcal{A}} \setminus E$ that, by definition, is exchangeable with E . Once again, homogeneity implies that either $E \succeq^C \widehat{\mathcal{A}} \setminus (E \cup e_2)$ or $\widehat{\mathcal{A}} \setminus (E \cup e_2) \succeq^C E$, and we can continue constructing events exchangeable with E and disjoint from each other in the obvious way. By Axiom 3 this construction must be finite and therefore constitutes a partition of $\widehat{\mathcal{A}}$ consisting of events at least as fine as E . Thus \succeq^C is fine.

In either the atomic or the fine and tight case, there exists a unique probability measure that agrees with \succeq^C . In the fine case, the fact that \mathcal{A} is closed under countable unions implies that the measure is convex-valued. Finally, whenever the measure of two events, $E, E' \in \mathcal{A}$, coincides, it must be that $E \succeq^C E'$ and $E' \succeq^C E$; in turn, Axiom 4 implies that $E \approx E'$.

We now prove that the decision maker is indifferent between all acts inducing the same distribution. If Σ is atomic, the outcomes can be permuted to generate one act from the other, proving the result. Consider, therefore, the case where Σ has no atoms. Consider any two acts that induce the same lottery with respect to μ : $f = x_1 A_1 \dots x_{n-1} A_{n-1} x_n$ and $f' = x_1 A'_1 \dots x_{n-1} A'_{n-1} x_n$, where n is finite and where for every $i = 1, \dots, n$ $\mu(A_i) = \mu(A'_i)$ (where A_n and A'_n are the preimages of x_n under f and f' , respectively). Let $A'_{1i} \equiv A'_1 \cap A_i$ for $i = 1, \dots, n$. Now,

$$f \sim x_1 A'_{11} x_1 (A_1 \setminus A'_{11}) x_2 A'_{12} x_2 (A_2 \setminus A'_{12}) f$$

Since $A_1 \setminus A'_{11}$ is comparable to A'_{12} , and $\mu(A_1 \setminus A'_{11}) \geq \mu(A'_{12})$, there is some $\xi \subset A_1 \setminus A'_{11}$ such that $\mu(\xi) = \mu(A'_{12})$; consequently, $\xi \approx A'_{12}$, and we can write

$$\begin{aligned} f &\sim x_1 A'_{11} x_1 A'_{12} x_1 (A_1 \setminus (A'_{11} \cup \xi)) x_2 \xi x_2 (A_2 \setminus A'_{12}) f \\ &\sim x_1 A'_{11} x_1 A'_{12} x_1 (A_1 \setminus (A'_{11} \cup \xi)) x_2 \xi x_2 (A_2 \setminus A'_{12}) x_3 A'_{13} x_3 (A_3 \setminus A'_{13}) f \end{aligned}$$

This time, $A_1 \setminus (A'_{11} \cup \xi)$ is comparable to A'_{13} , and $\mu(A_1 \setminus (A'_{11} \cup \xi)) \geq \mu(A'_{13})$. Thus one can exchange A'_{13} with a subset of $A_1 \setminus (A'_{11} \cup \xi)$. Since $\mu(A_1) = \sum_{i=1}^n \mu(A'_{1i})$, this can be continued until one arrives at:

$$f \sim x_1 A'_1 x_2 A_2^1 \dots x_{n-1} A_{n-1}^1 x_n$$

for some collection of events, A_j^1 , $j = 2, \dots, n$, where $\mu(A_j^1) = \mu(A_j) = \mu(A'_j)$ for $j = 2, \dots, n$.

This procedure can clearly be continued to write

$$\begin{aligned}
f &\sim x_1 A'_1 x_2 A_2^1 \dots x_{n-1} A_{n-1}^1 x_n \\
&\sim x_1 A'_1 x_2 A'_2 x_3 A_3^2 \dots x_{n-1} A_{n-1}^2 x_n \\
&\quad \vdots \\
&\sim x_1 A'_1 x_2 A'_2 x_3 A'_3 \dots x_{n-1} A'_{n-1} x_n
\end{aligned}$$

Thus $f \sim f'$, and the decision maker is indifferent between any two acts that induce the same lottery with respect to μ .

Proving necessity of the axioms is trivial. ■

Proof of Theorem 2: In light of Theorem 1, it remains to establish that adding Axiom 5 is necessary and sufficient for the existence of U . If Σ is atomic, then the result is obvious. Consider, therefore, the non-atomic case. The goal is to show that “given Axiom 5, \succeq is equivalent to a preference ordering, \succeq^{SL} , over simple lotteries on X that satisfies $p_n \succeq^{SL} q_n \forall n \Rightarrow p \succeq^{SL} q$ whenever (p_n, q_n) weakly converge to (p, q) ”. In other words, \succeq^{SL} is continuous with respect to the topology of weak convergence on the space of simple lotteries on X . One can then define \succeq^L as the unique weak closure of \succeq^{SL} in the set of Borel distributions of X . The topology of weak convergence on Borel measures of X is metric and separable (see Parthasarathy, 1967), thus by Debreu’s (1983) Theorem, continuity of \succeq^L in the same topology is equivalent to existence of a continuous utility representation.

Assume, therefore, Axioms 3-5, P1 and P5, and denote the measure from Theorem 1 as μ . Theorem 1 implies that \succeq is equivalent to a preference ordering, \succeq^{SL} , over simple lotteries. Given Axiom 5, it is sufficient to demonstrate that convergence in acts is equivalent to weak convergence of lotteries. Note that convergence in acts implies the weak convergence of the induced measure. To prove the converse consider a weakly convergent sequence of finite outcome lotteries $p_n \rightarrow p$. Let $(x_i, e_i)_k$ correspond to a finite act that induces p , such that the x_i ’s are distinct. The objective is to find a sequence of acts that induce $\{p_n\}$ and converge to $(x_i, e_i)_k$. Let $\bar{\delta}$ be smaller than the smallest distance between any two x_i ’s and note that for any $i \in 1, \dots, k$ and every $0 < \delta < \bar{\delta}$, the measure under p_n of the set $X_i(\delta) \equiv \{x \mid \delta > \|x - x_i\|\}$ converges to $\mu(e_i)$ as n increases. In particular, for every positive integer, m , there is a smallest positive integer, $N(m) > 0$, such that $|p_n(X_i(\frac{\bar{\delta}}{2^m})) - \mu(e_i)| < \frac{\bar{\delta}}{2^m}$ for every $n \geq N(m)$ and

every $i = 1, \dots, k$. It should be clear that $\{N(m)\}$ is a weakly increasing sequence. If $\{N(m)\}$ is bounded, then $p_n = p$ for every n greater than some finite N and one can easily construct a set of inducing acts that converges. If $\{N(m)\}$ is not bounded above, then for every $m \geq 1$ and $N(m) \leq n < N(m+1)$, define f_n to be an act that induces p_n in the following way. For each $i = 1, \dots, k$ let the outcomes in both the support of p_n and $X_i(\frac{\delta}{2^m})$ be arbitrarily listed as $y_{i,n}^1, \dots, y_{i,n}^l$. If $p_n(y_{i,n}^1) \leq \mu(e_i)$ then let $f_{i,n}$ be an act that pays $y_{i,n}^1$ in some event, $e_{i,n}^1 \subseteq e_i$ such that $\mu(e_{i,n}^1) = p_n(y_{i,n}^1)$ (this can be done since according to Theorem 1 μ is convex valued). Next, if $p_n(y_{i,n}^2) \leq \mu(e_i) - p_n(y_{i,n}^1)$, then let $f_{i,n}$ pay $y_{i,n}^2$ in some event, $e_{i,n}^2 \subseteq e_i \setminus e_{i,n}^1$ such that $\mu(e_{i,n}^2) = p_n(y_{i,n}^2)$. This can be continued until either all the $y_{i,n}^j$'s are assigned, or the measure of some $p_n(y_{i,n}^j)$ exceeds the measure of the 'unassigned' portion of e_i . In the latter case assign the residual event the outcome $y_{i,n}^j$. In the former case, denote the residual event as $E_{i,n}$ and assign it the outcome x_i ; note that $\mu(E_{i,n}) < \frac{\delta}{2^m}$. Finally let $f_{i,n}$ pay x_i everywhere outside of e_i . Note that p_n is induced by the act $\hat{f}_n E_n (f_{i,n}, e_i)_k$ for $E_n \equiv \bigcup_{i=1}^k E_{i,n}$ and some appropriate act \hat{f} . Moreover, $\mu(E_n) < \frac{\delta}{2^m} k$, thus by construction f_n converges to $(x_i, e_i)_k$. ■

Proof of Lemma 3: Suppose $E, A, E' \in \Sigma$ are disjoint, such that $E \approx E'$, and A is non-null. Assume it is not the case that $E \cup A \succ^C E'$. Since $E \cup A$ and E' are comparable, $E' \succeq^C E \cup A$. Thus E' contains a subset, ξ' , that is exchangeable with $E \cup A$. In particular, by exchanging ξ' for $E \cup A$, we have for any $x, x' \in X$ and $f \in \mathcal{F}$ that

$$x'(E \cup A)x E' f \sim x' \xi' x((E \cup A) \cup (E' \setminus \xi')) f = x' \xi' x(E \cup (A \cup E' \setminus \xi')) f$$

Similarly, by exchanging E with E' it follows that

$$x'(E \cup A)x E' f \sim x'(E' \cup A)x E f = x'(\xi' \cup (A \cup E' \setminus \xi')) x E f$$

Note that the set $A \cup E' \setminus \xi'$ is not null, since A is not null. Axiom 4' is therefore violated, meaning that it cannot be the case that $E' \succeq^C E \cup A$. Thus $E \cup A \succ^C E'$. ■

Proof of Theorem 3: First consider the following proposition:

Proposition A.2. *Assume P1, Axiom 3, Axiom 4' and let $\mathcal{A} \subseteq \Sigma$ be a homogeneous collection of events. Then for any $E, E', E'' \in \mathcal{A}$, $E \succeq^C E'$ and $E' \succeq^C E''$ imply $E \succeq^C E''$.*

Proof: Note that Lemma A.1 holds under Axiom 4'. For any $A, B \in \Sigma$, write $A \sim^C B$ whenever $A \succeq^C B$ and $B \succeq^C A$. As mentioned in the text, $A \sim^C B$ implies that $A \setminus B \approx B \setminus A$.

Lemma A.5. *Assume P1 and Axiom 4'. Suppose that $E, E', A, B \in \Sigma$ are such that $A \cap E = B \cap E' = \emptyset$, $E \sim^C E'$, A and B are comparable as are $E \cup A$ and $E' \cup B$. Then $E \cup A \sim^C E' \cup B$ if and only if $A \sim^C B$.*

Proof: Note that if the result holds for $A \cap B = E \cap E' = \emptyset$ then it holds otherwise as well. Assume, therefore, that A and B are disjoint as are E and E' .

For sufficiency, since A and B are comparable, assume without loss of generality that $A \succeq^C B$. Let $\hat{A} \subseteq A$ such that $\hat{A} \approx B$ and set $\xi \equiv A \setminus \hat{A}$. For any $x, x' \in X$ and $f \in \mathcal{F}$ write

$$\begin{aligned}
& x' \left(\xi \cup \hat{A} \cup (E \setminus B) \right) x \left(B \cup (E' \setminus A) \right) f \\
&= x' \left((E \setminus B) \cup (A \setminus E') \right) x \left((E' \setminus A) \cup (B \setminus E) \right) x(E \cap B) x'(E' \cap A) f \\
&\sim x \left((E \setminus B) \cup (A \setminus E') \right) x' \left((E' \setminus A) \cup (B \setminus E) \right) x(E \cap B) x'(E' \cap A) f \\
&= x E x' E' x(A \setminus E') x'(B \setminus E) f \\
&\sim x' E x E' x(A \setminus E') x'(B \setminus E) f \\
&= x \hat{A} x' B x(E' \setminus A) x'(E \setminus B) x(A \setminus \hat{A}) f \\
&\sim x' \hat{A} x B x(E' \setminus A) x'(E \setminus B) x(A \setminus \hat{A}) f \\
&= x' \left(\hat{A} \cup (E \setminus B) \right) x \left(\xi \cup B \cup (E' \setminus A) \right) f
\end{aligned}$$

Thus by Axiom 4', ξ is null, and $A \approx B$.

For necessity, suppose that $A \approx B$ and that $E \cup A$ and $E' \cup B$ are comparable. Assume without loss of generality that $E \cup A \succeq^C E' \cup B$, then there is some $\xi \subset E \cup A$ such that $\xi \cap (E' \cup B) = \emptyset$ and $(E \cup A) \setminus \xi \sim^C E' \cup B$. Let $\hat{E} \equiv E \setminus \xi$ and $\hat{A} \equiv A \setminus \xi$. Write:

$$\begin{aligned}
& x' \left((\hat{E} \setminus B) \cup (\hat{A} \setminus E') \right) x \left((E' \setminus \hat{A}) \cup (B \setminus \hat{E}) \right) x(B \cap E) x'(E' \cap A) x((A \cup E) \cap \xi) f \\
&\sim x \left((\hat{E} \setminus B) \cup (\hat{A} \setminus E') \right) x' \left((E' \setminus \hat{A}) \cup (B \setminus \hat{E}) \right) x(B \cap E) x'(E' \cap A) x((A \cup E) \cap \xi) f \\
&= x E x' E' x(A \setminus E') x'(B \setminus E) f \\
&\sim x' E x E' x(A \setminus E') x'(B \setminus E) f \\
&= x' B x A x(E' \setminus A) x'(E \setminus B) f \\
&\sim x B x' A x(E' \setminus A) x'(E \setminus B) f \\
&= x' \left((\hat{E} \setminus B) \cup (\hat{A} \setminus E') \right) x \left((E' \setminus \hat{A}) \cup (B \setminus \hat{E}) \right) x(B \cap E) x'(E' \cap A) x'((A \cup E) \cap \xi) f
\end{aligned}$$

Thus by Axiom 4', ξ is null, and $E \cup A \sim^C E' \cup B$. ■

Lemma A.6. *Assume P1, Axiom 3' and Axiom 4', and suppose E, E' and E'' are in a homogeneous collection. If $E \setminus E' \approx E' \setminus E$ and $E' \setminus E'' \approx E'' \setminus E'$, then $E \setminus E'' \approx E'' \setminus E$.*

Proof: Consider the diagram in Figure 2 where the leftmost, rightmost and bottom circles represent E, E' , and E'' , respectively. Without loss of generality assume that $E \setminus E'' \succeq^C E'' \setminus E$, so there must be some $E_1 \subseteq E$, and $\xi \subseteq E \setminus E''$ such that $E_1 \cup \xi = E$ and $E_1 \setminus E'' \approx E'' \setminus E$. Note that both ξ and E_1 are in \mathcal{A} . The idea is to construct a sequence of events akin to those in Axiom 3' by sequentially comparing pairs of events *clockwise* in Figure 2. To begin, let $\xi \equiv e_1 \cup d_1$ where $e_1 \subset E \setminus E'$ and

$$d_1 \subseteq E \cap E'.$$

Proceeding clockwise, compare now the events newly constructed from E with the event E' . Since $E' \succeq^C E$ it must be that $E' \succeq^C e_1 \cup d_1$, implying that there is some subset of $E' \setminus d_1$ that is exchangeable with e_1 . Denote this subset as $e_2 \cup d_2$, where $e_2 \subseteq E' \setminus E''$ and $d_2 \subseteq E' \cap E''$. Note that d_1, d_2, e_1 and e_2 are disjoint, and that $d_1 \cup e_2 \cup d_2 \in \mathcal{A}$ as is $E'_1 \equiv E' \setminus (d_1 \cup e_2 \cup d_2)$. Moreover, $e_1 \cup d_1 \cup E_1 = E \sim^C E' = (d_1 \cup e_2 \cup d_2) \cup E'_1$ implies, by Lemma A.5, that $E_1 \setminus E'_1 \approx E'_1 \setminus E_1$ (where $d_1 \cup E_1$ corresponds to A and $d_1 \cup E'_1$ corresponds to B in the Lemma).

Now move clockwise to compare E'' with the events constructed from E' . Specifically, since $E'' \succeq^C E'$ it must be that $E'' \succeq^C d_1 \cup e_2 \cup d_2$, implying that there is some subset of $E'' \setminus d_2$ that is exchangeable with $d_1 \cup e_2$ (recall that $d_2 \subseteq E''$ and $d_1 \cup e_2$ is disjoint from E''). Denote this subset as $e_3 \cup d_3$, where $e_3 \subseteq E'' \setminus E$ and $d_3 \subseteq E'' \cap E$. Note that $d_2 \cup e_3 \cup d_3 \in \mathcal{A}$ as is $E''_1 \equiv E'' \setminus (d_2 \cup e_3 \cup d_3)$. Moreover, $d_1 \cup e_2 \cup d_2 \cup E'_1 = E' \sim^C E'' = (d_2 \cup e_3 \cup d_3) \cup E''_1$ implies, by Lemma A.5, that $E''_1 \setminus E'_1 \approx E'_1 \setminus E''_1$. Again observe that the e_i 's and d_i 's are disjoint. Next, use the fact that $E_1 \succeq^C E''_1$ to construct $e_4 \cup d_4 \approx e_3$ and $E_2 \equiv E_1 \setminus (d_3 \cup e_4 \cup d_4)$ in the obvious way. Clearly, this can be continued indefinitely resulting in the sequence of disjoint events, $\{e_n, d_n\}_{n=1}^\infty$ in Σ , such that $d_{n-1} \cup e_n \approx e_{n+1} \cup d_{n+1}$ for every $n \geq 1$ (setting $d_0 \equiv \emptyset$). Axiom 3' implies that $e_1 \cup d_1$ is null. Thus $E \setminus E'' \approx E'' \setminus E$. ■

Now, given $E, E', E'' \in \mathcal{A}$, suppose that $E \succeq^C E'$ and $E' \succeq^C E''$. Let $e' \in \mathcal{A}$ be a comparison event between E' and E'' . By homogeneity of \mathcal{A} , $\hat{E}' \equiv e' \cup (E' \cap E'') \in \mathcal{A}$. Since $E \succeq^C E'$, it must be that $E \succeq^C \hat{E}'$. Letting $e \subseteq E$ be a comparison event between E and \hat{E}' , it must be that $\hat{E} \equiv e \cup (E \cap \hat{E}') \in \mathcal{A}$ as well. Note that by construction, $\hat{E} \setminus \hat{E}' \approx \hat{E}' \setminus \hat{E}$ and $\hat{E}' \setminus E'' \approx E'' \setminus \hat{E}'$, thus Lemma A.6 gives that $\hat{E} \setminus E'' \approx E'' \setminus \hat{E}$, or in other words, $E \supseteq \hat{E} \succeq^C E''$. Part (ii) of Proposition 1 implies that $E \succeq E''$. ■

Proposition A.2 and the definition of homogeneity establish that \succeq^C is a weak order (transitive and complete) over \mathcal{A} . Condition (ii) in the definition of a likelihood relation is satisfied by \succeq^C due to the presence of a non-null event and Proposition 1. The last condition is automatically satisfied by the definition of comparability.

The result in Lemma A.4 still goes through with trivial modifications, thus establishing that \succeq^C is a likelihood relation that is either atomic with equal likelihood over each atom, or atomless and tight. In the latter case, the proof that \succeq^C is fine is also identical to the argument in Theorem 1.

If \mathcal{A} is an algebra, then in either the atomic or the fine and tight case, there exists a unique probability measure that agrees with \succeq^C . Otherwise, in the atomic case, \mathcal{A} is clearly a union of algebras, each of which is generated by the same number of atoms. Assume, therefore that \mathcal{A} is not atomic and not an algebra. Then according to Zhang (1999) one needs to further prove that (i) for any two uniform partitions of $\widehat{\mathcal{A}}$ in \mathcal{A} , say $\{E_i\}_{i=1}^n$ and $\{E'_i\}_{i=1}^n$, if $|I| = |J|$ then $\bigcup_{i \in I} E_i \sim^C \bigcup_{i \in J} E'_i$; and (ii) for any decreasing sequence of events, $\{E_n\}_{n=1}^\infty$ in \mathcal{A} and

$E^*, E_* \in \mathcal{A}$, with $E^* \succ^C \bigcap_{n=1}^{\infty} E_n \succ^C E_*$ there is $N < \infty$ such that for every $n > N$, $E^* \succ^C E_n \succ^C E_*$.¹⁸

Proof of (i): Let $\{E_i\}_{i=1}^n$ and $\{E'_i\}_{i=1}^n$ be two uniform partitions of $\widehat{\mathcal{A}}$ in \mathcal{A} . Suppose $E_i \succ^C E'_j$ for some i, j . Then by Propositions 1 and A.2 this is true for every i, j . Moreover, Lemma A.5 implies that $e_1 \cup e_2 \sim^C E'_1 \cup E'_2$ for some $e_1 \subset E_1$ and $e_2 \subset E_2$ with $E_1 \setminus e_1$ and $E_2 \setminus e_2$ not null. Thus by Proposition 1, $E_1 \cup E_2 \succ^C E'_1 \cup E'_2$. The same reasoning also implies $\bigcup_{i \in I} E_i \succ^C \bigcup_{i \in J} E'_i$ for any $|I| = |J| \geq 1$. This, of course, produces a contradiction when $|I| = |J| = n$. Thus for any i, j , $E_i \sim^C E'_j$. Lemma A.5 then delivers the desired result.

Proof of (ii): Consider a decreasing sequence of events, $\{E_n\}_{n=1}^{\infty}$ in \mathcal{A} whose limit $E \equiv \bigcap_{n=1}^{\infty} E_n$ is not null. Assume there are $E^*, E_* \in \mathcal{A}$ such that $E^* \succ^C E \succ^C E_*$. Then by proposition 1 $E \subseteq E_n$ implies $E_n \succ^C E_*$. Now, define $E_0 \equiv \widehat{\mathcal{A}}$ and let $e_n \equiv E_n \setminus E_{n+1}$ for $n \geq 0$. Note that $\{e_n\}_{n=0}^{\infty} \cup E \equiv P_0$ forms a partition of $\widehat{\mathcal{A}}$. Since \mathcal{A} has no atoms and possesses the disjoint countable union property, it contains a sequence of partitions, $\{P_j\}_{j=0}^{\infty}$, such that P_{j+1} is finer than P_j , and each event in P_j is uniformly partitioned into two events in P_{j+1} . Consider the σ -algebra, \mathcal{A}_0 generated by $\bigcup_{j=0}^{\infty} P_j$. Clearly \succeq^C is complete and transitive in $\mathcal{A}_0 \subseteq \mathcal{A}$. Moreover, since every non-null event in \mathcal{A}_0 can be arbitrarily uniformly partitioned, \mathcal{A}_0 is fine. To establish that it is tight, consider $A \succ^C B$ for $A, B \in \mathcal{A}_0$, and assume without loss of generality that A and B are disjoint. Then there is some $A^* \subset A$ such that $A^* \approx B$ with $A \setminus A^*$ not null. The difficulty is that A^* may not be in \mathcal{A}_0 . Regardless, A can be uniformly partitioned into $\{\xi_i\}_{i=1}^k$, non-null sets such that $A \setminus A^* \succ^C \xi_i$ for $i = 1, \dots, k$. One can therefore write $A = (A \setminus \xi_1) \cup \xi_1 = A^* \cup \xi^* \cup (A \setminus (A^* \cup \xi^*))$, where $\xi^* \sim^C \xi_1$, $\xi^* \subset A \setminus A^*$ and $A \setminus (A^* \cup \xi^*)$ is not null. Lemma A.5 gives $A \setminus \xi_1 \sim^C A^* \cup (A \setminus (A^* \cup \xi^*)) \sim^C B \cup (A \setminus (A^* \cup \xi^*))$. Since all the events are in \mathcal{A} , a homogeneous collection, Lemma A.6 implies $A \setminus \xi_1 \sim^C B \cup (A \setminus (A^* \cup \xi^*))$ meaning via Proposition 1 that $A \setminus \xi_1 \succ^C B$ and $A \succ^C B \cup \xi_1$. Thus \succeq^C is a fine and tight likelihood relation on the algebra \mathcal{A}_0 . In particular, this means that it can be uniquely represented by a convex valued probability measure, μ_0 , on \mathcal{A}_0 . Note that $\mu_0(E_n \setminus E) \rightarrow 0$, thus for any non-null $\hat{a} \in \mathcal{A}_0$, there is some $N < \infty$ such that for all $n > N$, $\hat{a} \succ^C E_n \setminus E$.

Since $E^* \succ^C E$, Proposition 1 implies that there is some non-null $a \in \mathcal{A}$ such that $E^* \setminus a \sim^C E$. Moreover, since \mathcal{A}_0 is fine and contained in \mathcal{A} , there is some non-null $\hat{a} \in \mathcal{A}_0$ such that $a \succeq^C \hat{a}$. Thus Proposition A.2 and the result from the end of the last paragraph imply that there is some $N < \infty$ such that for all $n > N$, $a \succ^C E_n \setminus E$, and by Lemma A.5, $E^* \succ^C E_n$.

Summarizing, if \succeq^C is not atomic, then since it satisfies properties (i) and (ii) in Zhang (1999), and is a fine and tight likelihood relation that is closed under countable unions, there exists a unique agreeing convex-valued probability measure for \succeq^C on \mathcal{A} . Finally, whenever the

¹⁸If $\bigcap_{n=1}^{\infty} E_n$ is null, then it is implicitly assumed in Zhang (1999) that $E^* \succ^C \bigcap_{n=1}^{\infty} E_n$ implies there is $N < \infty$ such that for every $n > N$, $E^* \succ^C E_n$.

measure of two events, $E, E' \in \mathcal{A}$, coincides, it must be that $E \succeq^C E'$ and $E' \succeq^C E$; this is true if and only if $E \approx E'$.

Necessity easily follows from the fact that \mathcal{A} is a λ -system and the agreeing probability measure represents \succeq^C on \mathcal{A} . ■

Proof of Theorem 4: It suffices to show that the decision maker is indifferent between any two acts, say f and f' , that are identical outside a domain, while inducing the same lottery on the domain (the remainder of the proof follows that of Theorem 2). To this end, assume without loss of generality that $f = (x_i, e_i)_k$ and $f' = (x_i, e'_i)_k$, such that for some $l \leq k$, $\{e_i\}_{i=1}^l$ and $\{e'_i\}_{i=1}^l$ are in \mathcal{D} , partition $\widehat{\mathcal{D}}$, and $\mu_{\mathcal{D}}(e_i) = \mu_{\mathcal{D}}(e'_i)$ for $i = 1, \dots, l$; furthermore, if $k > l$, then $e_i = e'_i$ for the remaining $i = l+1, \dots, k$. If $\{e_i\}_{i=1}^l$ and $\{e'_i\}_{i=1}^l$ are uniform l -partitions of $\widehat{\mathcal{D}}$, then Axiom 6 establishes the result (in particular, this situation holds when \mathcal{D} is atomic). Otherwise, for every $m > l$ one can always find l non-negative integers $\{n_{i,m}\}$ such that for each $i = 1, \dots, l$, $\frac{n_{i,m}}{m} \leq \mu_{\mathcal{D}}(e_i) \leq \frac{n_{i,m}+1}{m}$; let $e_{i,l+1}$ be a subset of e_i such that $\mu_{\mathcal{D}}(e_{i,l+1}) = \frac{n_{i,l+1}}{l+1}$, and for every $m > l+1$ let $e_{i,m}$ be a superset of $e_{i,m-1}$ and subset of e_i such that $\mu_{\mathcal{D}}(e_{i,m}) = \frac{n_{i,m}}{m}$ – fineness of $\mu_{\mathcal{D}}$ guarantees that this definition is not vacuous. Now, consider the act $f_m = \bar{x} E_m (x_i, e_i)_k$ where $E_m = \bigcup_{i=1}^l e_i \setminus e_{i,m}$ and \bar{x} is some arbitrary but fixed outcome. A similar argument can be applied to construct $f'_m = \bar{x} E'_m (y_i, e'_i)_k$. Note that by Axiom 6, $f_m \sim f'_m$ since each induces a uniform m -partition of $\widehat{\mathcal{D}}$. Finally, $f_m \rightarrow f$, $f'_m \rightarrow f'$, Axiom 5 implies $f \sim f'$. ■

Proof of Proposition 2: Since there exists $x^*, x_* \in X$ with $x^* \succ x_*$, Axiom 4' holds. Assume $E \succeq^C E'$. For any $x^*, x_* \in X$ with $x^* \succ x_*$ and $f \in \mathcal{F}$, write

$$\begin{aligned} x^* E x_* E' f &= x^* \xi \cup \xi' x_* E' f \\ &\text{where } \xi \cup \xi' = E \text{ and } \xi' \approx E' \\ &\sim \underbrace{x^* \xi \cup E'}_{\text{By definition of } \approx} x_* \xi' f \\ &\succeq x^* E' \underbrace{x_* \xi \cup \xi' f}_{\text{By P3}} = x^* E' x_* E f \end{aligned}$$

Note that by Lemma 3 $E \succ^C E' \Rightarrow x^* E x_* E' f \succ x^* E' x_* E f$. The latter along with completeness of \succeq^C implies that $x^* E x_* E' f \succeq x^* E' x_* E f \Rightarrow E \succeq^C E'$. ■

Proof of Proposition 3: Suppose there are $E, E', A \in \Sigma$, $x(E \cup A)x'E'f \sim xEx'(E' \cup A)f$ for

every $x, x' \in X$ and $f \in \mathcal{F}$. Specializing to $f = x'$, it must be that $x(E \cup A)x' \sim xEx'$ for every $x, x' \in X$. Note that this is not consistent with the hypothesis when E is null unless A is null. Assuming E is not null, there are $y, y' \in X$ such that $x \succ y(E \cup A)x$ and $y'(E \cup A)x \succ x$. If $P3^{CU}$ is satisfied and A is not null, it must be that $xAyEx = yEx \succ y(E \cup A)x$, a contradiction. On the other hand, if $P3^{CL}$ is satisfied and A is not null, it must be that $y'(E \cup A)x \succ y'Ex = xAy'Ex$, also a contradiction. Thus A is null. This establishes that the hypothesis implies Axiom 4'. By Lemma 3, Axiom 4 is also implied. ■

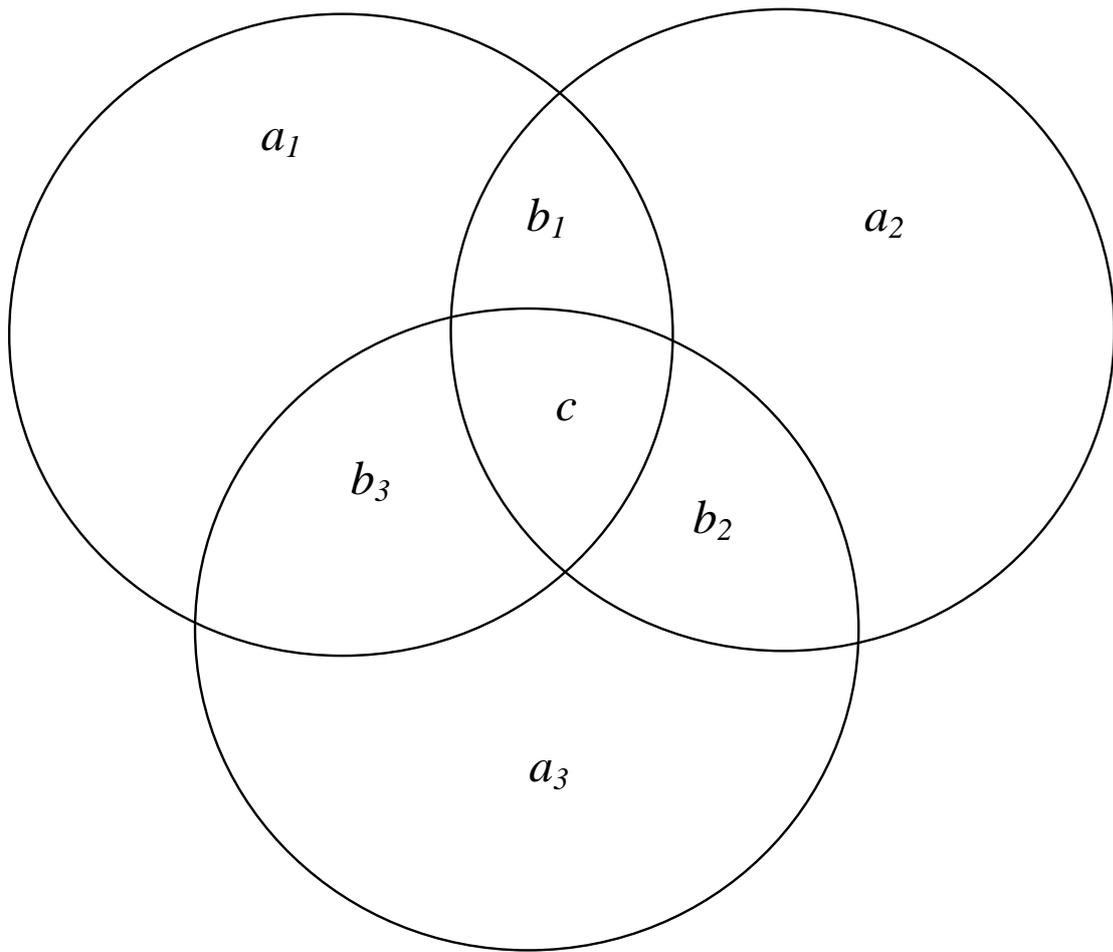


Figure 1: Venn diagram useful in proving Theorem 1 and Theorem 3.