# The Interaction Between Quality Control and Production

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**Abstract**

A real options model of resource extraction is considered where management controls both the extraction rates as well as the quality of extracted material earmarked for processing into final product. The minimum quality of material acceptable for processing is called the cutoff grade. If the cutoff is too high, much of the extracted material goes to waste exacting an opportunity cost. If the cutoff is too low, then input capacity constraints are fully utilized, but with poor revenues at output. The optimal strategy attains a balance between these considerations. A form of opportunity cost associated with extracting the marginal unit of resource seems to play a crucial role in finding the optimal strategic balance. This opportunity cost has a structure resembling a call option on the commodity, and its price sensitivity depends on the commodity price dynamics. In particular, mean reversion in the underlying lead to price-insensitive opportunity costs at high commodity prices, which can result in a more wasteful optimal extraction policy. A model is developed in the presence of input and output processing stage constraints, and the set of candidate optimal policies is identified. Simulations are used to expound on the nature of the optimal policies and the important role played by the opportunity cost.

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# 1 INTRODUCTION

It is often the case in a variety of resource industries that raw material has a distribution of economic value. In particular, some material may be deemed too poor in quality to be of economic use. The cutoff level between usable and rejected material is a function of the operational and economic circumstances. For example, consider a mining operation that can excavate 10 million tons of material per year but only has the capacity to process 4 million tons per year. If the mine management decides to extract more than it can process, the lowest grade material may go to waste.[1] Suppose that current prices are such that the mine management finds it optimal to extract the full 10 million tons per year; it would then choose to process only material in the top 40% of grade quality. Thus, economic conditions and operational constraints conspire to place a limit on the quality of material deemed suitable for processing and what is subsequently 'wasted'. In the mining industry, the choice of minimum quality level is known as the cutoff grade.

The aim of this paper is to illustrate the tradeoffs and interplay between resource utilization and quality control, as well as their impact on real production and waste policy. One contribution of the paper is the identification of the opportunity cost of extraction (OCE) as a crucial factor in determining the appropriate balance between extraction rates and quality cutoff.[2] This has previously been pointed out by Lane (1988) in a static price discounted cash flow model using a different notion of opportunity costs.[3] A more novel contribution, from an economic point of view, is the interpretation of the OCE as a derivative claim resembling a forward-start American call option written on the marginal extracted unit of material. The maturity of the forward component is bounded below by the minimum time to depletion of the reserves and the strike price is linked to the variable costs of production. The latter analogy is important since it characterizes the determinants of the OCE, including its price sensitivity. For instance, given a mean-reverting underlying resource price process, a forward start option with long forward maturity is insensitive to the commodity price when it is high. In certain instances, the resulting implication for the cutoff grade is dramatic: price-insensitive opportunity costs may lead to an extraction and cutoff grade policy that wastes increasingly more resource as the price increases; in an identical operational setup, the exact opposite can be true if the underlying price dynamics are non-stationary. The

[1] While subpar ore might be stockpiled, it is often more economical to dump it in a designated waste area along with excavated barren rock (overburden). The resulting mixture is typically of little economic potential.

[2] The opportunity cost of extraction measures the value of deferring production vis á vis the loss of value due to reserve depletion.

[3] Lane's discounted cash flow approach is generally ill-suited for calculating dynamic and time-consistent price-contingent strategies in a stochastic environment.

relevant conclusions are potentially manifold, applying both to the operation and design of facilities as well as government agencies that wish to regulate the practices of such facilities. In the latter case, profit and/or extraction tax linked to price movements can distort the optimal policy towards one that is 'less wasteful'.

Crucial to the analysis is the contingent claims, or real options, approach. Real options analysis is typically used to examine investment decisions in a simplified setting. The major insights from the literature are either methodological or concerned with conveying the message that the option to defer investment may be valuable. Moreover, the approach often faces criticism from practitioners in resource industries for being overly simplistic. By using the methodology of modern asset pricing, this paper offers new economic insights that extend beyond the value of deferring investment, as well as a realistic resource production model that compares favorably with what is considered best practice valuation in the resource industry.

### 1.1 What is an 'opportunity cost of extraction'?

To define and illustrate the economic role of the OCE, consider that the decision whether or not to extract a unit of raw material hinges on whether the profits from the unit together with the value of the remaining reserve exceed the value of the reserve with the unit left unexploited. In other words, one requires

$$\text{profit from unit} + PV\Big[J_E\Big] \geq PV\Big[J_U\Big]$$

where $J_E$ is the value of the 'exploited' reserve in the next period, $J_U$ is the value of the reserve in the next period assuming the unit is not extracted, and $PV[\cdot]$ denotes 'present value'. Note, first, that if the option to extract the unit is 'now or never', then $PV\Big[J_E\Big] = PV\Big[J_U\Big]$, and the unit is extracted whenever profit from the endeavor is positive. If extraction can be deferred costlessly, then each of $J_E$ and $J_U$ is a future value for the reserve, with the former having one unit less than the latter. In this case, one can loosely write $J_U = J(Q)$ and $J_E = J(Q - dQ)$ where $Q$ is the current size of the reserve, and $dQ$ is the marginal unit extracted. Thus, if production of a unit can be deferred costlessly, extraction of the marginal unit can only be justified if its profit exceed an opportunity cost whose magnitude equals the present value of $J(Q) - J(Q - dQ)$; in continuous time, the decision to extract reduces to comparing the variable profits per unit extracted with the variable opportunity cost of extraction: $\frac{\partial J}{\partial Q}$. It should be stressed that this discussion precludes the costs of varying production – if it is costly to change production levels, the marginal additional cost would enter on the appropriate side of the inequality.[4] A more involved model is presented later. Regardless, this simple analysis establishes two important ideas. First, when the production level can be changed, the option to defer the production of a marginal unit leads to a variable opportunity cost – the OCE; and second, the OCE is the sensitivity of the project to its size.

### 1.2 Quality control policy and the OCE

To illustrate the implications of the first point above for cutoff grade policy, consider the illustrative example of a resource extraction project (oil, forestry, mining, etc.) with finite reserve levels and maximum extraction capacity of 3 units per period. Suppose that, *after* extraction, material can be sorted into low, medium and high quality categories, fetching prices of $S_l, S_m$ and $S_h$ per unit, respectively. Naturally, $S_h > S_m > S_l$. The different grades of material are assumed to be present in equal amounts. Finally, assume that processing costs are $k$ per unit processed and that the OCE is $\xi$ per unit extracted. For the sake of simplicity, $\xi$ is assumed to be constant with respect to the number of units extracted.

Consider now a situation where there is no restriction on the amount of material that can be processed. In this case, if it is worthwhile to extract one unit, it is worthwhile to extract the maximum allowable. Production is therefore zero or maximum – the so-called 'bang-bang' solution. The cutoff decision now involves choosing the quality of material to be processed. If the cutoff is zero, all extracted material is processed, resulting in a cash flow of $S_l + S_m + S_h - 3k - 3\xi$. Table 1 shows the cash flows for this and two other cutoff choices. Notice that the quality control decision does not depend on the opportunity cost, $\xi$, since the same cost is incurred in all scenarios. When prices are low, so that medium and low quality material cannot pay for their respective processing costs, the optimal cutoff is high. When prices are high, so that processing low-grade material produces profit, the optimal cutoff is low. This type of situation is known as a 'break even' cutoff strategy. The cutoff is chosen so that the revenues from material at the margin are equated with the marginal costs of processing. Any material that can pay its own processing costs is acceptable.

To investigate another possibility, consider a similar situation to the above, except that the maximum amount of material that can be processed is 1 unit per period.[5] For simplicity, assume that either one unit is processed or none at all. If more material is

[4] If deferring extraction of a marginal unit involves large frictions (e.g., costs of a complete shutdown, etc.) then the decision is somewhat different and largely weighs the tradeoffs between maintaining negative cash flow production and avoiding such a liability by paying a lumpsum shutdown cost. The profit from a marginal unit is insignificant relative to these tradeoffs.

[5] This is an example of 'input processing constraints.' The model investigated later also considers output constraints.

extracted than there is processing capacity, only the highest price material is selected and the remainder is waste; the cutoff is the ratio of waste material to extraction level. Examples of such combined production and cutoff policies are given in Table 2. Because of the processing stage constraint, the cost of processing is the same for all the policies considered, namely, $k$. What differentiates the policies this time is the amount of opportunity cost incurred. If one assumes that $S_h = 3S_l$ and $S_m = 2S_l$, a simple arithmetic calculation shows that the first policy will be preferred to the others in the table when $S_l < \frac{3}{2}\xi$. The second policy is preferred to the other two when $\frac{3}{2}\xi < S_l < 3\xi$, and the third prevails when $S_l > 3\xi$. A lower opportunity cost at the margin implies a higher standard of minimum quality. This is, again, a 'break-even' policy, although it is the marginal *opportunity cost* that must be exactly covered by marginal revenues.

The relationship between opportunity costs and cutoff grades outlined above is also discussed in Lane (1988), albeit in a more complicated setup, using a different notion of opportunity costs, *and assuming static prices*. What is not understood is how the price sensitivity of the opportunity costs can affect the cutoff grade decision and associated policy for 'wasting' resource. To see the relevance of this, note that in the second example the cutoff, extraction, and waste levels increase as the per-unit profits increase in relation to the OCE. In other words, if $\xi$ is insensitive to changes in price, then as prices increase eventually $S_l$ exceeds $\frac{3}{2}\xi$ or $3\xi$, leading to a cutoff policy that increases with price. This should be contrasted with the first example in which a higher price means more material breaks even and thus the cutoff is *decreased*. At first blush, the second example is somewhat unintuitive, since the naive policy is to waste *less* material when prices increase. While the examples are highly stylized, they attempt to convey the message that the presence of operating constraints and the price sensitivity of the OCE is crucial in determining the most profitable extraction/cutoff policy and, subsequently, the amount of wasted resource.

### 1.3 Determinants of the OCE

The preceding discussion clarifies that processing stage constraints can induce a relationship between the OCE and cutoff decision, thus understanding the latter hinges on understanding the comparative statics of the OCE. Instrumental in attaining this goal is the realization that the variable OCE is the marginal value of the project achieved by adding to it an additional unit of resource (i.e., the OCE is the sensitivity of the project to its size, $\frac{\partial J}{\partial Q}$).

Given this, three main determinants of the OCE come immediately to mind: the size of the reserve, the projected cost of extracting and processing the marginal unit added to the reserve, and the spot price of the resource. The roles of these factors can be understood by noting that the value of an additional unit (and therefore the OCE) is the discounted and risk-adjusted expected profit from that unit.

**Reserve Size** – The earliest time at which an additional unit can be processed is $T = Q/q$, where $Q$ is the quantity of reserves remaining and $q$ is the maximum capacity rate of resource extraction. Moreover, depending on the projected operational flexibility available, processing of the additional unit may be further deferred even when it is all that remains. Thus the OCE resembles a forward-start American style option with stochastic forward maturity bounded below by $T$. Increasing $Q$ is tantamount to increasing the maturity of the forward part of the option and can affect its value in several competing ways. First, at low prices and low convenience yield the option value benefits from an increase in the forward maturity, $T$, since the strike price dominates the intrinsic value of the option and suffers more than the commodity price from the additional discounting. At high prices the reverse is true since the intrinsic value of the option is dominated by the commodity price. Finally, as the forward maturity is increased, there is a higher cumulative likelihood that the project (along with the option) will be abandoned before $T$ and this reduces the value of the forward-start option. Summarizing, a larger reserve tends to exhibit a larger OCE when profit margins (i.e., moneyness of the option) is small and smaller OCE when profit margins are high.

**Costs** – Increasing costs of extraction and processing tend to reduce the intrinsic option value and increase the likelihood of abandonment before $T$ thus resulting in a lower OCE.

**Price Dynamics** – First, like any call option, the OCE is increasing in the underlying commodity price. The price sensitivity of the option, on the other hand, depends on the resource price process. For instance, if the underlying is mean-reverting, then for $T$ larger than the half-life of mean-reversion, the forward-start option value is insensitive to changes in the *current* spot price (unless the project is close to being abandoned). Thus in the constrained processing example above, the optimal cutoff increases with price. By contrast, it can be shown that this behavior is not necessarily exhibited when the price process is geometric Brownian-motion. The different optimal cutoff grades associated with the different price dynamics can have important implications for project design and public policy on resource management.

While the topic is not explored in this paper, it should be emphasized that even in the absence of quality control (and processing capacity constraints) considerations, opportunity cost plays an important role when extraction costs are convex. Consider, for example, a production model with convex costs, $k(q)$. At each point in time, the optimal production policy, $q^*$ maximizes:

$$Z(q) = qS_t - k(q) - q\frac{\partial J}{\partial Q} \tag{1}$$

where $S_t$ is the per-unit output price and $\frac{\partial J}{\partial Q}$ is the per-unit OCE. If $k(q)$ is differentiable, the solution is given by

$$q^\star = k'^{\,-1}(S_t - \frac{\partial J_O}{\partial Q}) \tag{2}$$

Because $k(q)$ is convex, $k'^{\,-1}$ is monotonically increasing. Thus the optimal extraction policy is an increasing function of $(S_t - \frac{\partial J}{\partial Q})$. Clearly, here too, the price-sensitivity of the opportunity costs plays a crucial role. The reaction of production to price depends on $1 - \frac{\partial^2 J}{\partial S \partial Q}$, affecting whether production increases or decreases in response to increasing prices.

The remainder of the paper develops further the insights discussed above. In particular, the interaction between resource production (i.e., extraction) and quality control (i.e., cutoff) is analyzed in more detail in the presence of input and output constraints. The realization that a proper evaluation of opportunity costs requires a contingent claims approach is used to develop a model which combines the real options framework used by Brennan and Schwartz (1985) with the discounted cash flow cutoff grade model proposed by Lane (1988). Brennan and Schwartz (1985) were the first to apply contingent claims analysis to resource extraction. Many other treatments have followed since, including Paddock, McDonald, and Siegel. (1985), Siegel and Smith (1988), Dixit and Pindyck (1994), Trigeorgis (1996), and Smith and McCardle (1999). While Lane (1988) was the first to realize the significance of opportunity costs in the determination of mining cutoff grades, the idea can be extended (as is done in this paper) to other resource industries. The rest of this paper is organized as follows. Section 2 develops the valuation model and derives the nature of the optimal production and cutoff policies in the presence of capacity constraints on the input and output processing stages. Section 3 applies the model to the forest and mining industries. While some intuition is obtained by qualitative analysis, the model must generally be solved numerically and project parameters are chosen to highlight some of the interesting features already discussed. Section 4 concludes.

## 2 THE MODEL

To model contingent extraction and cutoff policies, one has to quantify the effects of policy change on future cash flows and reserve levels. An extracted resource can contain several types of similar commodities in varying concentrations. For example, diamond ore may contain various qualities of diamonds present in different amounts; an acre of timber can contain different species of trees in varying density and quality. To simplify matters and focus the analysis, the quality notion is restricted to a single commodity present in extracted material.

Consider a project, as shown schematically in Figure 1, where a resource is removed at a rate $q_t \in [0, H]$ from reserves at time $t$. The variable cost of extraction is $k_w(S_t)$, where $S_t$ is the commodity price at time $t$; extraction costs may depend on the commodity price when, for example, the resource must be purchased (e.g., price-linked stumpage fees for timber). The extracted material has a distribution of quality measured by $c \in [\underline{c}, \bar{c}]$, which can be used for sorting. It is taken for granted that a unit of material with a higher value of $c$ fetches a higher price after processing. From here on, $c$ is referred to as the 'grade' of the material. Let the density for the distribution of possible grades be $f(c)$. In general, controlling for quality corresponds to processing only material that is within some 'acceptable' interval. A gold mine, for example, may choose to process only material which has more than 1 gram of gold per ton; another example is a sawmill where only logs with diameter between 12 and 36 inches are chosen for processing into lumber. At the risk of sacrificing some realism, it is assumed that the acceptable interval of quality at time $t$ is $[c_t, \bar{c}]$. In other words, higher quality material is never rejected for the purpose of processing.[6] *After* extracting $q_t$, it is assumed that material can be separated into components above and below the cutoff grade, $c_t$. The proportion of $q_t$ that is below the cutoff grade can be disposed as waste or sold to another facility (stockpiling for future processing is discussed later in the paper). Suppose that the variable cash flow from the rejected material, not including extraction costs, is given by $k_R(c_t, S_t)$. The amount of material rejected is $F(c_t)q_t$, where $F(c)$ is the cumulative distribution function for the grade distribution. Thus the rejected material at time $t$ earns a cash flow of $k_R(c_t, S_t)F(c_t)q_t$.

It is assumed that the maximum *input* capacity rate for processing extracted material

[6] This assumes that there is only a single control variable, namely, $c_t$. Some operations may require a dynamic quality 'ceiling' because processing facilities cannot handle the 'higher quality' material. For instance, a sawmill may be reluctant to process very large logs because they lead to a slower production line; the larger logs are therefore sold to another facility instead. This additional control variable is ignored in this paper.

is $G_I$, giving $\min(q_t P(c_t), G_I)$ as the amount of material processed at time $t$, where

$$P(c) \equiv 1 - F(c) = \int_c^{\bar{c}} f(x)\, dx \tag{3}$$

is the fraction of extracted material above the cutoff grade.[7] Assuming a cost per unit processed of $k_I(c)$, the added cost of processing acceptable material at time $t$ is $k_I(c_t)\min(q_t P(c_t), G_I)$.

The material chosen for processing must of course be converted to a final product. The conversion depends on the grade, the degree of loss during handling, and facility output capacity constraints. To capture this, it is assumed that one unit of material destined for processing yields $g(c)$ units of final output (the conversion factor between input and output depends on the cutoff grade). In mining, for example, the relationship between input and output is determined by the *mean grade above cutoff*; e.g., a higher value of $c$ increases the metal yield per unit input. In a sawmill, processing larger sized logs likewise increases output production per unit input. Assuming a maximum output capacity rate of $G_O$, the output of finished product at time $t$ is therefore given by $\min(q_t P(c_t)g(c_t), G_I g(c_t), G_O)$.

There are also costs associated with the amount of final output. Denote these as $k_O(c)$ per unit output. In mining, these costs are associated with smelting, refining and/or shipping. The total real variable costs (per unit time) incurred before realizing revenues can be collected to give:

$$\begin{aligned} \text{Total Real Variable Costs} = {} & k_O(c_t) \min(q_t P(c_t) g(c_t), G_I g(c_t), G_O) \\ & + k_I(c_t)\min(q_t P(c_t), G_I) + q_t k_w(S_t) \end{aligned} \tag{4}$$

The amount of revenue from final output is given by $S_t\min(q_t P(c_t)g(c_t), G_I g(c_t), G_O)$. Adding all of the costs and revenues at time $t$ gives the total cash flow at time $t$:

$$\begin{aligned} \text{Cash Flow}_t = {} & (S_t - k_O(c_t)) \min(q_t P(c_t) g(c_t), G_I g(c_t), G_O) - k_I(c_t)\min(q_t P(c_t), G_I) \\ & - q_t k_w(S_t) + q_t F(c_t) k_R(c_t, S_t) \end{aligned} \tag{5}$$

Assume that the interest rate is constant and that the commodity price is the only source of uncertainty evolving as an exponentiated Ornstein-Uhlenbeck (i.e., mean reverting)

[7] The optimal strategy ensures that $\min(q_t P(c_t), G_I) + q_t F(c) = q_t$.

Itô diffusion:

$$dS_t = S(\mu - \delta - \kappa \ln S_t)dt + S\sigma dB_t \tag{6}$$

where $dB_t$ is a Brownian motion increment, $\mu$ is the growth rate of the price, $\delta$ is a constant net convenience yield proxying for the consumption benefits of having the commodity on-hand, $\kappa$ is the rate of mean-reversion of the price to its long-run median, and $\sigma$ is the diffusion coefficient.[8] In the absence of mean-reversion (i.e., $\kappa = 0$), the process is standard geometric Brownian motion. To calculate the present value of cash flows under the optimal production policy, one must discount their *risk-adjusted* expected value using the risk-free rate. Given the assumptions over the price process, risk-adjustment is accomplished by substituting the risk-free rate, $r$, for the drift, $\mu$, in the stochastic differential equation, (6) (see, for instance, Harrison and Kreps (1979)).

**Proposition 1.** *The value of the project with reserve $Q_t$ while it is earning cash flows is*

$$\begin{aligned}\Big(\frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} + (\mu - \delta - \kappa \ln S_t)S\frac{\partial}{\partial S}\Big)J_C(S_t, Q) \\ + \max_{c_t, q_t}\Big\{ - q_t \frac{\partial J_O(S_t, Q)}{\partial Q} + Cash\ Flow_t\Big\} - MN_O = rJ_O(S_t, Q)\end{aligned} \tag{7}$$

*and the value of the project while production is temporarily shut down is*

$$\Big(\frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} + (\mu - \delta - \kappa \ln S_t)S\frac{\partial}{\partial S}\Big)J_C(S_t, Q) - MN_C = rJ_C(S_t, Q) \tag{8}$$

*where $MN_O$ is the rate of maintenance costs incurred while the project is operating and $MN_C < MN_O$ is the schedule of maintenance costs for the project while shut. A sufficient condition guaranteeing that the shut-down, re-start and abandonment policies are optimal is given by the six 'smooth pasting conditions:'*

$$\begin{aligned} J_C(S_C(Q), Q) - K_C &= J_O(S_C(Q), Q), & \frac{\partial J_C(S_C(Q), Q)}{\partial S} &= \frac{\partial J_O(S_C(Q), Q)}{\partial S} \\ J_C(S_O(Q), Q) &= J_O(S_O(Q), Q) - K_O, & \frac{\partial J_C(S_O(Q), Q)}{\partial S} &= \frac{\partial J_O(S_O(Q), Q)}{\partial S} \\ J_C(S_a(Q), Q) &= V_a, & \frac{\partial J_C(S_a(Q), Q)}{\partial S} &= 0 \end{aligned}$$

*where $S_C(Q), S_O(Q)$ and $S_a(Q)$ are, respectively, the optimal price boundaries for temporary*

[8] At the cost of increased computational burden, one can easily supplement this with additional random variables or use an alternative price-process specification.

*shut down at cost $K_C$, re-start at cost $K_O$, and abandonment with salvage value of $V_a$.*

*Proof.* The derivation of the partial differential equations follows from the absence of arbitrage (see, for example, Brennan and Schwartz (1985)). Sufficiency of the smooth pasting conditions follows from Brekke and Øksendal (1991). □

The partial differential (or diffusion) operator in Eqns. (7) and (8) corresponds to the growth in value of the project due to capital gains - some of which are stochastic while some due to compensation for the time value of capital that can be invested elsewhere (i.e., time value of money). The next set of terms on the left hand side of each Equation correspond to the cash flow generated by the project as well as the opportunity cost incurred during production. The maximization over $q$ is constrained to give solutions between 0 and the maximum production rate, $H$, while $c$ is constrained to be in $[\underline{c}, \bar{c}]$. Equating the two sides of the equation is tantamount to asserting that *after adjusting for risk* the increase in value due to cash flow and capital gains must equal the gains that can be attained by selling the project and investing the proceeds in a risk-free security - this is the insight of the Black-Scholes-Merton approach.

The smooth pasting conditions allow one to solve for the price boundaries; for instance, production is temporarily ceased if prices fall below $S_C(Q)$, and is re-started if prices increase past $S_O(Q)$. The costs of shut-down and re-start imply hysterises in the optimal policy. If prices fall below $S_a(Q)$ the project is abandoned in favor of the salvage value, $V_a$. Note that the optimal policy is contingent on the size of the reserve, $Q_t$. Note that a necessary (but not sufficient) condition for a non-trivial shutdown policy is that $MN_O > MN_C$. Otherwise, management is better off allowing production to drop to zero while paying the lower operating overhead costs. In fact, as long as $MN_O > MN_C + K_C r$ where $K_C$ is the shutdown cost, production never falls to zero while the project is in operation; shutting down is the preferred alternative.

To solve Eq. (7) one must maximize the cash flow in Eq. (5) with respect to $q_t$ and $c_t$, then substitute the results back into Eq. (7) and solve the (possibly non-linear) PDE. To this end, consider the generalized cash flow,

$$\begin{aligned} Z(q,c) = (S - k_O(c)) \min(qP(c)g(c), G_I g(c), G_O) - k_I(c)\min(qP(c), G_I) \\ - q(k_w(S) + \frac{\partial J_O}{\partial Q}) + qF(c)k_R(c,S) \end{aligned} \tag{9}$$

Consistent with the discussion in the Introduction, the generalized cash flow includes the OCE, $\frac{\partial J_O}{\partial Q}$, associated with extraction. The opportunity cost behaves as a premium on the variable extractions cost, $k_w(S_t)$.

**Proposition 2.** *The optimal production and cutoff policy maximizes $Z(q_t, c_t)$ and is one of:*

| *Policy #* | $q^\star$ | $c^\star$ *solves* | *Type* |
|---|---|---|---|
| *0* | *0* | *NA* | |
| *1* | $H$ | $Z'(H, c^\star) = 0$ | *Extraction Limiting* |
| *2* | $\frac{G_I}{P(c^\star)}$ | $Z'(\frac{G_I}{P(c^\star)}, c^\star) = 0$ | *Input Limiting* |
| *3* | $\frac{G_O}{g(c^\star)P(c^\star)}$ | $Z'(\frac{G_O}{g(c^\star)P(c^\star)}, c^\star) = 0$ | *Output Limiting* |
| *4* | $H$ | $P(c^\star) = \frac{G_I}{H}$ | *Balancing Extraction-Input* |
| *5* | $H$ | $g(c^\star)P(c^\star) = \frac{G_O}{H}$ | *Balancing Extraction-Output* |
| *6* | $\frac{G_I}{P(c^\star)}$ | $g(c^\star) = \frac{G_O}{G_I}$ | *Balancing Input-Output* |

*Proof.* First note that the maximization in Eq. (7) is tantamount to maximizing $Z(q_t, c_t)$. Now, since $Z(q_t, c_t)$ is quasi-linear in the extraction rate, $q_t$, maximization with respect to $q_t$ binds one of the capacity constraints. The optimal extraction rate, $q^\star(c_t)$, is one of 0, $\frac{G_I}{P(c_t)}$, $\frac{G_O}{g(c_t)P(c_t)}$ or $H$.[9] In the absence of input and output capacity constraints, $Z(q, c)$ is linear in $q$ and maximization produces the familiar 'bang-bang' solution: extraction is zero or $H$ (consistent with the first example from the Introduction). The second example in the Introduction corresponds to the case where $G_O$ is infinite, $k_w = 0 = k_R$, and $G_I < H$.

One way to proceed is to substitute each of the possible optimal extraction rates, $q_t^\star$, back into the generalized cash flow and maximize each resulting expression with respect to $c$. This gives a set of possible optimal policies. The chosen policy is the global maximum. Given $q^\star(c_t)$, the maximum of $Z(q^\star(c_t), c_t)$ with respect to $c_t$ corresponds either to an interior optimum or to binding another of the capacity constraints. An interior optimum for $c_t$ satisfies a first-order condition that sets marginal profits equal to marginal costs. The table lists the candidate policies. □

In general, there are six nontrivial candidates for the optimal policy: three policies with only one binding capacity constraint (and an interior maximum with respect to $c_t$), and three policies that bind two of the capacity constraints. The first three have been termed *limiting* policies by Lane (1988), while the last three policies are known as *balancing* policies. These arise due to the effects of the three capacity constraints on the cash flow. The constraints are over extraction, processing input and final output capacities. Three of the

[9]This result arises because the variable costs are assumed independent of $q_t$. Relaxing the assumption can result in an additional critical point (an interior optimum).

candidate policies dictate that production and cutoff should be adjusted so that *two* of the capacities are fully utilized (binding or balancing strategies). The other three policies state that one of the capacities be maximally utilized while production and cutoff be adjusted so that the minimum level of acceptable material 'breaks even' at the margin. Opportunity costs are relevant to policies 2 and 3 (the first-order condition for policy 1 is independent of opportunity costs). This implies that even if all variable cost parameters are constant, the maximization over both $c$ and $q$ potentially yields a non-linear partial differential equation – in contrast with the usual bang-bang solution that obtains in the absence of the cutoff control. Ideally, when designing a facility it is desirable to allow for interior solutions (Policies 1-3) to exist over much of the life of the project and probable support of the commodity price process. In other words, if the *expected* policy is of the binding type over most of the course of the project, then it is likely that the facility is poorly designed and perhaps should be appropriately expanded. On the other hand, if a balancing strategy is hardly ever expected to be optimal, then it is likely that management over-invested in capacity.

It is difficult to say much more about the comparative statics of the optimal policies without specializing further by making explicit assumptions over the different exogenous functions of $c$ and $S$ in the generalized cash flows. The next section does this by applying the model to several realistic settings. Before doing this, it is useful to make a more careful comparison with the closest model in the literature. The 'best-practice' model for the determination of optimal mining cutoff grades is that developed by Lane (1988). In Lane's model, based on a discounted cash flow analysis, random fluctuations in the spot price are not considered explicitly, thus his model lacks the ability to consider time-consistent price-contingent strategies. To explore the relationship between Lane's work and this paper, it is instructive to consider a variant on his derivation. The value of the project at time $t$ is a function of the remaining reserves, $Q$, and the spot price, $S_t$. Let the mine value be $V(S_t, Q)$ and assume that a *constant* unit of reserve, $q_t dt$, is depleted over some variable time period, $dt$. The value of the mine is therefore

$$V(S_t, Q) = \max_{c_t, q_t} \left( D_t q_t dt - MN_O dt + V(S_{t+dt}, Q - q_t dt) e^{-\Gamma dt} \right) \tag{10}$$

where $\Gamma$ is the firm's discount rate, $D_t = \text{Cash Flow}(q_t, c_t)/q_t$ is the revenue from the unit extracted, and $MN_O$ is the rate for fixed costs. Keeping in mind that $q_t dt$ is held constant, this can be expanded in a Taylor series and rearranged to give

$$0 = q_t dt \max_{c_t, q_t} \left( D_t + \frac{1}{q_t} \left[ \frac{dV}{dS} \frac{dS}{dt} - MN_O - \Gamma V \right] - \frac{\partial V}{\partial Q} \right) \tag{11}$$

Noting that $\frac{\partial V}{\partial Q}$ does not explicitly depend on $q_t$ and $c_t$, this results in Lane's equation:

$$\frac{\partial V}{\partial Q} = \max_{c_t, q_t} \left( D_t - \frac{1}{q_t} F \right) \tag{12}$$

where the maximization is subject to the capacity constraints and $F \equiv -\frac{dV}{dS}\frac{dS}{dt} + MN_O + \Gamma V$. Lane calls $F$ an opportunity cost. A re-examination of the derivation in a no-arbitrage setting and use of Itô's Lemma show that $F = -(A - r_t)V + MN_O$, where $A$ is the diffusion operator for price movements and $r$ is the riskless interest rate. Thus, it is appropriate to identify $F$ with capital gains forgone. Although not immediately obvious, it is not hard to check that Eqs. (7) and (12) give equivalent optimal policies and valuations.[10] Since Lane ignores random price components and uses a constant discount factor instead, to reproduce his results one simply needs to set the diffusion parameter to zero and write

$$\frac{\partial V}{\partial S}\frac{dS_t}{dt} - \Gamma V - MN_O = -\max_{c_t, q_t} \left\{ -q_t \frac{\partial V}{\partial Q} + \text{Cash Flow}_t \right\} \tag{13}$$

One problem with Lane's approach of ignoring price fluctuations is that the option aspect of the opportunity cost, $\frac{\partial J_O}{\partial Q}$, is not transparent or even useful in the analysis. There has been an attempt to place Lane's theory in the context of contingent claims analysis by Mardones (1991, 1993). He develops a model where the cutoff decision is calculated using Lane's discounted cash flow approach and the resulting 'optimal' cutoff level is used in a contingent claims valuation of the mine. This treatment is not fully consistent because the cutoff grade is not optimized along with the objective function. Mardones does not discuss the call option aspect of opportunity costs.

# 3 APPLICATIONS

## 3.1 Forestry: A Simple Sawmill Model

The harvesting of timber from tree stands and consequent conversion into marketable forest products is a complicated process involving numerous stages where economic decisions are made. What follows is a vulgarized model of such an operation where a firm can log on a government-owned timber tract and supply its sawmill with logs for conversion into lumber. In particular, the model ignores the renewable nature of the resource. Given this limitation, the model only strictly applies to projects whose expected duration is much shorter than

[10]To show this, solve for $F$ in Eq. (12) and substitute the results into the first-order condition. This should produce the first-order conditions for Eq. (7).

the time required for new growth to be ready for harvest. On the other hand, much of the intuition and the nature of the optimal policies is robust.[11] For references on the use of option pricing in the forestry sector, see Plantinga (1998).

Consider a firm that can harvest up to some capacity rate, $H$, paying stumpage fees for every unit volume extracted. Stumpage fees are paid by logging companies to governments for extracting forest resources from public land. The fees may be adjusted by local governments to reflect prevailing market and political conditions. The input capacity constraint is $G_I$. There are no output capacity constraints on the sawmill operation, and the amount of lumber at the output end is assumed to be dependent on the average diameter of logs at the input side. It is assumed that a higher average diameter at input results in higher output yield.[12] A given log is chosen for milling on the basis of its diameter. The quality measure, $c$, is therefore the diameter of harvested material. Timber considered too small in diameter is sold at a price that covers the stumpage fees and logging/transportation costs. It is assumed that markets are sufficiently competitive so that profits made from rejected material are marginal.

Let $f(c)$ be the distribution of log diameters for the population of timber units in the stand (measured in volume). The costs of logging are $k_w^o$ per cubic meter and the stumpage fees are assumed proportional to the price of lumber. In other words, $k_w(S) = k_w^o + \chi_{Stump} S$, where $\chi_{Stump}$ is a constant. Since rejected material covers its own costs, it does not contribute to the cash flow.[13] This implies that $k_R(S) = k_w(S)$. If the cost of milling timber of *uniform* diameter, $c$, is $\kappa(c)$, then the input variable cost, $k_I(c_t)$, is an average cost of milling for the distribution of logs chosen for processing: $k_I(c) = \frac{1}{P(c)} \int_c^{\bar{c}} \kappa(x) f(x)\ dx$. Likewise, if the output yield from material of uniform diameter, $c$, is $\gamma(c)$, then the total output for the timber above the cutoff is an average, $g(c) = \frac{1}{P(c)} \int_c^{\bar{c}} \gamma(x) f(x)\ dx$. It is usually the case that the cost of milling, $\kappa(c)$, is decreasing in diameter, $c$, for the same volume of timber; further, as mentioned above, $\gamma(c)$ is monotonically increasing. Finally, it is assumed that the variable cost at output is constant, $k_O(c) = k_O$. Using previous notation, the model can

[11] Eq. (7) assumes that the project under consideration extracts a non-renewable resource. If the resource is renewable at some costs, as in forestry, then the maximization on the right-hand side of the equation has the additional terms: $\max\limits_{c,q,\tau} \left\{ (\tau - q) \frac{\partial J_O}{\partial Q} + \text{Cash Flow}_t - k_\tau \tau \right\}$, where $\tau$ is the rate of renewal and $k_\tau$ is the variable cost of renewal. This addition may reduce the opportunity costs by affecting the size of resource reserves over time. It is easy to check, however, that the form of the solutions for the optimal strategies governing $q_t$ and $c_t$ is not changed.

[12] This is not entirely a realistic assumption. A production line may slow down if supplied with very large logs. Accounting for this requires an additional cutoff control for larger diameter material, and making the input capacity a decreasing function of large diameter logs. It is far form clear that such a complication can add much insight to the analysis. It is therefore assumed that the sawmill can handle any log size.

[13] The quality control problem is most acute when it is *not* profitable for the firm to act as a timber supplier. To emphasize this aspect, it is assumed that the firm cannot be financed solely by its harvesting technology. Given that, there is little loss of generality in setting the cash flow from rejected harvested material to zero.

be summarized by

$$k_w(S) = k_w^o + \chi_{Stump}\, S \quad \text{where } \chi_{Stump} \text{ is constant} \tag{14}$$

$$k_R(S,c) = k_w^o + \chi_{Stump}\, S \tag{15}$$

$$k_I(c) = \frac{1}{P(c)} \int_c^{\bar{c}} \kappa(x) f(x)\, dx \qquad \text{with } \kappa'(c) < 0 \tag{16}$$

$$g(c) = \frac{1}{P(c)} \int_c^{\bar{c}} \gamma(x) f(x)\, dx \qquad \text{with } \gamma'c) > 0 \tag{17}$$

where $S$ is the prevailing price for lumber.

It is straight forward to show that the generalized cash flow, $Z(q,c)$, takes the following form:

$$\begin{aligned} Z(q,c) = {} & ((S - k_O)g(c) - k_I(c)) \min(qP(c), G_I) \\ & - (k_w^o + \chi_{Stump}\, S)qP(c) - q\frac{\partial J_O}{\partial Q} \end{aligned} \tag{18}$$

The optimal policies for harvesting and cutoff at time $t$ are derived by maximizing $Z(q_t, c_t)$ subject to $0 \leq q_t \leq H$ and $\underline{c} \leq c \leq \bar{c}$. The analysis in section 2 applies, but the set of possible optimal policies is reduced to three – two limiting and one balancing – since there are only two capacity constraints.

**Lemma 1.** *The candidate optimal policies for the sawmill from Proposition 2 are calculated to be,*

| *Policy #* | $q^\star$ | $c^\star$ *solves* | *Necessary conditions* |
|---|---|---|---|
| *0* | *0* | *NA* | |
| *1* | $H$ | $\gamma(c^\star)(S_t - k_O) - (\kappa(c^\star) + k_w^o + \chi_{Stump}\, S_t) = 0$ | $\gamma'(c^\star)(S_t - k_O) - \kappa'(c^\star) > 0$ *and* $P(c^\star) \leq \frac{G_I}{H}$ |
| *2* | $\frac{G_I}{P(c^\star)}$ | $(S_t - k_O)(g(c^\star) - \gamma(c^\star)) +$ $(\kappa(c^\star) - k_I(c^\star)) - \frac{1}{P(c^\star)}\frac{\partial J_O}{\partial Q} = 0$ | $\gamma'(c^\star)(S_t - k_O) - \kappa'(c^\star) > 0$ *and* $P(c^\star) \geq \frac{G_I}{H}$ |
| *3* | $H$ | $P(c^\star) = \frac{G_I}{H}$ | $\frac{G_I}{H} < 1$ |

*Proof.* $c^\star$ is the solution to the first order condition in Proposition 2 for policies 1 and 2. The necessary conditions are second order conditions required for $c^\star$ to be a local maximum. □

Policy 1 corresponds to a strategy where the logging rate is maximum but there is slack in the processing capacity. Thus the situation is analogous to the first example of the Introduction (Table 1). Here too total revenues from marginal quality timber exactly cover the costs (a break-even cutoff policy). Policy 3 is the balanced strategy that binds the production and input capacity constraints (i.e., both are fully utilized). The second and most interesting policy binds the processing but not the extraction capacity. The first term in the first-order condition is the marginal revenue per unit input at the lowest acceptable grade; the second term is the marginal input costs for the said unit. The final term is the opportunity cost incurred per unit input. The policy therefore equates marginal revenues less input costs to the opportunity cost, $\frac{\partial J_O}{\partial Q}$. This is also a break-even policy analogous to the second example discussed in the Introduction (see Table 2).

**Lemma 2.** *The comparative statics of policies 1 and 2 are given by,*

$$Sign\left[\frac{\partial c^\star}{\partial S}\right] = Sign\left[\chi_{Stump} - \gamma(c^\star)\right] \quad \textit{policy 1} \tag{19}$$

$$Sign\left[\frac{\partial c^\star}{\partial S}\right] = Sign\left[g(c^\star) - \gamma(c^\star) - \frac{1}{P(c_t)}\frac{\partial^2 J_O}{\partial S \partial Q}\right] \quad \textit{policy 2} \tag{20}$$

*Proof.* Follows immediately from differentiating $c^\star$. ☐

Policy 1 concerns a scenario in which the harvesting capacity is at a maximum, but there is slack in the processing capacity. In some sense, the mill is 'starving' for material. The adjustment of the cutoff in reaction to a price increase depends on whether the output yield of marginal material, $\gamma(c^\star)$, is greater than the government's share in profits, $\chi_{Stump}$. Thus the government can set the stumpage fees, $\chi_{Stump} S$, to encourage an increase or decrease of $c^\star$ in response to lumber price movements; this also impacts overall production output (and indirectly, employment and the corporate tax bill). If $\chi_{Stump} > \gamma(c^\star)$, the firm does not mill lower quality timber in times of high lumber prices; instead, production from lower grade material is outsourced. If $\chi_{Stump} < \gamma(c^\star)$, then the opposite happens; production capacity is increasingly utilized during times of high prices. Since tracts of different quality distribution exhibit different optimal cutoffs, it may be sensible to set the price-contingent stumpage fee to reflect the overall quality of timber.

Policy 2, which fully exploits the processing capacity, reacts to price movements in a way that crucially depends on the opportunity cost. Since $g(c^\star) - \gamma(c^\star) > 0$, if the OCE, $\frac{\partial J_O}{\partial Q}$, is insensitive to price, then the cutoff is increasing with price. Since the processing stage is binding, increasing the cutoff requires an increase in the harvesting of timber, preferring to supply the mill with higher quality wood than to prolong the life of the resource reserve. In some instances this may lead to increasing underutilization of lumber quality timber (if, say, it is only economical to sell rejected material to a pulp mill). The argument here, as in the stylized introductory example, hinges on the price sensitivity of the opportunity cost. In particular, if lumber price quickly reverts to some long-run mean, then the OCE (i.e., the value of adding a unit of timber) of a moderate size reserve will be relatively price insensitive (since the unit will not enter production until the reserve is depleted).

To qualitatively assess the effect of price dynamics on the second strategy when prices are not mean-reverting, one can approximate the OCE as the forward-start American style option discussed in the Introduction and consider geometric Brownian-motion price dynamics (i.e., $\kappa = 0$ in Eq. (6)). Let $T$ be the estimated time to reserve depletion and notice that when $S_T$ is high one can further assume that approximately all timber is processed to lumber yielding a profit of $((S_T - k_O)g(\underline{c}) - k_I(\underline{c})) - (k_w^o + \chi_{Stump} S_T)$. Discounting this to the present gives a high price approximation for the OCE of

$$\frac{\partial J_O}{\partial Q} \approx e^{-\delta T}(g(\underline{c}) - \chi_{Stump})S - e^{-rT}\Big(k_O g(\underline{c}) + k_I(\underline{c})) + k_w^o\Big)$$

Where $S$ is the current spot price, $\delta$ is the net convenience yield of the underlying, and $r$ is the risk-free rate. Using this and the first order condition for $c^\star$ in Policy 2, one can manipulate the comparative static equation, (20), to yield:

$$\text{Sign}\left[\frac{\partial c^\star}{\partial S}\right] \approx \text{Sign}\left[k_O(g(c^\star) - \gamma(c^\star)) + (k_I(c^\star) - \kappa(c^\star)) - \frac{e^{-rT}}{P(c^\star)}\Big(k_O g(\underline{c}) + k_I(\underline{c})) + k_w^o\Big)\right]$$

Given the assumptions on $g(\cdot), k_I(\cdot)$ and $\kappa(\cdot)$, if $T$ (i.e., reserve life) is not too large the expression on the right hand side is positive. Thus in sharp contrast with the mean-reverting case, a non-stationary price process induces a conservative harvesting policy on the part of small (or possibly moderate) scale operations during times of high lumber price.

An important question is when might Policy 2 be prevalent. While a precise answer depends on many operational and economic variables, the most important determinant is the relative size of harvesting versus milling capacities. Clearly, if there are no harvesting (resp. milling) constraints Policy 2 (resp. 1) dominates. It does not seem unrealistic to assume that milling constraints are relatively more important since that component is often more capital intensive.

A casual analysis of lumber futures prices suggests that returns on lumber are better described by a random walk with drift than a mean-reverting process.[14] This can have

[14]The author analyzed futures contracts on random-length lumber from the Chicago Mercantile Exchange (January 1998 to October 1998). A fit to an Ornstein-Uhlenbeck process gives a mean reversion parameter that is essentially zero. A fit to a random walk with drift gives an annual diffusion parameter of 34.3% and a proportional annual

significant policy implications at the firm level. Specifically, there is an economic argument for a more conservative harvesting approach whenever processing capacity is dwarfed by harvesting capacity. In such a case, higher NPV is implied by a policy that prolongs the life of the resource rather than one that depletes it opportunistically during times of increasing prices.

The analysis above disregards taxes. Such frictions are important to add in realistic applications but they do not change the essence of the points made. While opportunity costs are not real expenses and they cannot be deducted from the tax bill, a higher flat tax will result in a roughly proportional decrease in the firm value and thus the opportunity costs. The optimal cutoff policy should therefore remain largely unchanged. On the other hand, if prices are mean reverting, charging a higher profit tax when prices are well above their long-run mean has a relatively smaller impact on the OCE than the actual cash flows. Thus the OCE in comparison with the profits becomes effectively magnified by a factor of $\frac{1}{1-\tau_m}$, where $\tau_m$ is the additional high-price profit tax. When policy 2 is in effect, the general impact is to reduce the amount of waste by reducing the cutoff from where it might be in the absence of contingent taxes.

To illustrate some of these points, consider a numerical example of a sawmill with operating parameters as given in Table 3. The numerical solution is not conventional because the optimal cutoff, and thus the cash flows, are nonlinear functions of the opportunity cost. Moreover, smooth pasting conditions, as in Brennan and Schwartz (1985) and Brekke and Øksendal (1991), must be imposed for the optimal stopping boundaries corresponding to shutting, restarting or abandoning the mill. Solutions are generated by a convergent iterative scheme, that obeys a variational inequality associated with the smooth pasting conditions.[15]

For varying levels of resource reserves, the optimal harvesting and log (i.e., timber and not 'logarithm') diameter policy are given in Figure 2, as is the opportunity cost. Since the milling capacity constraint is more acute, the optimal policies shift between 2 and 3. As suggested by the qualitative analysis, non-stationarity of the price process leads to a decrease of the production/cutoff policies as prices increase. Note that the OCE resembles a call option on a dividend-paying asset following a log-normal price process. As discussed in the Introduction, at low prices the OCE *increases* with reserve size due to the higher relative importance of discounting the 'strike price'. The opportunity cost only begins to decrease with reserve size at very high prices (not seen in the graph) because the convenience yield used in the calculation is fairly small.

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convenience yield of 1.5% (assuming an annual risk-free rate of 6%). Given the short observation period, however, it may be presumptuous to infer that lumber prices necessarily follow a geometric Brownian motion process.

[15] Interested readers may contact the author for more details on the numerical algorithm.

At larger reserve levels, the harvesting and log diameter cutoff policies change nearly discontinuously between policy 3, which harvests more intensively, and policy 2, which is more conservative. For instance when the reserve has 5 million cubic meters of timber remaining, a change of less than 10% in lumber price can lead to a 20% increase in harvesting, most of which is subsequently not milled. Although this may seem somewhat technical, there may be important economic and policy implications. In practice, the interior and boundary optima may be very close in cash flow value. The boundary optimum may be, however, less ecologically efficient. It may therefore be possible, through judicious use of taxes or tax breaks, to distort the policy while keeping firm revenues almost unchanged.

### 3.2 Mining: An Open Pit Copper Mine

For an operating mine, there are typically three stages of production: (i) the *mining* stage, where units of various grade are extracted up to some capacity. (ii) the *treatment* stage, where ore is milled and concentrated, again up to some capacity constraint; and (iii) the *marketing* stage, where the concentrate is smelted and/or refined to a final product which is shipped and sold; the last stage is also subject to capacity constraints. All three stages have associated costs. After the first stage, it is possible to direct mined units either to a waste dump (or stockpile) or to the treatment mill. The grade above which extracted material is milled is called the *milling cutoff grade*. It is thus clear that management has at least two dynamic controls to optimize: mining level and milling cutoff grade.

There is another type of cutoff grade, called the pit cutoff grade, which determines the optimal limits of the pit walls, and thus the ultimate size of the reserve. This also defines the grade distribution of material that will be removed during the life of the mine. In principle, the mine plan can change in reaction to economic conditions. However, because of the scale and costs entailed, design changes are akin to strategic expansion decisions. Recognizing this, it can further be argued that the predominant effect of pit cutoff grade adjustments is to reduce the opportunity cost of extraction at high prices; this is because high prices might signal a reserve size expansion, which can consequently reduce the opportunity cost. The qualitative behavior of the opportunity cost should, however, remain unchanged.

Although an important planning problem, a full pit design model is beyond the scope of this paper. A useful extension of the model can, however, incorporate deposit expansion into the set of management flexibility. The example to be discussed below assumes that the pit size is fixed and that it contains a single ore zone that can be described by an appropriate grade tonnage distribution. Although it is common for deposits to consist of multiple zones, single zone analysis is standard (see Lane (1988)). In a large-scale operation, material is extracted in basic Selected Mining Units (SMU's) – say, truckloads – of 'known' grade that can be individually diverted to the mill or the waste dump. Although there is flexibility in how the SMU's are scheduled for extraction, this is usually severely limited by the geometry of the pit and the pit plan. It is assumed that during the relevant time scale for planning (i.e., months) enough SMU's are extracted to fully sample the grade distribution in the pit. It is therefore sensible to speak of a milling cutoff grade applied to the extracted units.[16]. From here on all references to the cutoff grade should be interpreted as referring to the milling cutoff grade.

To specifically apply the results from Section 2, assume that the variable cost parameters are constant and rejected SMU's are disposed without the option to stockpile or provide revenues by some other means. These assumptions are discussed later in this subsection. As before, $f(x)$ denotes the statistical grade distribution of SMU's in the reserve. The mean of this distribution is denoted by $\mu$. The conversion factor between input and output, $g(c)$, is simply the mineral content of processed material times a recovery factor,

$$g(c) = \frac{R}{P(c)} \int_c^{\bar{c}} x f(x) dx \tag{21}$$

That is, $g(c)$ is a recovery factor, $R$, times the mean grade above the cutoff. Given the above assumptions, it is possible to show that the maximization in Eq. (7) is equivalent to a maximization of the generalized cash flow,

$$Z(q, c) = (S_t - k_O)\ qP(c)g(c) - k_I qP(c) - q(k_w + \frac{\partial J_O}{\partial Q}) \tag{22}$$

subject to the constraints

$$\begin{aligned} 0 \le q &\le H \\ qP(c) &\le G_I \\ qP(c)g(c) &\le G_O \end{aligned}$$

[16]One can think of the individual SMU's as samples from the grade tonnage distribution. Any SMU extracted which has a grade above the milling cutoff is processed. In the time scales of interest here, sufficient numbers of SMU's move through the mill to make averaging over the grade tonnage distribution, $f(x)$, sensible. Further, it is assumed that $f(x)$ doesn't change through time – i.e., $f(x)$ is spatially stationary.

**Lemma 3.** *The candidate optimal policies for the mine from Proposition 2 are calculated to be,*

| *Policy #* | $q^\star$ | $c^\star$ *solves* | *Necessary conditions* |
|---|---|---|---|
| *0* | *0* | *NA* | |
| *1* | $H$ | $c^\star = \frac{k_I}{R(S_t - k_O)}$ | $S_t - k_O > 0,\ P(c^\star) \leq \frac{G_I}{H}$ *and* $P(c^\star)g(c^\star) \leq \frac{G_O}{H}$ |
| *2* | $\frac{G_I}{P(c^\star)}$ | $P(c^\star)(g(c^\star) - Rc^\star)(S_t - k_O) = k_w + \frac{\partial J_O}{\partial Q}$ | $S_t - k_O > 0,\ P(c^\star) \geq \frac{G_I}{H}$ *and* $g(c^\star) \leq \frac{G_O}{G_I}$ |
| *3* | $\frac{G_O}{\bar{g}(c^\star)P(c^\star)}$ | $P(c^\star)(g(c^\star) - c^\star)k_I = (k_w + \frac{\partial J_O}{\partial Q})c^\star$ | $S_t - k_O > 0,\ g(c^\star)P(c^\star) \geq \frac{G_O}{H}$ *and* $g(c^\star) \geq \frac{G_O}{G_I}$ |
| *4* | $H$ | $P(c^\star) = \frac{G_I}{H}$ | $\frac{G_I}{H} \leq 1$ *and* $g(c^\star)P(c^\star) \leq \frac{G_O}{H}$ |
| *5* | $H$ | $g(c^\star)P(c^\star) = \frac{G_O}{H}$ | $\frac{G_O}{H} \leq R\mu$ *and* $P(c^\star) \leq \frac{G_I}{H}$ |
| *6* | $\frac{G_I}{P(c^\star)}$ | $g(c^\star) = \frac{G_O}{G_I}$ | $\frac{G_O}{G_I} \geq R\mu$ *and* $P(c^\star) \geq \frac{G_I}{H}$ |

*Proof.* $c^\star$ is the solution to the first order condition in Proposition 2 for policies 1-3. The necessary conditions are second order conditions required for $c^\star$ to be a local maximum. □

The last three policies are balancing strategies where two of the three capacity constraints are binding. Policy 4 saturates the mining and input treatment capacities. Policy 5 corresponds to choosing a cutoff that maximally utilizes mining and marketing capacities, while policy 6 maximally utilizes treatment and marketing capacities. The first three policies are break-even or capacity-limiting policies where one capacity constraint is binding and marginal costs are equated to marginal revenues. In policy 1, any material that earns enough to cover input and output costs is acceptable. Policy 2 demands that at the cutoff, the marginal revenues per unit *input* be equated to the marginal costs incurred (both Policy 1 and 2 are familiar from the sawmill example). A new strategy, Policy 3, sets the marginal revenues per unit *output* to equal their marginal variable costs. Both policy 2 and 3 involve the opportunity cost, $\frac{\partial J_O}{\partial Q}$.

**Lemma 4.** *The comparative statics of policies 1-3 are given by,*

$$\frac{\partial c^\star}{\partial S} = -\frac{k_I}{R(S-k_O)^2} \qquad \textit{policy 1} \tag{23}$$

$$\frac{\partial c^\star}{\partial S} = \frac{P(c^\star)(g(c^\star)-Rc^\star) - \frac{\partial^2 J_O}{\partial S \partial_Q}}{P(c^\star)R(S-k_O)} \qquad \textit{policy 2} \tag{24}$$

$$\frac{\partial c^\star}{\partial S} = -\frac{c^\star \frac{\partial^2 J_O}{\partial S \partial_Q}}{k_w + \frac{\partial J_O}{\partial Q} + P(c^\star)k_I} \qquad \textit{policy 3} \tag{25}$$

*Proof.* Follows immediately from differentiating $c^\star$. □

Policies 1 and 3 are conservative in that the cutoff decreases with increasing prices. The second, treatment capacity limiting policy, is sensitive to the curvature of the opportunity cost. Here, as in the sawmill example and the second example from the Introduction, a price-insensitive OCE (associated, for instance, with mean-reverting price dynamics) implies a policy that *increases* the cutoff at high prices; the idea is to profit from the high prices, which will not last, while sacrificing some of the life of the reserve. As with the sawmill example, it is possible to show that geometric Brownian-motion price dynamics lead to a high-price Policy 2 cutoff exhibiting,

$$\text{Sign}\left[\frac{\partial c^\star}{\partial S}\right] = \text{Sign}\left[(g(c^\star) - Rc^\star)k_O + k_w - \frac{e^{-rT}}{P(c^\star)}\Big(g(\underline{c})k_O + k_I + k_w\Big)\right]$$

Thus at small (and even moderate) reserve sizes the latter equation is positive, promoting a conservative attitude towards waste; positive price shocks are permanent, rendering poor grade material profitable if not wasted. Once again, the optimal approaches to production and quality control dramatically depend on the underlying price dynamics as mediated through the opportunity cost of extraction.

The assumption of constant variable costs is fairly standard, even though it may be violated in practice. Cutoff-dependent costs can be handled as in the sawmill example; the main impact is to add marginal savings to the marginal revenues in the first-order conditions. As discussed in Section 2, variable costs that increase with mining level will potentially introduce interior optima (e.g., solutions where none of the capacity constraints are binding). Although tedious, it is possible to show that adding quadratic components, $q^2$ and $q^2P^2(c)$, to the costs does not qualitatively change the results even when an interior solution for $q^\star$ obtains. In general, the comparative statics are such that the sign of $\frac{\partial c}{\partial S}$ in Policies 2 and 3 depends on the behavior of the opportunity cost in a similar manner to that derived under the simpler assumptions.

Another feature ignored in the analysis is a stockpiling option. Essentially, one expects that the future value of a unit designated for the stockpile will reduce the opportunity cost incurred in extraction. In the presence of a stockpiling option, the opportunity cost only accounts for the loss of material that has been processed (i.e., a fraction of the unit extracted). For example, assuming no stockpiling, at very high prices (when call options can be approximated by discounted forwards) the opportunity cost per unit mined is $\approx \mu \exp(-rT) F_T$ where $r$ is a constant riskless rate, $T$ is the amount of life left in the reserve, $F_T$ is the forward price of one unit of the commodity, and $\mu$ is the mean grade of the mined unit. If there is an option to stockpile, then one only loses the opportunity to exploit the processed fraction. Thus the opportunity cost becomes $\approx P(c)g(c)\exp(-rT)F_T$. Note that this does *not* alter the behavior of the opportunity cost with respect to price or much of the comparative statics. The analysis should thus be fairly robust to the inclusion of complications such as level dependent variable costs and stockpiling.

Consider next an open pit Copper mine project with the parameters given in Table 4. Figures 3a and 3b plot the value of the mine when operating and while temporarily shut. Shown are the curves for reserve levels varying from 10 to 50 million tons of raw material. If the mine is *currently* operating, then the optimal policy is given by the curves in Figure 3a. If the mine is currently shut, then it opens or abandons according to the curves in Figure 3b. When spot prices warrant, the mine opens or remains open (solid lines); otherwise, it is temporarily shut down or remains shut (dashed lines). Startups always happen at higher prices than shutdowns because of the costs involved (hysteresis). Site abandonment occurs when the value of the project approaches the abandonment costs. Note that the approach to abandonment is steep; the 'elbow' is an artifact of the Ornstein-Uhlenbeck process which features very strong mean reversion at low prices (the strength of the mean reversion diverges as prices fall to zero). Mine operation is conducted by following a continuous series of such curves as the mine is depleted and spot prices change. Thus a policy exists for every possible price path.

More interestingly, Figures 4a and 4b show the optimal extraction and cutoff policies for the open mine. Because the marketing stage represents the most severe constraint, optimal policies tend to saturate output capacity. At low prices the dominant strategy is of the break-even type (policy 3) whether reserves are high or low. As spot prices increase, a shift in policy occurs when little reserves are left (i.e., opportunity costs are high). Large reserve projects, however, remain on a break-even policy. The low reserve projects are more sensitive to price movements. This is demonstrated by the more dramatic reaction of the optimal policies to changes in spot prices. The figures show that at high prices the cutoff decreases as the reserve is depleted. Because the output capacity is kept at a maximum, decreases in the cutoff result in a lower extraction rate.

Figure 4c plots the opportunity cost, $\frac{\partial J}{\partial Q}$. Note that the vertical axis corresponds to \$ per ton. There are roughly 11 lbs. of copper per ton extracted when the cutoff is zero and less for higher cutoff. For comparison, Figure 5 shows the term structure for European options on the commodity. Although not exact copies, there is a remarkable similarity, verifying that opportunity costs are analogous to forgone options that expire when the reserve is depleted. More importantly, it is clear from the plots that the OCE is insensitive to the price when the price is high. This is not true when project is close to being abandoned. One might therefore expect somewhat different cutoff grade behavior in this region than at high prices.

Much was said earlier about the possibility that minimum acceptable quality can actually *increase* as prices increase. To see this, consider the project described in Table 4, only now the choke in production occurs at the input end: $G_I = 2.0$ Million Tons/Year, and $G_O = 60$ Million Lbs. of Copper/Year. The optimal extraction and cutoff policies for the open mine are shown in Figures 6a and 6b. Notice that, as argued, both the extraction rate and the cutoff increase as prices rise. To see why this seemingly wasteful policy is sensible, consider that the opportunity cost (shown in Figure 6c) decreases at the margin with increasing prices. In other words, the opportunity cost is nearly insensitive in price at moderate and high price levels for the larger reserve curves. Recalling the argument in the introduction and Eq. (23), it is clear that the benefits of using more of the slack in the final output capacity outweigh the loss of more reserves. Moreover, because the marginal opportunity cost decreases with larger reserve sizes, larger projects more readily increase the cutoff at lower prices, as seen in the figures. The increase in cutoff and production is saturated when both the input and extraction capacities are fully utilized; this corresponds to the Treatment-Mining capacity balancing policy 4.

From this analysis, it should be that the price dynamics play a crucial role vis á vis the OCE in understanding the interaction between production and quality control. The mean-reverting price process considered thus far leads to sublinear growth in the value of a forward-start American option and thus in the opportunity cost. A log-normal model, on the other hand, leads to price-sensitive option values at all prices and throughout much of the term structure. This can be expected to radically change the decision to increase minimum quality standards with increasing prices. To investigate this in an identical setting, the project with $G_I = 2.0$ Million Tons/Year, and $G_O = 60$ Million Lbs. of Copper/Year is re-visited but in the context of a log-normal price process with constant proportional convenience yield. The risk-adjusted price evolves according to

$$dS = S(r - \delta)dt + S\sigma dZ \tag{26}$$

with $r = 0.05$, $\delta = 0.02$ and $\sigma = 0.233$ (annualized). The optimal cutoff and extraction policies are shown in Figures 7a and 7b. Notice that this time, the cutoff and extraction controls are monotonically decreasing in spot price even though the strategies used are exclusively of the type associated with policy 2. The reason for the reversal is explained by the opportunity cost function, which is now far more price sensitive to spot prices. The opportunity cost is shown in Figure 7c; once again, the interpretation of the opportunity cost as a call option forgone is useful.

It is worth noting that when Lane's model is used to analyze the last example (Figure 7), there are two general patterns. The mine value is lower, particularly at lower prices. This is the well-known option component added in the contingent claims approach. The cutoff and extraction policies have the same shape and trends, but are generally *higher*. In other words, with a log-normal price process, the contingent claims valuation gives a more conservative strategy for resource exploitation. This is due to a higher potential for future profitability ascribed to 'waste' by the real options analysis. The difference can be as large as 10% at intermediate prices (at sufficiently high prices there is not much difference). Such differences can make a serious impact on the duration of the operation. Naturally, in the presence of a mean reverting price process, Lane's model gives even more dramatically different recommendations.

# 4 CONCLUSIONS

The most significant contributions of this paper are: (i) the identification of the opportunity cost of extraction as a key variable in constructing optimal production policies in the presence of capacity constraints; and (ii) the realization that the OCE resembles a call option written on the underlying commodity which expires when reserves are depleted. The joint insight is important for the construction of optimal production policies and, potentially, for public policy makers who wish to affect production policies especially in the resource industry.

There are many important applications that can extend the analysis presented here. These include project and facility planning, social welfare and waste management. In particular, price-contingent taxes increase the relative magnitude of the opportunity costs, thereby promoting a more efficient usage of resources. This poses an interesting possibility for governments to balance ecological and economic considerations by optimally adjusting the tax code for resource based industries. Finally, the call option nature of opportunity costs can allow for qualitative insights when analyzing complicated control problems.

Another issue is that the *long-term* behavior of commodity prices is crucial for a proper understanding of opportunity cost. This is still a fledgling area of research with some promising work appearing recently by Routledge, Seppi and Spatt (2000) and Schwartz (1997, 1998). In particular, Schwartz's work suggests that commodities, such as copper and oil, have both mean-reverting and persistent components. This can impact the opportunity cost in a non-trivial way; since the high-price behavior depends on the reserve size (i.e., time to expiry), the implication is that differences in optimal policies between operations with small versus large reserves may be enhanced.


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Table 1:
Cash Flows for Different Cutoff Policies When the Processing Stage Is Not Constrained

| Cutoff | Cash |
|---|---|
| 0 – All extracted material is processed | $S_l + S_m + S_h - 3k - 3\xi$ |
| $\frac{1}{3}$ – Only top 67% of extracted material is processed | $S_m + S_h - 2k - 3\xi$ |
| $\frac{2}{3}$ – Only top 33% of extracted material is processed | $S_h - k - 3\xi$ |

Table 2:
Cash Flows for Different Cutoff Policies When Processing Stage Is Restricted to 1 Unit Per Period

| Units Extracted | Cutoff | Cash |
|---|---|---|
| 1 | 0 – All extracted material is processed | $\frac{1}{3}(S_l + S_m + S_h) - k - \xi$ |
| 2 | $\frac{1}{2}$ – Only top 50% of extracted material is processed | $\frac{1}{3}S_m + \frac{2}{3}S_h - k - 2\xi$ |
| 3 | $\frac{2}{3}$ – Only top 33% of extracted material is processed | $S_h - k - 3\xi$ |

Table 3:
Sawmill Example Parameters

| Model Input | Assumptions |
|---|---|
| Diameter Distribution | Diameters of timber units (measured in cubic meters) are distributed with mean and standard deviation of 18 and 6 inches, respectively. The population of timber units is assumed to be log-normal. |
| Capacity Constraints | Harvesting: $H = 5$ Million $m^3$/Year<br>Milling: $G_I = 1.0$ Million $m^3$/Year |
| Variable Costs | Logging: $k_w^o = 5$ \$/$m^3$<br>Stumpage: $\chi_{Stump} = 0.15$<br>Input: $\kappa(c) = 15 + \frac{1500in^2}{c^2}$ \$/$m^3$ ($c$ is measured in inches)<br>Output: $k_O = 5$ \$/$m^3$<br>Production Ratio: $g(c) = \max\left\{0, 0.7 - \frac{6.5in}{c}\right\}$ |
| Other Costs | $MN_O = 15$ \$Million / Year operating overhead<br>$MN_C = 10$ \$Million / Year overhead during temporary shutdown<br>$K_C = K_O = 1.0$ \$Million temporary shut down / startup costs<br>$V_a = 0$ \$ Salvage value |
| Taxes | Symmetric profit taxes of 30%. |
| Price Process | Risk adjusted Random Walk: $dS = S(r-\delta)dt + S\sigma dZ$<br>$r = 0.06$, $\delta = 0.015$ $\sigma = 0.343$ (annual rates) |

Table 4:

Open Pit Copper Project Parameters

| Model Input | Assumptions |
|---|---|
| Grade Distribution | Copper grade is distributed log-normally with mean grade of 0.5% and standard deviation of .27%. |
| Capacity and Recovery Constraints | Mining: $H = 10.0$ Million Tons / Year<br>Treatment: $G_I = 3.0$ Million Tons / Year<br>Marketing: $G_O = 34$ Million Lbs. Copper / Year ($R = 0.85$) |
| Variable Costs | Mining: $k_w = 0.25$ \$ / Ton<br>Treatment: $k_I = 3.00$ \$ / Ton<br>Marketing: $k_O = 0.15$ \$ / Lb. Copper |
| Other Costs | $MN_O = 8.0$ \$Million / Year operating overhead<br>$MN_C = 4.0$ \$Million / Year overhead during temporary shutdown<br>$K_C = K_O = 2.0$ \$Million temporary shut down / startup costs<br>$V_a = 15$ \$Million clean up / abandonment costs |
| Taxes | Symmetric profit taxes of 40% and proportional taxes of 2%. |
| Price Process | Risk adjusted Ornstein-Uhlenbeck: $dS = S(r - \delta - \kappa \ln S)dt + S\sigma dZ$<br>$r = 0.05, \;\; \delta = 0.0833 \;\; \kappa = 0.369 \;\; \sigma = 0.233$ (annualized)<br>Consistent with the parameter fits of Schwartz (1997). |

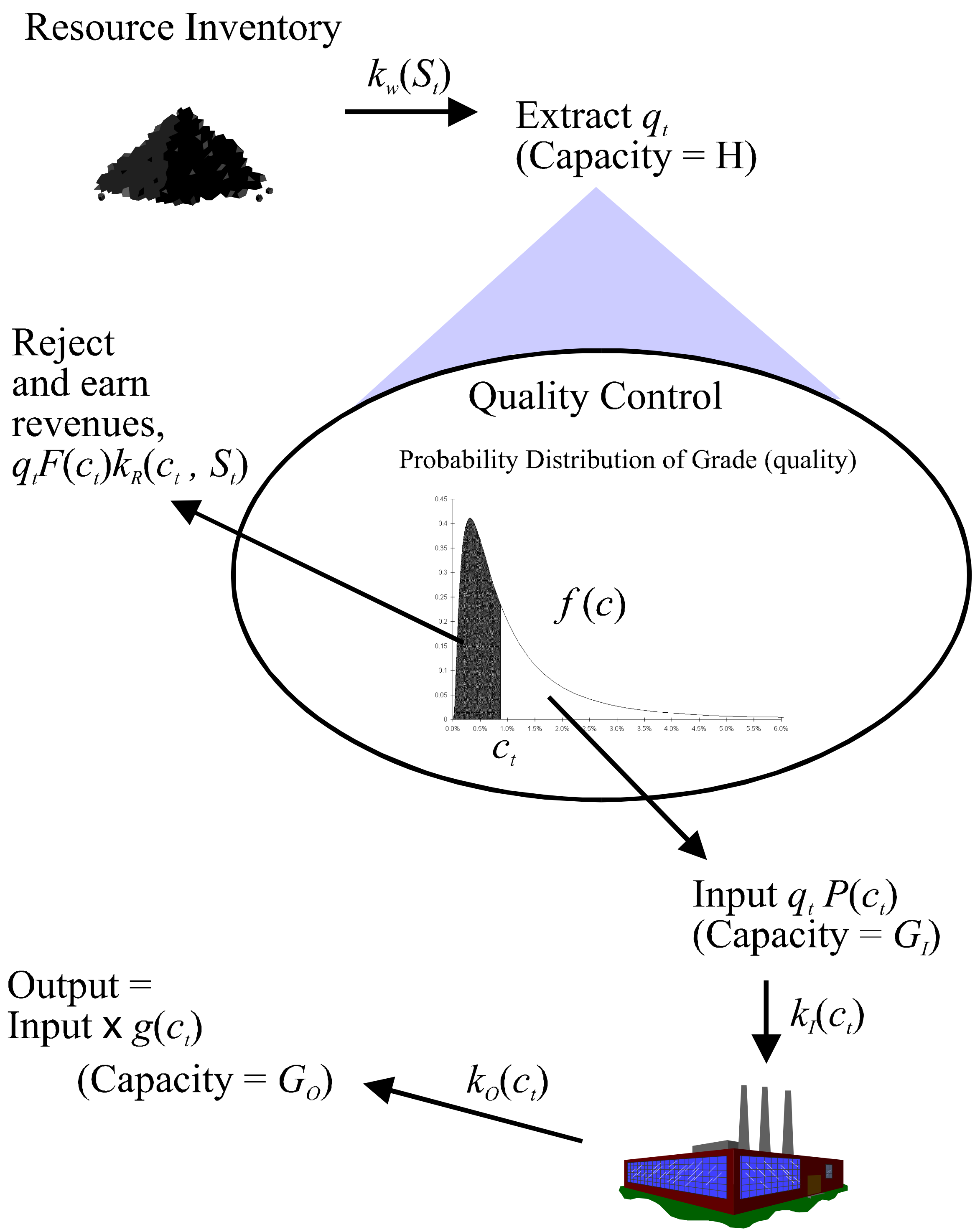


Figure 1: Illustration of the role of quality control in the production process.

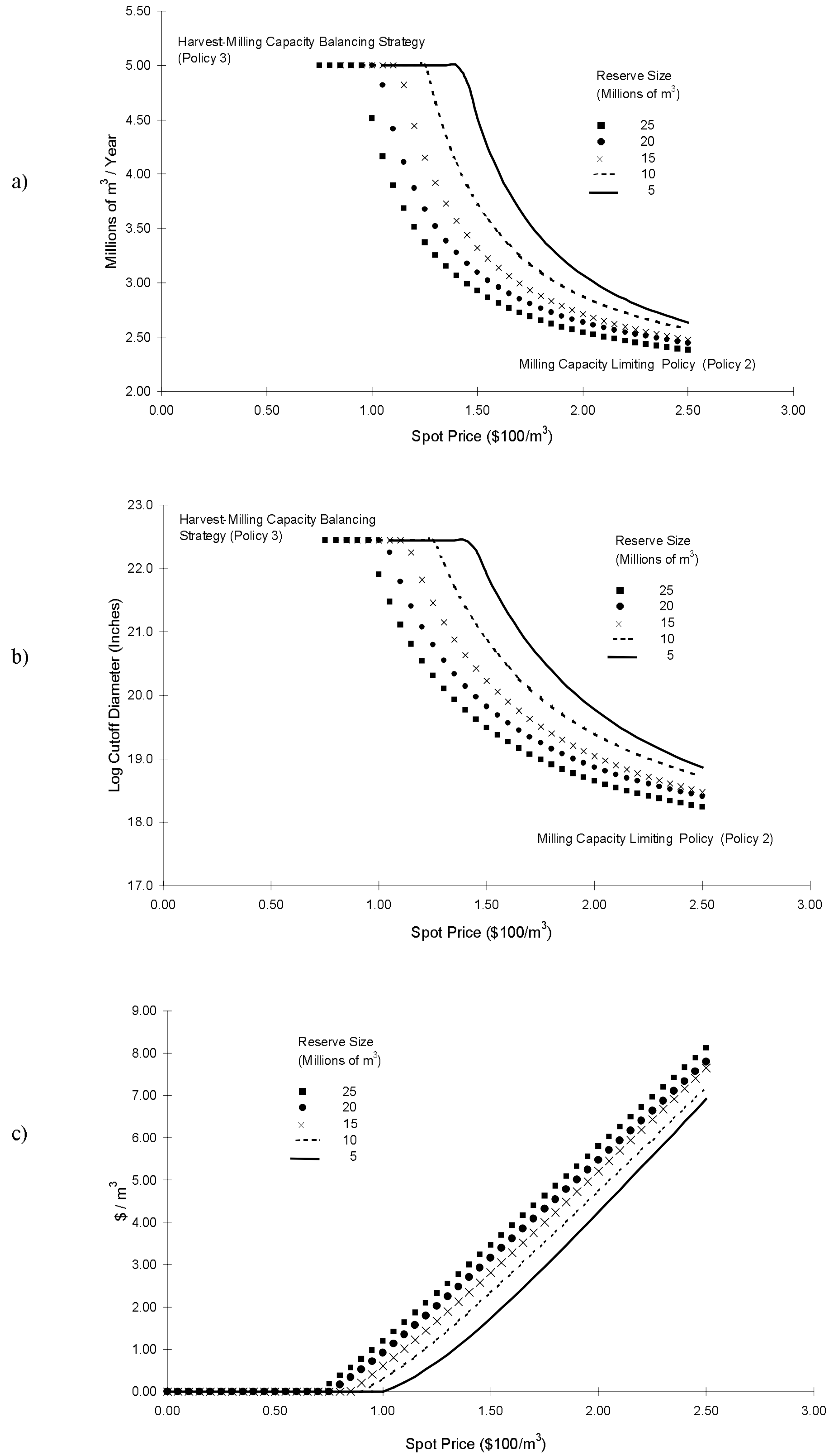


Figure 2: (a) Harvesting policy, (b) diameter cutoff grade policy, and (c) opportunity costs for an open sawmill, assuming log-normal lumber prices; $H = 5$ and $G_I = 1$ million $m^3$ per year. The different curves correspond to various reserve sizes.

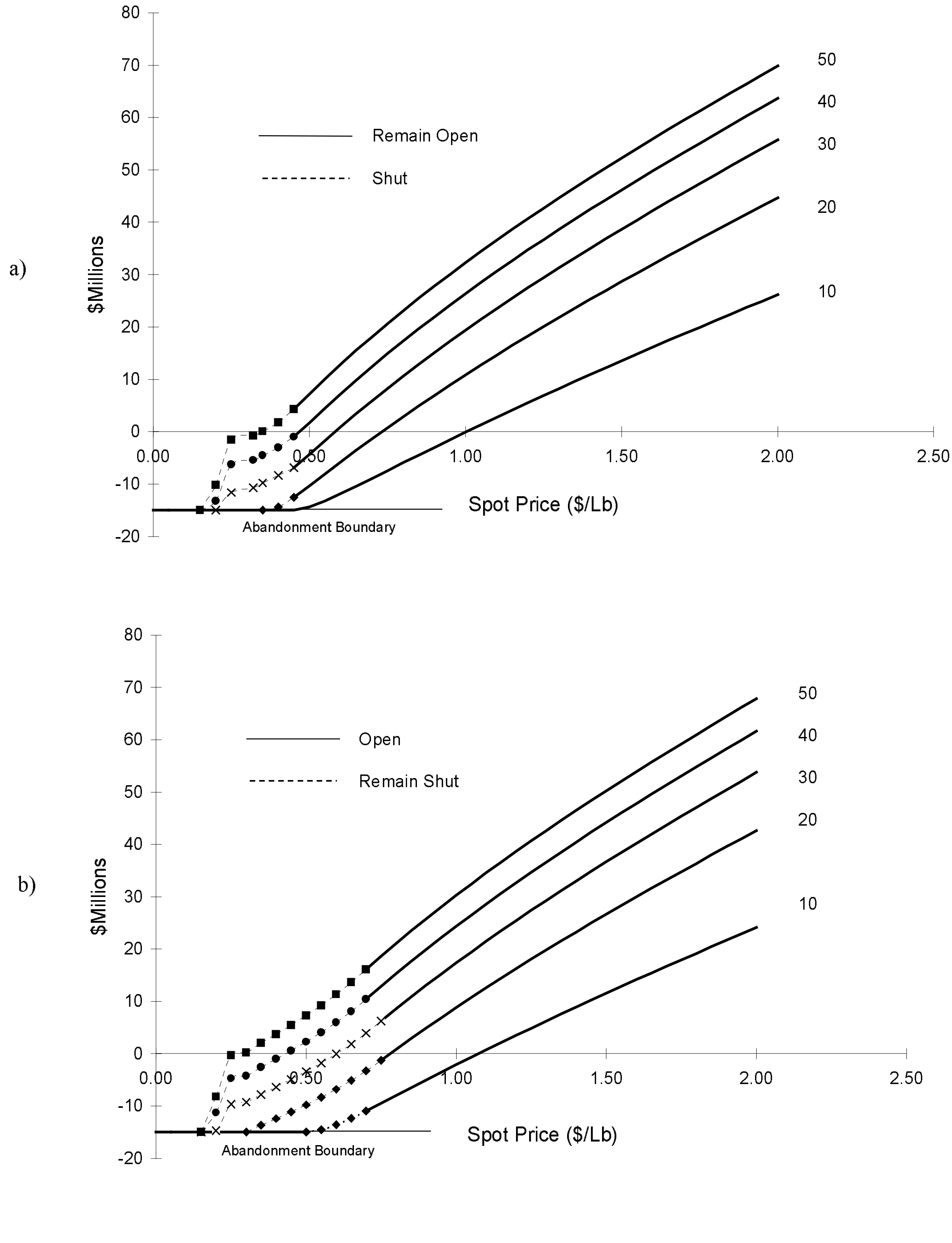


Figure 3: Mine value and policy for (a) an operating mine, and (b) a closed mine. Mine characteristics are as in Table 4. The different curves correspond to various deposit sizes (in millions of tons of material).

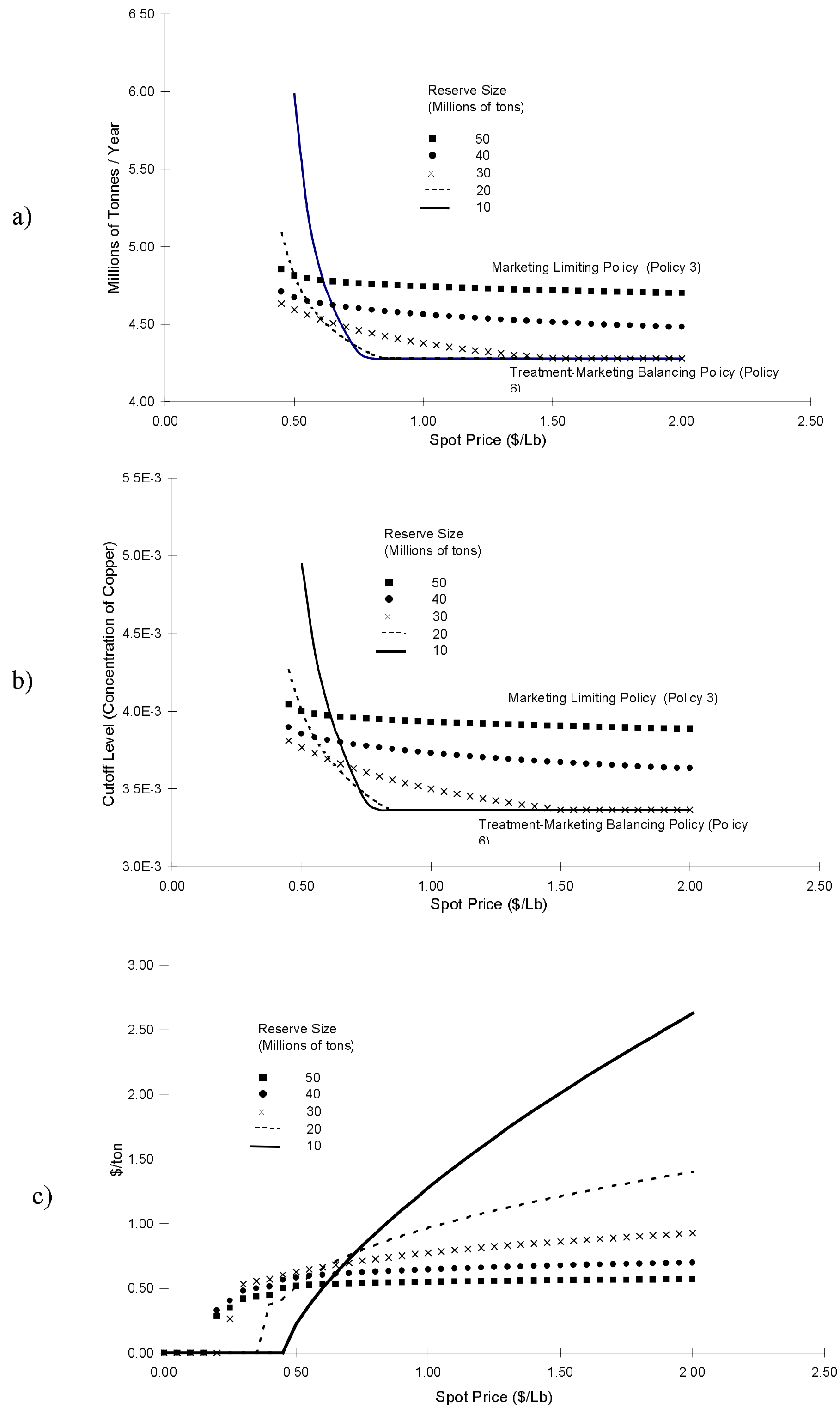


Figure 4: (a) Extraction policy, (b) cutoff grade policy, and (c) opportunity costs for an open mine with mean-reverting commodity prices; $H = 10$ and $G_I = 3$ million tons per year and $G_O = 34$ million Lbs. per year. The different curves correspond to various deposit sizes.

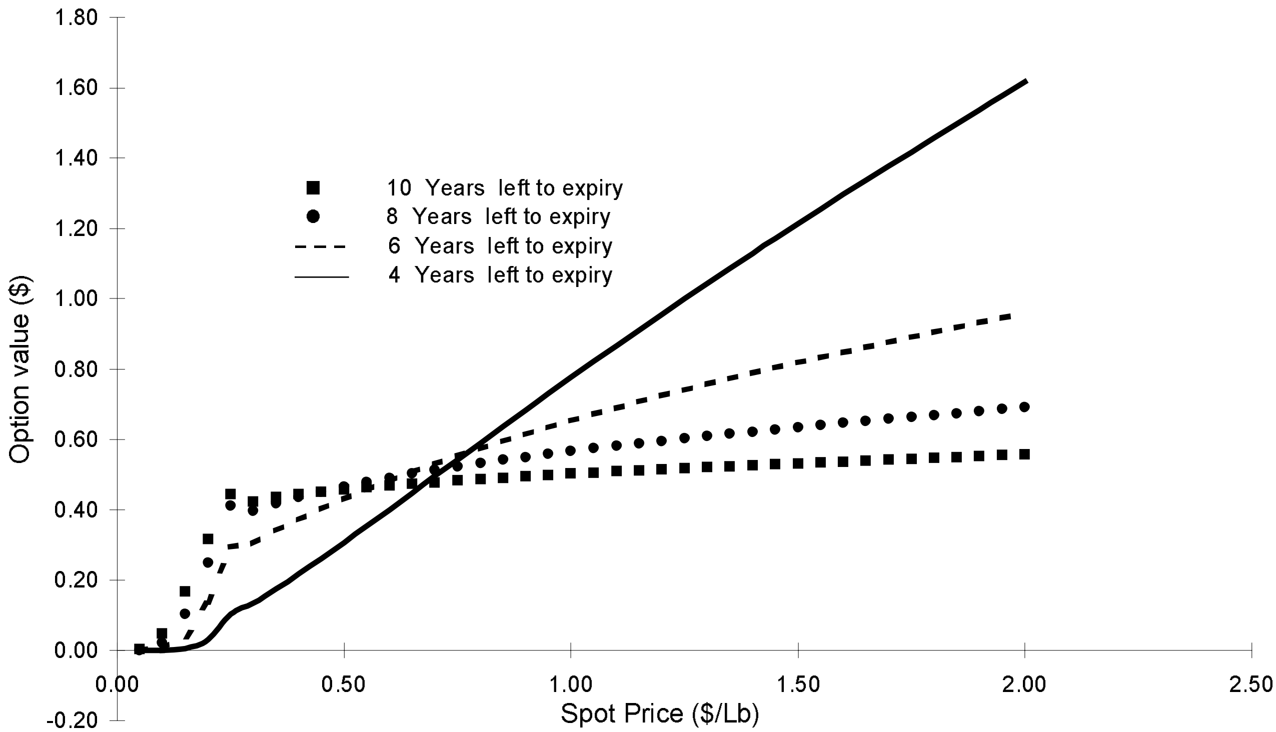


Figure 5: European call option written on the copper content of a ton of ore from the deposit described in Table 4. The strike price is $0.90.

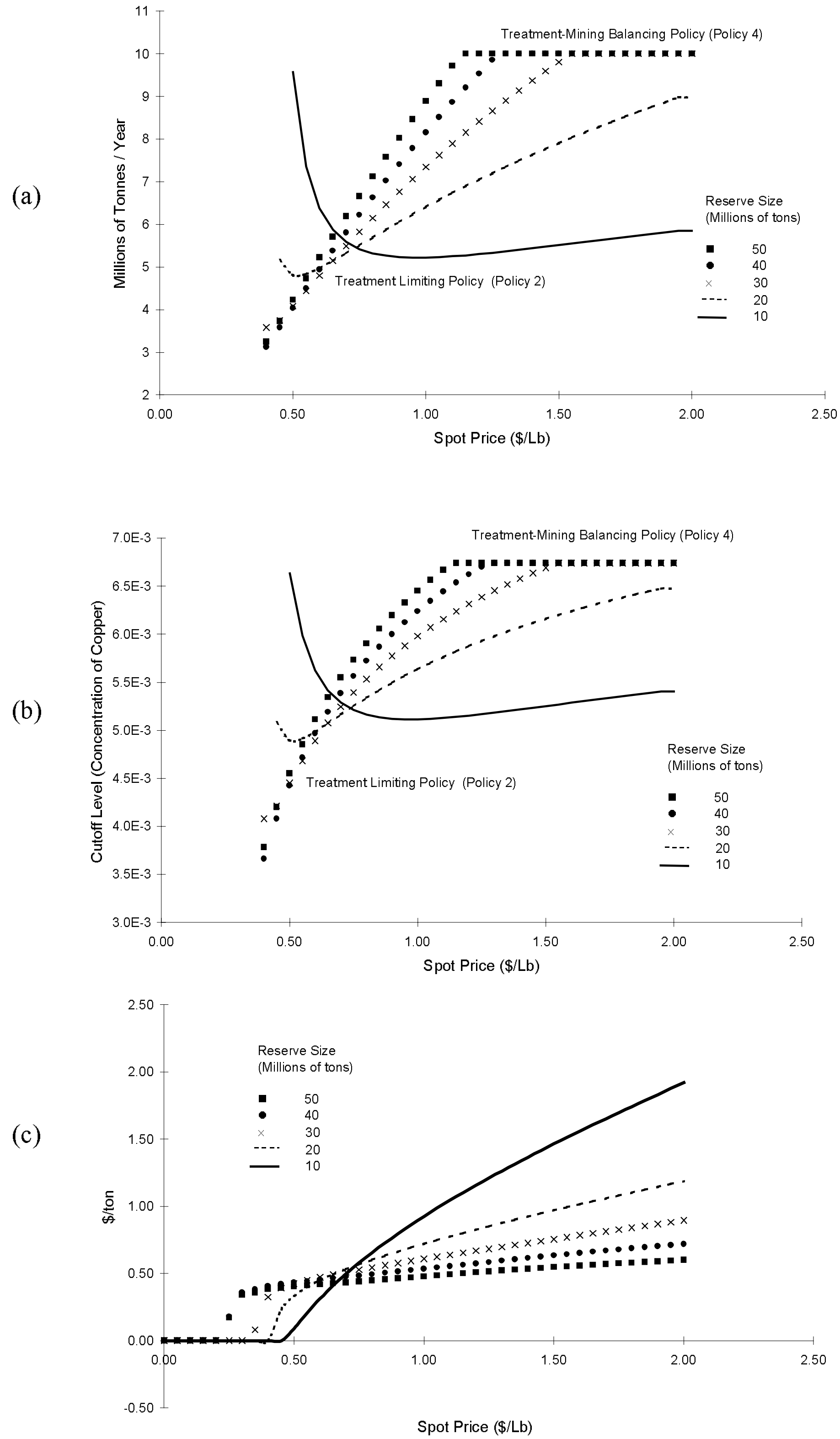


Figure 6: (a) Extraction policy, (b) cutoff grade policy, and (c) opportunity costs for an open mine with mean-reverting commodity prices; $H = 10$ and $G_I = 2$ million tons per year and $G_O = 60$ million Lbs. per year. The different curves correspond to various deposit sizes.

(a)

(b)

(c)

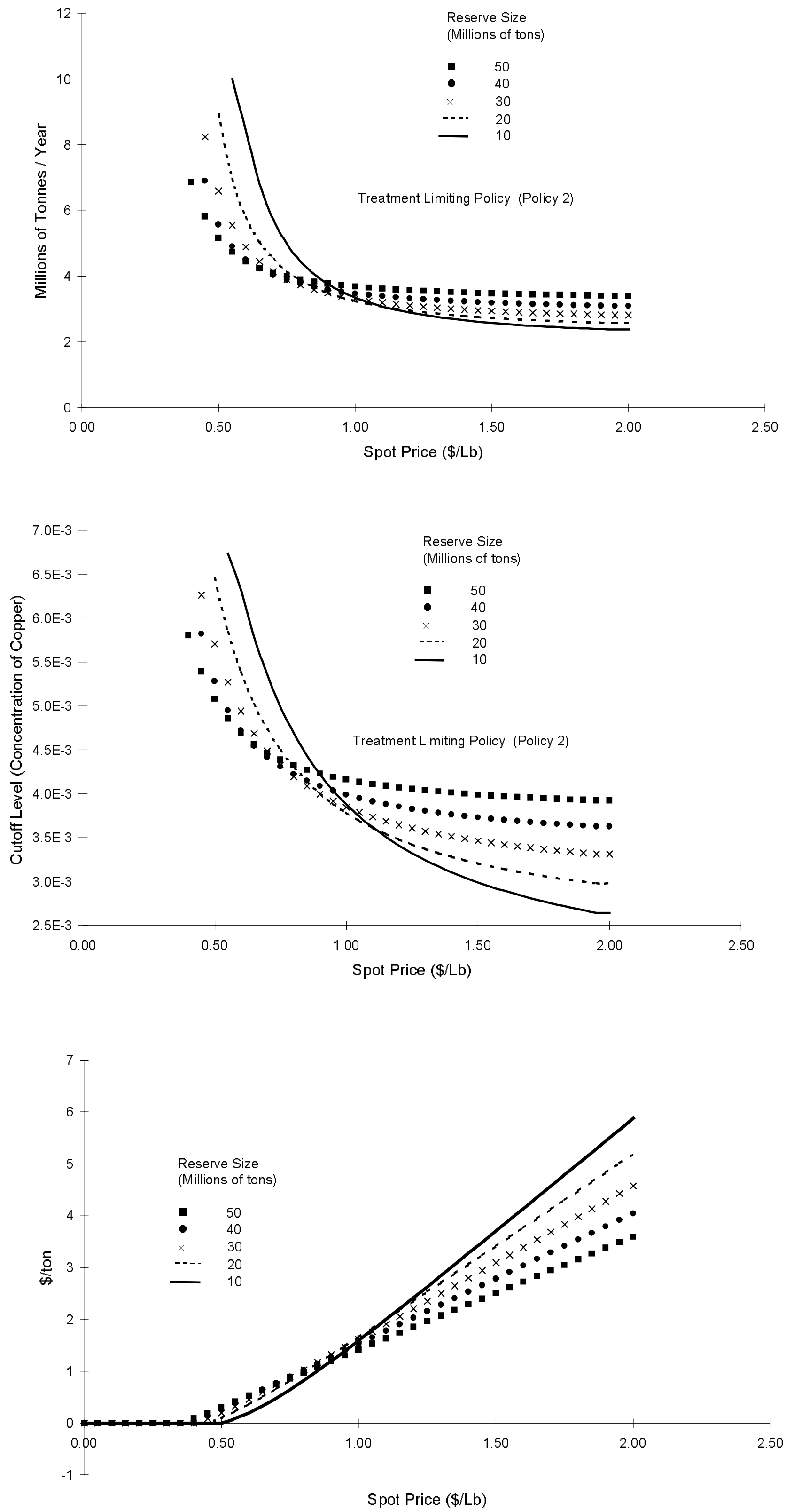


Figure 7: (a) Extraction policy, (b) cutoff grade policy, and (c) opportunity costs for an open mine with log-normal commodity prices; $H = 10$ and $G_I = 2$ million tons per year and $G_O = 60$ million Lbs. per year. The different curves correspond to various deposit sizes.