Technical Appendix for "Uncertainty Aversion and Systemic Risk" by David L. Dicks and Paolo Fulghieri

Proof of Theorem 1. The problem is modelled as a sequential game. At $t = 0$, banks offer contracts, $\left\{r_{1\tau}, r_{2\tau}^{l}, r_{2\tau}^{h}\right\}$ to investors, committing to capital allocation $\{S_\tau, K_\tau\}$. Next, investors decide their investment strategy, $\{S_a, d_A, d_B\}$. At $t = 1$, investors decide whether to withdraw from each bank, $w_\tau = 1$, or to stay in each bank, $w_\tau = 0$. At $t = 2$, the risky project is realized and all assets are divided among investors remaining in the banks.[1] We will solve the game by backward induction. Finally, we will show that banks have no incentive to change their capital allocation ex post, banks implement the socially optimal allocation, and that banks are exposed to the risk of runs.

To fix notation, suppose that the bank invests in $\{S_{\tau 0}, K_{\tau 0}\}$ at $t = 0$. After paying to investors who withdraw at $t = 1$, the bank has $\{S_{\tau 1}, K_{\tau 1}\}$ remaining. Thus, if $(1 - \eta_\tau)$ percentage of deposits remain in bank τ, each unit of deposit receives $\frac{1}{1-\eta_\tau}\left[S_{\tau 1} + 1_\tau R K_{\tau 1}\right]$, where 1_τ is the indicator of success for type τ assets. Thus, the expected value of remaining in bank τ at $t = 1$ is $\frac{1}{1-\eta_\tau}\left[S_{\tau 1} + p\left(\theta_T\right) R K_{\tau 1}\right]$. Because budget constraints for the bank will bind in equilibrium, $S_{\tau 0} = \lambda r_{1\tau} + (1-\lambda) r_{2\tau}^{l}$, $K_{\tau 0} = (1-\lambda)\frac{r_{2\tau}^{h}}{R}$, and $S_{\tau 0} + K_{\tau 0} = 1$.[2]

At $t = 1$, investors decide whether to withdraw, setting $w_\tau = 1$, or stay in the bank, setting $w_\tau = 0$. Suppose that, in equilibrium, share η_τ of deposits are withdrawn at $t = 1$. Payouts depend on the solvency of the bank. If the bank is solvent, investors who withdraw receive $r_{1\tau}$ from bank τ at $t = 1$; investors who stay in the bank receive a share of remaining assets. Late investors who remain in the bank thus have an expected payoff of $\frac{1}{1-\eta_\tau}\left[S_{\tau 1} + p\left(\theta_T\right) R K_{\tau 1}\right]$. In contrast, if the bank is insolvent, some investors who withdraw receive $r_{1\tau}$ at $t = 1$, but some receive nothing (due to the sequential service constraint); investors who stay in the bank receive nothing because $S_{\tau 1} = K_{\tau 1} = 0$ when the bank is insolvent.

Because banks are benevolent, they will liquidate assets efficiently, so they will use the safe asset first, then the risky second (because $\ell < 1 < p\left(\theta_T\right) R$). Therefore, for $\eta_\tau \leq \frac{S_{\tau 0}}{r_{1\tau}}$, $S_{\tau 1} = S_{\tau 0} - \eta_\tau r_{1\tau}$ and $K_{\tau 1} = K_{\tau 0}$, but for $\eta_\tau \in \left(\frac{S_{\tau 0}}{r_{1\tau}}, \frac{S_{\tau 0} + \ell K_{\tau 0}}{r_{1\tau}}\right)$, $S_{\tau 1} = 0$ and $K_{\tau 1} = K_{\tau 0} - \frac{1}{\ell}\left(\eta_\tau r_{1\tau} - S_{\tau 0}\right)$. Finally, if $\eta_\tau > \frac{S_{\tau 0} + \ell K_{\tau 0}}{r_{1\tau}}$, the bank is insolvent: $S_{\tau 1} = K_{\tau 1} = 0$. Investors who withdraw receive $r_{1\tau}$ for sure if the bank is solvent; if the bank is insolvent, they receive a lottery that pays $r_{1\tau}$ with probability $\frac{S_{\tau 0} + \ell K_{\tau 0}}{\eta_\tau r_{1\tau}}$ and 0 otherwise. Thus, when η_τ other investors withdraw at $t = 1$, if a late investor stays in bank τ, he expects to receive $y_\tau\left(\eta_\tau, \theta_T\right)$ per unit of investment, where

$$y_\tau\left(\eta_\tau, \theta_\tau\right) = \begin{cases} \frac{1}{1-\eta_\tau}\left[S_{\tau 0} - \eta_\tau r_{1\tau} + p\left(\theta_T\right) R K_{\tau 0}\right] & \eta_\tau \leq \frac{S_{\tau 0}}{r_{1\tau}} \\ \frac{p\left(\theta_T\right) R}{1-\eta_\tau}\left[K_{\tau 0} - \frac{1}{\ell}\left(\eta_\tau r_{1\tau} - S_{\tau 0}\right)\right] & \eta_\tau \in \left(\frac{S_{\tau 0}}{r_{1\tau}}, \frac{S_{\tau 0} + \ell K_{\tau 0}}{r_{1\tau}}\right) \\ 0 & \eta_\tau > \frac{S_{\tau 0} + \ell K_{\tau 0}}{r_{1\tau}} \end{cases} \tag{1}$$

Alternatively, if the late investor withdraws from the bank at $t = 1$, he receives, per unit of investment,

$$z_\tau\left(\eta_\tau\right) = \begin{cases} r_{1\tau} & \eta_\tau \leq \frac{S_{\tau 0} + \ell K_{\tau 0}}{r_{1\tau}} \\ \frac{S_{\tau 0} + \ell K_{\tau 0}}{\eta_\tau} & \eta_\tau > \frac{S_{\tau 0} + \ell K_{\tau 0}}{r_{1\tau}} \end{cases}. \tag{2}$$

Define $u\left(w_A, w_B, \eta_A, \eta_B\right)$ as the expected payoff to late investors as a function of their withdrawal decisions at each bank, w_A and w_B, and the withdrawal decisions of all other investors at each bank, η_A and η_B. Thus,

$$u\left(w_A, w_B, \eta_A, \eta_B\right) = \sum_{\tau \in \{A,B\}} \left\{w_\tau z_\tau\left(\eta_\tau\right) + \left(1 - w_\tau\right) y_\tau\left(\eta_\tau, \theta_T\right)\right\} d_\tau.$$

Because the payoffs from the two banks are additively separable, the decision to remain in a bank is independent

[1] This assumption could alternatively be stated as assuming that any surplus earned is returned to depositors, and investors who remain in the bank bear the cost of a shortfall.

[2] Though it would be feasible to provide safe assets for $r_{1\tau}$ and $r_{2\tau}^{l}$ by investing in the risky asset and liquidating it at $t = 1$, it would be inefficient to do so, because $\ell < 1$. Therefore, $S_{\tau 0}$ and $K_{\tau 0}$ optimally implement r_τ.

of the other bank. As in Diamond and Dybvig (1983), if a sufficient number of other investors withdraw early from bank A, $\eta_A \geq \frac{S_{A0}+\ell K_{A0}}{r_{1A}}$, all investor find it optimal to withdraw early as well: $u(1,j,\eta_A,\eta_B) > u(0,j,\eta_A,\eta_B)$ for all $j \in \{0,1\}$ and for all η_B when η_A is large enough. This is a panic run, which is not the focus of our paper. Because we only consider fundamental runs, we suppose late investors remain in banks unless it is optimal to withdraw when only early investors withdraw. Formally, investors stay in bank A iff $u(0,j,\lambda,\lambda) \geq u(1,j,\lambda,\lambda)$, and investors stay in bank B iff $u(j,0,\lambda,\lambda) \geq u(j,1,\lambda,\lambda)$. Conversely, if $u(1,j,\lambda,\lambda) > u(0,j,\lambda,\lambda)$, all investors run bank A, and if $u(j,1,\lambda,\lambda) > u(j,0,\lambda,\lambda)$, all investors run bank B. Therefore, investors stay in bank τ, setting $w_\tau = 0$, $y_\tau(\lambda,\theta_T) \geq z_\tau(\lambda)$. Because $S_{\tau 0} = \lambda r_{1\tau} + (1-\lambda) r_{2\tau}^l$, $\lambda \leq \frac{S_{\tau 0}}{r_{1\tau}}$, so $y_\tau(\lambda,\theta_T) \geq z_\tau(\lambda)$ iff $r_{2\tau}^l + p(\theta_T) r_{2\tau}^h \geq r_{1\tau}$. Alternatively, if $r_{1\tau} > r_{2\tau}^l + p(\theta_T) r_{2\tau}^h$, investors will run bank τ, which implies that investors receive $r_{1\tau}$ with probability $\chi_\tau = \frac{S_{\tau 0}+\ell K_{\tau 0}}{r_{1\tau}}$. Define C_τ as the set of incentive compatible contracts, and C_τ^c as the set of contracts that induce a run.

Now we have determined the optimal withdrawal decisions of investors at $t=1$, consider optimal investment decisions by investors at $t=0$. The investors' problem is

$$\begin{aligned} V = \max_{S_a, d_\tau} \quad & U_0 \\ s.t. \quad & S_a + d_A + d_B \leq 2 \end{aligned} \tag{3}$$

Let κ be the multiplier for investor's budget constraint, let L be the Lagrangian function, and let $D^*(r)$ be the set of solutions to this problem, given the contract offered by banks $r = \{r_A, r_B\}$. We will now characterize the set of solutions to this problem.

If $r_\tau \in C_\tau^c$, $\tau \in \{A,B\}$, both banks offer contracts that induce investors to run. Investors' expected payoff is from investing $\{S_a, d_A, d_B\}$ is $U_0 = \Upsilon_{RR}$, where

$$\Upsilon_{RR} = \lambda \Upsilon_{ERR} + (1-\lambda)[S_a + \chi_A r_{1A} d_A + \chi_B r_{1B} d_B], \tag{4}$$

$$\begin{aligned} \Upsilon_{ERR} \quad = \quad & \chi_A \chi_B u(S_a + r_{1A} d_A + r_{1B} d_B) + \chi_A (1-\chi_B) u(S_a + r_{1A} d_A) \\ & + (1-\chi_A) \chi_B u(S_a + r_{1B} d_B) + (1-\chi_A)(1-\chi_B) u(S_a), \end{aligned}$$

and $\chi_\tau = \frac{S_{\tau 0}+\ell K_{\tau 0}}{r_{1\tau}}$. Note $\frac{\partial L}{\partial S_a} = \lambda \frac{\partial \Upsilon_{ERR}}{\partial S_a} + (1-\lambda) - \kappa$, where

$$\begin{aligned} \frac{\partial \Upsilon_{ERR}}{\partial S_a} \quad = \quad & \chi_A \chi_B u'(S_a + r_{1A} d_A + r_{1B} d_B) + \chi_A (1-\chi_B) u'(S_a + r_{1A} d_A) \\ & + (1-\chi_A) \chi_B u'(S_a + r_{1B} d_B) + (1-\chi_A)(1-\chi_B) u'(S_a), \end{aligned}$$

and $\frac{\partial L}{\partial d_\tau} = \lambda \frac{\partial \Upsilon_{ERR}}{\partial d_\tau} + (1-\lambda) \chi_\tau r_{1\tau} - \kappa$, where

$$\frac{\partial \Upsilon_{ERR}}{\partial d_\tau} = \chi_{\tau'} u'(S_a + r_{1\tau} d_\tau + r_{1\tau'} d_{\tau'}) \chi_\tau r_{1\tau} + (1-\chi_{\tau'}) u'(S_a + r_{1\tau} d_\tau) \chi_\tau r_{1\tau}.$$

Note $\chi_\tau r_{1\tau} = S_{\tau 0} + \ell K_{\tau 0} < 1$ because $S_{\tau 0} + K_{\tau 0} = 1$ and $\ell < 1$. Because $u'' < 0$, $\frac{\partial \Upsilon_{ERR}}{\partial d_\tau} < \frac{\partial \Upsilon_{ERR}}{\partial S_a}$. Therefore, $d_\tau = 0$ and $S_a = 2$: if both banks offer contracts that induce investors to run at $t=1$, investors will refuse to invest in that bank at $t=0$. Investors receive $V_{Autarky} = \lambda u(2) + 2(1-\lambda)$.

If $r_\tau \in C_\tau$ but $r_{\tau'} \in C_{\tau'}^c$, bank τ writes a contract inducing investors to remain in the bank, while bank τ' writes a contract inducing investors to run. Investors' expected payoff is $U_0 = \Upsilon_\tau(\theta_T)$, where

$$\Upsilon_\tau(\theta_\tau) = \lambda \Upsilon_{E\tau} + (1-\lambda)\left[S_a + \chi_{\tau'} r_{1\tau'} d_{\tau'} + \left(r_{2\tau}^l + p(\theta_\tau) r_{2\tau}^h\right) d_\tau\right], \tag{5}$$

and

$$\Upsilon_{E\tau} = \chi_{\tau'} u(S_a + r_{1\tau} d_\tau + r_{1\tau'} d_{\tau'}) + (1-\chi_{\tau'}) u(S_a + r_{1\tau} d_\tau).$$

Note $\frac{\partial L}{\partial S_a} = \lambda \frac{\partial \Upsilon_{E\tau}}{\partial S_a} + (1-\lambda) - \kappa$, where $\frac{\partial \Upsilon_{E\tau}}{\partial S_a} = \chi_{\tau'} u'(S_a + r_{1\tau} d_\tau + r_{1\tau'} d_{\tau'}) + (1-\chi_{\tau'}) u'(S_a + r_{1\tau} d_\tau)$. $\frac{\partial L}{\partial d_{\tau'}} = \lambda u'(S_a + r_{1\tau} d_\tau + r_{1\tau'} d_{\tau'}) \chi_{\tau'} r_{1\tau'} + (1-\lambda) \chi_{\tau'} r_{1\tau'} - \kappa$. Thus, $\frac{\partial L}{\partial d_{\tau'}} < \frac{\partial L}{\partial S_a}$, so $d_{\tau'} = 0$. Therefore, investors will refuse to invest in any bank that writes a contract inducing runs. Because $d_{\tau'} = 0$, $\frac{\partial L}{\partial S_a} = \lambda u'(S_a + r_{1\tau} d_\tau) + (1-\lambda) - \kappa$ and $\frac{\partial L}{\partial d_\tau} = \lambda u'(S_a + r_{1\tau} d_\tau) r_{1\tau} + (1-\lambda)\left(r_{2\tau}^l + p(\theta_T) r_{2\tau}^h\right) - \kappa$.

Finally, if $r_\tau \in C_\tau$ for $\tau \in \{A, B\}$, investors will run neither bank. Investors' expected payoff is $U_0 = \Upsilon_{AB}(\theta_T, \theta_T)$, where

$$\Upsilon_{AB}(\theta_A, \theta_B) = \lambda u(S_a + r_{1A} d_A + r_{1B} d_B) + (1-\lambda)\left[S_a + \left(r_{2A}^l + p(\theta_A) r_{2A}^h\right) d_A + \left(r_{2B}^l + p(\theta_B) r_{2B}^h\right) d_B\right]. \tag{6}$$

Thus, $\frac{\partial L}{\partial S_a} = \lambda u'(S_a + r_{1A} d_A + r_{1B} d_B) + (1-\lambda) - \kappa$ and
$\frac{\partial L}{\partial d_\tau} = \lambda u'(S_a + r_{1A} d_A + r_{1B} d_B) r_{1\tau} + (1-\lambda)\left(r_{2\tau}^l + p(\theta_T) r_{2\tau}^h\right) - \kappa$, for $\tau \in \{A, B\}$.

Therefore, we have characterized the optimal investment decision by investors given contracts offered by the banks (with FOCs). Let us now turn to the optimal contracts offered by banks. We will solve the bank's problem, ignoring the IC constraints, then verify that the IC constraints are satisfied. Thus, $U_0(r, d) = \Upsilon_{AB}(\{\theta_T, \theta_T\})|_{r,d}$ on this region. Banks are benevolent, so bank τ solves

$$\begin{aligned} \max_{r_\tau} \quad & V \\ s.t. \quad & \lambda r_{1\tau} + (1-\lambda) r_{2\tau}^l + (1-\lambda)\frac{r_{2\tau}^h}{R} \leq 1. \end{aligned} \tag{7}$$

Suppose to the contrary that $r_{2\tau}^l > 0$. Consider an alternative contract $\tilde{r}_\tau$ such that $\tilde{r}_{1\tau} = r_{1\tau}$, $\tilde{r}_{2\tau}^l = 0$ and $\tilde{r}_{2\tau}^h = r_{2\tau}^h + R r_{2\tau}^l$. Pick $d \in D^*(r)$ and $\tilde{d} \in D^*(\tilde{r})$. By switching from r_τ to $\tilde{r}_\tau$, the bank provides investors with utility $V(\tilde{r}) = U_0\left(\tilde{r}, \tilde{d}\right)$. By definition of the maximum, $U_0\left(\tilde{r}, \tilde{d}\right) \geq U_0(\tilde{r}, d)$, and $U_0(\tilde{r}, d) = U_0(r, d) + (1-\lambda)(p(\theta_T) R - 1) r_{2\tau}^l d_\tau$. Also, $U_0(r, d) = V(r)$. Therefore, $V(\tilde{r}) \geq V(r)$, with strict inequality if $d_\tau > 0$. Therefore, $r_{2\tau}^l = 0$ in any optimal contract.

The budget constraint binds in equilibrium, so $r_{2\tau}^h = \frac{R}{1-\lambda}(1 - \lambda r_{1\tau})$. Thus, $V(r) = U_0(r, d)$, and we can express

$$U(r, d) = \lambda u(S_a + r_{1\tau} d_\tau + r_{1\tau'} d_{\tau'}) + (1-\lambda) S_a + (1-\lambda) p(\theta_T) r_{2\tau'}^h d_{\tau'} + p(\theta_T) R (1 - \lambda r_{1\tau}) d_\tau.$$

We will now prove that banks set $S_a + r_{1\tau} d_\tau + r_{1\tau'} d_{\tau'} = c_1^*$, where $u'(c_1^*) = p(\theta_T) R$. Suppose to the contrary that $S_a + r_{1\tau} d_\tau + r_{1\tau'} d_{\tau'} < c_1^*$. Consider the alternative contract $\tilde{r}_{1\tau} = r_{1\tau} + \delta$, where δ is small. $V(\tilde{r}) = U\left(\tilde{r}, \tilde{d}\right)$ and $U\left(\tilde{r}, \tilde{d}\right) \geq U(\tilde{r}, d)$. By first-order approximation, $U(\tilde{r}, d) - U(r, d) = \lambda [u'(c_1) - p(\theta_T) R] d_\tau \delta$, so $U(\tilde{r}, d) \geq U(r, d)$, which implies that $V(\tilde{r}) \geq V(r)$, with strict inequality if $d_\tau > 0$. Similarly, if $S_a + r_{1\tau} d_\tau + r_{1\tau'} d_{\tau'} > c_1^*$, consider $\tilde{r}_{1\tau} = r_{1\tau} - \delta$ for small δ: $U(\tilde{r}, d) - U(r, d) = \lambda [p(\theta_T) R - u'(c_1)] d_\tau \delta$, so $U(\tilde{r}, d) \geq U(r, d)$ and $V(\tilde{r}) \geq V(r)$, with strict inequality for $d_\tau > 0$. It is optimal for investors to set $S_a = 0$ if $p(\theta_T) r_{2\tau}^h \geq r_{1\tau} \geq 1$ and one inequality is strict, so $d_\tau > 0$ for at least one bank. Therefore, banks set $r_{1\tau}$ so that $u'(c_1) = p(\theta_T) R$. Because $u'(2) > p(\theta_T) R$, $S_a + r_{1A} d_A + r_{1B} d_B > 2$, so one of the banks offers $r_{1\tau} > 1$, and investors set $S_a = 0$. Note that it is WLOG optimal for both banks to set $r_{1\tau}^{\rho *} = \frac{1}{2} c_1^*$, where $u'(c_1^*) = p(\theta_T) R$, $r_{2\tau}^{l\rho *} = 0$, and $r_{2\tau}^{h\rho *} = \frac{R}{1-\lambda}(1 - \lambda r_{1\tau}^{\rho *})$ and, from investor FOCs, it is WLOG optimal for investors to set $d_\tau = 1$ (investors are indifferent between all allocations such that $d_A + d_B = 2$).

Because the budget constraint binds at each bank and households invest all wealth at banks, $d_A + d_B = 2$, we can express the value late investors receive by staying in the bank as $U_1(\theta_T) = p(\theta_T) R \frac{2 - \lambda c_1^*}{1-\lambda}$. By (20), $2 < c_1^* < \frac{2p(\theta_T)R}{\lambda p(\theta_T) R + (1-\lambda)}$, $U_1(\theta_T) > \frac{2p(\theta_T)R}{\lambda p(\theta_T) R + (1-\lambda)}$. Thus, $U_1(\theta_T) > c_1^*$, so (12) is lax. Similarly, by offering symmetric contracts, both (13) and (14) are lax as well. Banks implement the optimal contract with capital allocation $S_{\tau 0} = \lambda r_{1\tau}^{\rho *}$ and $K_{\tau 0} = 1 - \lambda r_{1\tau}^{\rho *}$.

Banks are dynamically consistent – if they had the option to secretly change their capital allocation ex post, banks would ex post select the same allocation. If the bank deviated to $S_{\tau 0} + \varepsilon$ and $K_{\tau 0} - \varepsilon$, where $\varepsilon > 0$, investor payoff would change by $(1-\lambda)(1 - p(\theta_T) R)\varepsilon < 0$. Similarly, if the bank deviated to $S_{\tau 0} - \varepsilon$ and $K_{\tau 0} + \varepsilon$, investor payoff would change by $\frac{p(\theta_T)R}{1-\lambda}\left(1-\frac{1}{\ell}\right)\varepsilon<0$. Thus, any ex post deviation from the ex ante optimal capital allocation harms the bank's objective.

We will now show that banks implement the socially optimal allocation with linear contracts. Consider the social planner's problem. The social planner can allocate resources to the early type through the storage technology, and can allocate resources to the portfolio of late investors: the social planner allocates c_1 to early consumers and portfolio $\left\{c_2^S, c_2^A, c_2^B\right\}$ to late consumers. This allocation provides investors with utility $\mathcal{U}\left(\{\theta_T,\theta_T\}\right)$, where

$$\mathcal{U}\left(\overrightarrow{\theta}\right)=\lambda u\left(c_1\right)+(1-\lambda)\left[c_2^S+p\left(\theta_A\right)c_2^A+p\left(\theta_B\right)c_2^B\right]. \tag{8}$$

The planner's problem is

$$\begin{aligned} \max_{c} \quad & \mathcal{U}\left(\{\theta_T,\theta_T\}\right) \\ s.t. \quad & \lambda c_1+(1-\lambda)\left(c_2^S+\frac{c_2^A}{R}+\frac{c_2^B}{R}\right)\leq 2 \end{aligned}$$

Let κ be the multiplier for the budget constraint and L be the Lagrangian function. The FOCs are $\frac{\partial L}{\partial c_1}=\lambda u'\left(c_1\right)-\lambda\kappa$, $\frac{\partial L}{\partial c_2^S}=(1-\lambda)-(1-\lambda)\kappa$, and $\frac{\partial L}{\partial c_2^\tau}=(1-\lambda)p\left(\theta_T\right)-\frac{1-\lambda}{R}\kappa$. Because $\frac{\partial L}{\partial c_2^S}<R\frac{\partial L}{\partial c_2^\tau}$, $c_2^S=0$. Further, $\frac{\partial L}{\partial c_2^\tau}=\frac{\partial L}{\partial c_1}=0$ implies $\kappa=u'\left(c_1\right)=p\left(\theta_T\right)R$. Thus, the social planner allocates c_1^* to the early type. Also, $\kappa>0$, so the budget constraint binds, and thus, $c_2^A+c_2^B=\frac{R}{1-\lambda}\left(2-\lambda c_1^*\right)$. It is WLOG optimal for the social planner to set $c_2^A=c_2^B$. Note that this allocation is implemented when both banks offer contracts $r_\tau^{\rho*}$ and investors set $d_\tau=1$.

Finally, because $r_{1\tau}^{\rho*}>1$, $y_\tau(1)=0<z_\tau(1)$, so banks are exposed to the risk of runs. That is, runs are feasible because $S_{\tau 0}+\ell K_{\tau 0}<r_{1\tau}^{\rho*}$, but runs are off-equilibrium, because investors would refuse to invest in the bank if they expected to run. ■

Proof of Theorem 2. The timing of the problem is the same as that in Theorem 1, as are the cash flows. At $t=1$, investor payoff given withdrawal decision $\{w_A,w_B\}$, is now

$$u\left(w_A,w_B,\eta_A,\eta_B\right)=\min_{\overrightarrow{\theta}\in C}\sum_{\tau\in\{A,B\}}\left\{w_\tau z_\tau\left(\eta_\tau\right)+\left(1-w_\tau\right)y_\tau\left(\eta_\tau,\theta_\tau\right)\right\}d_\tau.$$

where $y_\tau\left(\eta_\tau,\theta_\tau\right)$ is from (1) and $z_\tau\left(\eta_\tau\right)$ is from (2). Because we focus on fundamental runs, we suppose late investors remain in banks unless it is optimal to withdraw when only early investors withdraw. Formally, investors remain in both banks iff $u(0,0,\lambda,\lambda)=\max_{i,j\in\{0,1\}}u(i,j,\lambda,\lambda)$, so letting $r=\{r_\tau\}_{\tau\in\{A,B\}}$, let $\tilde{C}_{AB}(r)$ be the set of contracts that induce late investors to stay in both banks. Similarly, let $\tilde{C}_A(r)$ be the set of contracts that induce late investors to stay in only bank A but withdraw from bank B, $u(0,1,\lambda,\lambda)=\max_{i,j\in\{0,1\}}u(i,j,\lambda,\lambda)$, and let $\tilde{C}_B(r)$ be the set of contracts such that $u(1,0,\lambda,\lambda)=\max_{i,j\in\{0,1\}}u(i,j,\lambda,\lambda)$. Finally, let $\tilde{C}_0(r)$ be the set of contracts that induce late investors to withdraw from both banks, so that $u(1,1,\lambda,\lambda)=\max_{i,j\in\{0,1\}}u(i,j,\lambda,\lambda)$. We will assume that investors who are indifferent will stay in the bank (ties go toward stability), so define $C_{AB}=\tilde{C}_{AB}$, $C_\tau=\tilde{C}_\tau\backslash C_{AB}$, and $C_0=\tilde{C}_0\backslash\left(C_{AB}\cup C_A\cup C_B\right)$.

We have characterized the optimal withdrawal decisions of investors; consider optimal investment decision by investors at $t=0$. Similar to the proof of Theorem 1, investors solve (3). If banks offer contracts in C_0, investors earn utility $U_0=\Upsilon_{RR}$, as defined in (4). Because Υ_{RR} does not depend on θ_τ, by identical logic to the proof of Theorem 1, investors refuse to invest in either bank, setting $S_a=2$ and earning $V_{Autarky}$.

If banks offer contracts in C_τ, investors earn utility $U_0=\Upsilon_\tau$, defined in (5). By identical logic to the proof of Theorem 1, investors refuse to invest in bank τ', $d_{\tau'}=0$. However, on C_τ, investors will only have exposure to type τ assets, so $\theta_\tau^a=\theta_L$. Thus, $r\in C_\tau$ requires that $r_{2\tau}^l+p\left(\theta_L\right)r_{2\tau}^h\geq r_{1\tau}$. Combining budget constraints, $\lambda r_{1\tau}+(1-\lambda)\left(r_{2\tau}^l+\frac{r_{2\tau}^h}{R}\right)=1$, and $p\left(\theta_L\right)R<1$, if $r_{1\tau}>1$, $r_{2\tau}^l+p\left(\theta_L\right)r_{2\tau}^h<1$, so C_τ is empty if $r_{1\tau}>1$. Because $d_{\tau'}=0$, $\frac{\partial L}{\partial S_a}=\lambda u'\left(S_a+r_{1\tau}d_\tau\right)+(1-\lambda)-\kappa$ and $\frac{\partial L}{\partial d_\tau}=\lambda u'\left(S_a+r_{1\tau}d_\tau\right)r_{1\tau}+(1-\lambda)\left[r_{2\tau}^l+p\left(\theta_L\right)r_{2\tau}^h\right]-\kappa$. Because $\lambda r_{1\tau}+(1-\lambda)\left(r_{2\tau}^l+\frac{r_{2\tau}^h}{R}\right)=1$, $p\left(\theta_L\right)R<1$, $r_{1\tau}\leq 1$, and $u'(2)>1$, $\frac{\partial L}{\partial S_a}\geq\frac{\partial L}{\partial d_\tau}$, with strict inequality if $r_{1\tau}<1$ or $r_{2\tau}^h>0$. Thus, it is WLOG optimal for investors to set $S_a=2$ when banks offer contracts in C_τ.

If banks offer contracts in C_{AB}, investors earn $U_0 = \Upsilon_{AB}$, defined in (6). This implies that $\frac{\partial L}{\partial S_a} = \lambda u'(c_1) + (1-\lambda) - \kappa$, $\frac{\partial L}{\partial d_\tau} = \lambda u'(c_1) r_{1\tau} + (1-\lambda)\left(r_{2\tau}^l + p(\theta_\tau^a) r_{2\tau}^h\right) - \kappa$, for $\tau \in \{A, B\}$, where $c_1 = S_a + r_{1A} d_A + r_{1B} d_B$ and θ_τ^a is from Lemma 2. Suppose to the contrary that investors have corner beliefs and exposure to risky assets: $\exists \tau$ s.t. $r_{2\tau}^h d_\tau > 0$ and $\theta_\tau^a = \theta_L$. Because $r \in C_{AB}$, $r_{1\tau} \leq r_{2\tau}^l + p(\theta_L) r_{2\tau}^h$. Because $r_{2\tau}^h > 0$, and the budget constraint must be satisfied, this implies that $\frac{\partial L}{\partial d_\tau} < \frac{\partial L}{\partial S_a}$. Contradiction. Thus, investors are only willing to invest in risky assets if they have interior beliefs. Also, $r_{2\tau}^h d_\tau > 0$ only if $r_{2\tau'}^h d_{\tau'} > 0$.

Now that we have solved the optimal investment behavior by investors, we will solve the bank's problem. As we have shown, if either bank violates the IC constraints, investors will refuse to invest in either bank, so banks will satisfy the IC constraints. We will guess that IC constraints (13) and (14) are lax, then verify that these constraints are satisfied. The banks are benevolent, so bank τ solves

$$
\begin{aligned}
\max_{r_\tau} \quad & V \\
s.t. \quad & \lambda r_{1\tau} + (1-\lambda) r_{2\tau}^l + (1-\lambda)\frac{r_{2\tau}^h}{R} \leq 1 \\
& r_{1\tau} d_\tau + r_{1\tau'} d_{\tau'} \leq \min_{\vec{\theta} \in C} U_1\left(\vec{\theta}\right)
\end{aligned}
$$

where V and d are the value function and solutions to (3), respectively.

Safe Equilibrium: If the other bank sets $r_{2\tau'}^h = 0$, it will be optimal for bank τ to set $r_{2\tau}^h = 0$. Suppose to the contrary that $r_{2\tau}^h > 0$. By Lemma 2, $\theta_\tau = \theta_L$ for all $r_{2\tau}^h > 0$. Consider an alternative contract $\tilde{r}_\tau$ such that $\tilde{r}_{1\tau} = r_{1\tau}$, $\tilde{r}_{2\tau}^l = r_{2\tau}^l + \frac{r_{2\tau}^h}{R}$ and $\tilde{r}_{2\tau}^h = 0$. Pick $d \in D^*(r)$ and $\tilde{d} \in D^*(\tilde{r})$. By switching from r_τ to $\tilde{r}_\tau$, the bank provides investors with utility $V(\tilde{r}) = U_0\left(\tilde{r}, \tilde{d}\right)$. By definition of the maximum, $U_0\left(\tilde{r}, \tilde{d}\right) \geq U_0(\tilde{r}, d)$, and $U_0(\tilde{r}, d) = U_0(r, d) + (1-\lambda)(1 - p(\theta_L) R)\frac{r_{2\tau}^h}{R} d_\tau$, where $U_0(r, d) = V(r)$. Therefore, $V(\tilde{r}) \geq V(r)$, with strict inequality if $d_\tau > 0$. Thus, $r_{2\tau}^h = 0$ in any optimal contract. Therefore, if one bank does not invest in the risky assets, the best response of the other bank is to not invest either. Because $u'(2) > 1$, banks provide as much insurance against the liquidity shock as possible, so the IC binds. Thus, $r_{1\tau} = r_{2\tau}^l = 1$ and $r_{2\tau}^h = 0$. Note that (13) simplifies to $r_{2\tau}^l \geq r_{1\tau}$, so it is satisfied. Similarly, (14) is also satisfied. When faced with this contract, investors find it weakly optimal to set $d_\tau = 1$. Because $y_\tau(\eta, \theta) = 1 = z_\tau(\eta)$ for all η, banks are not exposed to the risk of runs.

Risky Equilibrium: We will show that the equilibrium from Theorem 1 is also an equilibrium here. If both banks offer $r_\tau^{\rho*}$, note (12) is lax at both banks, so we will guess that it is lax. Let κ_τ be the multiplier for the budget constraint of bank τ, and let L_τ be the Lagrangian function for bank τ. Let $c_1 = S_a + r_{1A} d_A + r_{1B} d_B$. If both banks invest in risky assets, $r_{2\tau}^h > 0$, investors optimally invest so that they have interior beliefs in equilibrium. Because beliefs are interior, it can easily be shown that the objective is strictly concave, so there is a unique continuously differentiable $\{S_a, d_A, d_B\}$. $\frac{\partial L_\tau}{\partial r_{1\tau}} = \lambda u'(c_1) d_\tau - \lambda \kappa_\tau$, $\frac{\partial L_\tau}{\partial r_{2\tau}^l} = (1-\lambda) d_\tau - (1-\lambda)\kappa_\tau$, and $\frac{\partial L_\tau}{\partial r_{2\tau}^h} = (1-\lambda) p(\theta_\tau^a) d_\tau - (1-\lambda)\frac{\kappa_\tau}{R}$. $\frac{\partial L_\tau}{\partial r_{1\tau}} = 0$ iff $\kappa_\tau = u'(c_1) d_\tau$. Because $\frac{\partial L_\tau}{\partial r_{2\tau}^h} = 0$, $\kappa_\tau = p(\theta_\tau^a) R d_\tau$, so $u'(c_1) = p(\theta_\tau^a) R$. Because this holds for both banks, this implies that $\theta_A^a = \theta_B^a$, so by Lemma 2, $r_{2A}^h d_A = r_{2B}^h d_B$. Also, $\theta_\tau^a = \theta_T$, and $p(\theta_T) R > 1$, which implies that $R\frac{\partial L_\tau}{\partial r_{2\tau}^h} > \frac{\partial L_\tau}{\partial r_{2\tau}^l}$, so $r_{2\tau}^l = 0$. Note the FOCs are satisfied if each bank offers $r_\tau^{\rho*}$ where $r_{1\tau}^{\rho*} = \frac{1}{2} c_1^*$, $u'(c_1^*) = p(\theta_T) R$, $r_{2\tau}^{l\rho*} = 0$, and $r_{2\tau}^{h\rho*} = \frac{R}{1-\lambda}\left(1 - \lambda r_{1\tau}^{\rho*}\right)$. Because $r_{1\tau}^{\rho*} > 1$ and $p(\theta_T) R > 1$, $S_a = 0$. Because banks offer symmetric contracts, $r_\tau^{\rho*} = r_{\tau'}^{\rho*}$, investors optimally set $d_\tau = 1$. Thus, we can express $U_1(\theta_T) = p(\theta_T) R \frac{2 - \lambda c_1^*}{1-\lambda}$. By (20), $2 < c_1^* < \frac{2p(\theta_T)R}{\lambda p(\theta_T) R + (1-\lambda)}$, which implies that $U_1(\theta_T) > \frac{2p(\theta_T)R}{\lambda p(\theta_T) R + (1-\lambda)}$, so (12) is lax. By identical logic to that in the proof of Theorem 1, banks have no incentive to change $S_{\tau 0}$ and $K_{\tau 0}$ ex post. Because $r_{1\tau} > 1$, C_τ is empty: it is better to run both banks rather than just one, so (13) and (14) are lax when banks offer symmetric contracts. Further, because $r_{1\tau}^{\rho*} > 1$, $y_\tau(1, \theta_\tau) = 0 < z_\tau(1)$, so banks are exposed to the risk of runs in this equilibrium. Because $p(\theta_T) r_{2\tau}^h > r_{1\tau} > 1$, $r_\tau^{\rho*}$ Pareto dominates $r_\tau^{\sigma*}$.

Finally, we will show the risky equilibrium implements the socially optimal allocation. Consider the social planner's problem, similar to the Proof of Theorem 1. When the social planner allocates c_1 to early consumers and

$\{c_2^S, c_2^A, c_2^B\}$ to late, investors receive $\min_{\overrightarrow{\theta}\in C}\mathcal{U}\left(\overrightarrow{\theta}\right)$, where $\mathcal{U}$ is defined in (8). The planner's problem is

$$\begin{aligned} \max_{c} \quad & \min_{\overrightarrow{\theta}\in C}\mathcal{U}\left(\overrightarrow{\theta}\right) \\ s.t. \quad & \lambda c_1 + (1-\lambda)\left(c_2^S + \frac{c_2^A}{R} + \frac{c_2^B}{R}\right) \leq 2 \end{aligned}$$

Let κ be the multiplier for the budget constraint and L be the Lagrangian. The FOCs are $\frac{\partial L}{\partial c_1} = \lambda u'(c_1) - \lambda\kappa$, $\frac{\partial L}{\partial c_2^S} = (1-\lambda) - (1-\lambda)\kappa$, and $\frac{\partial L}{\partial c_2^\tau} = (1-\lambda)p(\theta_\tau^a) - \frac{1-\lambda}{R}\kappa$, where θ_τ^a is from Lemma 2 (substituting in c_2^τ for $r_{2\tau}^h d_\tau$). Because $\frac{1}{2}(\theta_A+\theta_B) = \theta_T$, $\exists\tilde{\tau}$ s.t. $\theta_{\tilde{\tau}} \geq \theta_T$, so $\frac{\partial L}{\partial c_2^S} < R\frac{\partial L}{\partial c_2^{\tilde{\tau}}}$, which implies $c_2^S = 0$. Further, $\frac{\partial L}{\partial c_2^\tau} = 0$ implies $\kappa = p(\theta_\tau^a)R$, which implies $\theta_A^a = \theta_B^a$, so (by Lemma 2) $c_2^A = c_2^B$ and $\theta_\tau^a = \theta_T$. $\frac{\partial L}{\partial c_1} = 0$ implies $u'(c_1) = p(\theta_T)R$. Note this allocation is implemented when banks offer $r_\tau^{\rho*}$ and investors set $d_\tau = 1$. ■

Proof of Theorem 3. The proof is similar to the proof of Theorem 1 and Theorem 2: the problem is similarly modelled as a sequential game. At $t=0$, banks offer contracts $\{r_{1\tau}, r_{2\tau}^l, r_{2\tau}^h\}$ to investors, committing to capital allocation $\{S_\tau, K_\tau\}$. Next, investors decide their investment strategy, $\{S_a, d_A, d_B\}$. At $t=1$, investors decide whether to withdraw from each bank, $w_\tau = 1$, or to stay in each bank, $w_\tau = 0$. Distinct from Theorem 1 and Theorem 2, however, is that investors can condition their withdrawal decisions on the signal s_τ. The signal s_τ give the payoff given success of projects at $t=2$: $R_\tau = s_\tau R$. Recall the structure of s_τ: with probability ε, there is bad news about type τ assets, so $s_\tau = \phi$ and $s_{\tau'} = 1$, for $\tau \in \{A, B\}$ and $\tau' \neq \tau$. With probability $1-2\varepsilon$, $s_\tau = 1$ for both banks. At $t=2$, the risky project is realized and all assets are divided among investors remaining in the banks. We solve the game by backward induction.

Uncertainty-Neutral Investors: The cashflows are similar to that in the proof of Theorem 1, except that if the payoff of remaining in the bank depends on how bad the shock is. If the shock is not sufficiently bad, $\phi \geq \frac{\ell}{p(\theta_T)R}$, banks prefer to liquidate the safe asset first, then the risky asset. Thus, for $s_\tau \in \{\phi, 1\}$,

$$y_\tau(\eta_\tau, \theta_\tau, s_\tau) = \begin{cases} \frac{1}{1-\eta_\tau}\left[S_{\tau 0} - \eta_\tau r_{1\tau} + p(\theta_\tau)s_\tau R K_{\tau 0}\right] & \eta_\tau \leq \frac{S_{\tau 0}}{r_{1\tau}} \\ \frac{p(\theta_\tau)s_\tau R}{1-\eta_\tau}\left[K_{\tau 0} - \frac{1}{\ell}(\eta_\tau r_{1\tau} - S_{\tau 0})\right] & \eta_\tau \in \left(\frac{S_{\tau 0}}{r_{1\tau}}, \frac{S_{\tau 0}+\ell K_{\tau 0}}{r_{1\tau}}\right) \\ 0 & \eta_\tau > \frac{S_{\tau 0}+\ell K_{\tau 0}}{r_{1\tau}} \end{cases}. \tag{9}$$

Alternatively, if the shock is very bad, $\phi < \frac{\ell}{p(\theta_T)R}$, the bank finds it optimal to liquidate their entire position in risky assets, whether investors run the bank or not. This implies that the expected cashflow of remaining in the bank following bad news is

$$y_\tau(\eta_\tau, \theta_\tau, \phi) = \begin{cases} \frac{1}{1-\eta_\tau}\left[S_{\tau 0} + \ell K_{\tau 0} - \eta_\tau r_{1\tau}\right] & \eta_\tau \leq \frac{S_{\tau 0}+\ell K_{\tau 0}}{r_{1\tau}} \\ 0 & \eta_\tau > \frac{S_{\tau 0}+\ell K_{\tau 0}}{r_{1\tau}} \end{cases}. \tag{10}$$

Because we focus on fundamental runs, similar to the proof of Theorem 1, investors find it optimal to remain in bank τ iff $y_\tau(\lambda, \theta_T, s_\tau) \geq z_\tau(\lambda)$, where z_τ is given in (2). Thus, $r \in C_\tau^1$ iff late investors optimally set $w_\tau = 0$ if $s_\tau = 1$, which holds iff $r_{2\tau}^l + p(\theta_T)r_{2\tau}^h \geq r_{1\tau}$. Similarly, if $\phi \geq \frac{\ell}{p(\theta_T)R}$, $r \in C_\tau^\phi$ iff $r_{2\tau}^l + p(\theta_T)\phi r_{2\tau}^h \geq r_{1\tau}$. Alternatively, if $\phi < \frac{\ell}{p(\theta_T)R}$, $r \in C_\tau^\phi$ iff $r_{2\tau}^l + \frac{\ell}{R}r_{2\tau}^h \geq r_{1\tau}$. Note that $C_\tau^\phi \subset C_\tau^1$. Thus, bank τ can decide whether to write a contract in C_τ^ϕ, by setting $r_{1\tau} \leq r_{2\tau}^l + p(\theta_T)\tilde{\phi}r_{2\tau}^h$, where $\tilde{\phi} = \max\left\{\phi, \frac{\ell}{p(\theta_T)R}\right\}$, a contract in $C_\tau^1 \backslash C_\tau^\phi$, by setting $r_{2\tau}^l + p(\theta_T)\tilde{\phi}r_{2\tau}^h < r_{1\tau} \leq r_{2\tau}^l + p(\theta_T)r_{2\tau}^h$, or a contract in C_τ^{1c}, by setting $r_{1\tau} > r_{2\tau}^l + p(\theta_T)r_{2\tau}^h$.

By identical logic to that in Theorem 1, investors would refuse to invest in a bank that will be run with probability 1, setting $d_\tau = 0$ for that bank. Thus, we can restrict attention to contracts in C_τ^1. Because bad news about each bank occurs with disjoint probability ε, investor utility is

$$\Upsilon = (1-2\varepsilon)\Upsilon_N + \varepsilon\Upsilon_A + \varepsilon\Upsilon_B,$$

where Υ_N is the expected utility of investors when there is no news, and Υ_τ is the expected utility of investors when there is bad news about bank τ'. Because banks will offer contracts in C_τ^1, investors remain in both banks if there is no bad news. Thus, if there is no bad news, $s_\tau = 1$ for $\tau \in \{A, B\}$, investors earn $\Upsilon_N(\{\theta_T, \theta_T\})$, where

$$\Upsilon_N\left(\overrightarrow{\theta}\right) = \lambda u\left(S_a + r_{1\tau}d_\tau + r_{1\tau'}d_{\tau'}\right) + (1-\lambda)\left[S_a + \left(r_{2\tau}^l + p\left(\theta_\tau\right)r_{2\tau}^h\right)d_\tau + \left(r_{2\tau'}^l + p\left(\theta_{\tau'}\right)r_{2\tau'}^h\right)d_{\tau'}\right]. \tag{11}$$

In contrast, if there is bad news about bank τ', investor utility depends on the type of contract bank τ' wrote. If bank τ' wrote a contract in $C_{\tau'}^\phi$, investors will stay in both banks, so their expected payoff is $\Upsilon_\tau(\{\theta_T, \theta_T\})$, where

$$\Upsilon_\tau\left(\overrightarrow{\theta}\right) = \lambda u\left(S_a + r_{1\tau}d_\tau + r_{1\tau'}d_{\tau'}\right) + (1-\lambda)\left[S_a + \left(r_{2\tau}^l + p\left(\theta_\tau\right)r_{2\tau}^h\right)d_\tau + \left(r_{2\tau'}^l + p\left(\theta_{\tau'}\right)\tilde{\phi}r_{2\tau'}^h\right)d_{\tau'}\right],$$

and $\tilde{\phi} = \max\left\{\phi, \frac{\ell}{p(\theta_T)R}\right\}$. If bank τ' wrote a contract in $C_{\tau'}^1 \backslash C_{\tau'}^\phi$, investors will run bank τ', so their expected payoff is $\Upsilon_\tau(\{\theta_T, \theta_T\})$, where

$$\begin{aligned} \Upsilon_\tau\left(\overrightarrow{\theta}\right) &= \lambda\left[\chi_{\tau'}u\left(S_a + r_{1\tau}d_{1\tau} + r_{1\tau'}d_{\tau'}\right) + \left(1-\chi_{\tau'}\right)u\left(S_a + r_{1\tau}d_\tau\right)\right] \\ &\quad + (1-\lambda)\left[S_a + \left(r_{2\tau}^l + p\left(\theta_\tau\right)r_{2\tau}^h\right)d_{1\tau} + \chi_{\tau'}r_{1\tau'}d_{1\tau'}\right], \end{aligned}$$

$\chi_{\tau'} = \frac{S_{\tau'0} + \ell K_{\tau'0}}{r_{1\tau'}}$, $S_{\tau'0} = \lambda r_{1\tau} + (1-\lambda)r_{2\tau}^l$, and $K_{\tau'0} = (1-\lambda)\frac{r_{2\tau}^h}{R}$. We can express investor utility as

$$\Upsilon = \Upsilon_N + \varepsilon\left(\Upsilon_A + \Upsilon_B - 2\Upsilon_N\right).$$

Also, all left-hand and right-hand derivatives of Υ_N, Υ_A, and Υ_B exist and are finite.

Thus, we have the optimal withdrawal decisions by late investors; consider optimal investment by investors. Investors solve

$$\begin{aligned} V &= \max_{S_a, d_A, d_B} \Upsilon \\ s.t. &\quad S_a + d_A + d_B \leq 2 \end{aligned}$$

Let κ be the multiplier for the constraint, and let L be the Lagrangian function for investors. For signs $x \in \{-, +\}$,

$$\frac{\partial_x L}{\partial S_a} = \lambda u_x'\left(S_a + r_{1\tau}d_{1\tau} + r_{1\tau'}d_{\tau'}\right) + (1-\lambda) + \varepsilon\left(\frac{\partial_x \Upsilon_A}{\partial S_a} + \frac{\partial_x \Upsilon_B}{\partial S_a} - 2\frac{\partial_x \Upsilon_N}{\partial S_a}\right) - \kappa,$$

$$\begin{aligned} \frac{\partial_x \Upsilon}{\partial d_\tau} &= \lambda u_x'\left(S_a + r_{1\tau}d_{1\tau} + r_{1\tau'}d_{\tau'}\right)r_{1\tau} + (1-\lambda)\left(r_{2\tau}^l + p\left(\theta_T\right)r_{2\tau}^h\right) \\ &\quad + \varepsilon\left(\frac{\partial_x \Upsilon_A}{\partial d_\tau} + \frac{\partial_x \Upsilon_B}{\partial d_\tau} - 2\frac{\partial_x \Upsilon_N}{\partial d_\tau}\right) - \kappa \end{aligned}$$

If $r_{1\tau} > 1$ and $r_{2\tau}^l + p\left(\theta_T\right)r_{2\tau}^h > 0$, $\frac{\partial_x L}{\partial d_\tau} > \frac{\partial_x L}{\partial S_a}$, so $S_a = 0$. Let $D^*(r)$ be the set of optimal investment policies for investors, given contracts $r = \{r_A, r_B\}$.

Now we have the optimal investment policy by investors, $D^*(r)$, consider the optimal contracts offered by banks. Guess that IC constraints are lax, then verify later. Bank τ solves

$$\begin{aligned} &\max_{r_\tau} \quad V \\ &s.t. \quad \lambda r_{1\tau} + (1-\lambda)\left(r_{2\tau}^l + \frac{r_{2\tau}^h}{R}\right) \leq 1 \end{aligned}$$

Investor utility is increasing in r, so the budget constraint binds. Suppose to the contrary that $r_{2\tau}^l > 0$. Consider an alternative contract $\tilde{r}_\tau$ such that $\tilde{r}_{1\tau} = r_{1\tau}$, $\tilde{r}_{2\tau}^l = r_{2\tau}^l - \delta$ and $\tilde{r}_{2\tau}^h = r_{2\tau}^h + \delta R$, for δ small and positive. Pick $d \in D^*(r)$ and $\tilde{d} \in D^*(\tilde{r})$. By switching from r_τ to $\tilde{r}_\tau$, the bank provides investors with utility $V(\tilde{r}) = \Upsilon\left(\tilde{r}, \tilde{d}\right)$. By definition of the maximum, $\Upsilon\left(\tilde{r},\tilde{d}\right) \geq \Upsilon\left(\tilde{r},d\right)$. Also,

$$\Upsilon\left(\tilde{r},d\right) - \Upsilon\left(r,d\right) = \Delta\Upsilon_N + \varepsilon\left(\Delta\Upsilon_A + \Delta\Upsilon_B - 2\Delta\Upsilon_N\right),$$

where $\Delta\Upsilon_\tau = \Upsilon_\tau\left(\tilde{r},d\right) - \Upsilon_\tau\left(r,d\right)$. If $d_\tau = 0$, $\Delta\Upsilon_\tau = 0$ for $\tau \in \{A,B,N\}$, so the $r_{2\tau}^l = 0$ is WLOG optimal. If $d_\tau > 0$ $\Delta\Upsilon_N = (1-\lambda)\left(p\left(\theta_T\right)R - 1\right)d_\tau\delta$, and $p\left(\theta_T\right)R > 1$, so $\Delta\Upsilon_N > 0$. Because all left-hand and right-hand derivatives exist and are finite, $\Delta\Upsilon_\tau$ is finite, for $\tau \in \{A,B,N\}$. Thus, for ε small enough, $\Upsilon\left(\tilde{r},d\right) > \Upsilon\left(r,d\right)$. Because $\Upsilon\left(r,d\right) = V\left(r\right)$, $V\left(\tilde{r}\right) \geq V\left(r\right)$, with strict inequality if $d_\tau > 0$. Therefore, $r_{2\tau}^l = 0$ in any optimal contract. Similarly, Bank τ sets $r_{1\tau}$ so that $r_{1\tau}d_\tau + r_{1\tau'}d_{\tau'} = \tilde{c}$. Suppose to the contrary that $r_{1\tau}d_\tau + r_{1\tau'}d_{\tau'} < \tilde{c}$ and $d_\tau > 0$. Consider deviation $\tilde{r}_{1\tau} = r_{1\tau} + \delta$, for small positive δ, so $\tilde{r}_{2\tau}^h = r_{2\tau}^h - R\frac{\lambda}{1-\lambda}\delta$ (the budget constraint binds). In this case, $\Delta\Upsilon_N = \lambda\left(\psi - p\left(\theta_T\right)R\right)d_\tau\delta > 0$. Because ε is small, this implies $V\left(\tilde{r}\right) > V\left(r\right)$, so contract r cannot be optimal. Suppose to the contrary that $r_{1\tau}d_\tau + r_{1\tau'}d_{\tau'} > \tilde{c}$ and $d_\tau > 0$. Consider deviation $\tilde{r}_{1\tau} = r_{1\tau} - \delta$, for small positive δ, so $\tilde{r}_{2\tau}^h = r_{2\tau}^h + R\frac{\lambda}{1-\lambda}\delta$, which implies $\Delta\Upsilon_N = \lambda\left(p\left(\theta_T\right)R - 1\right)d_\tau\delta > 0$. Because ε is small, this implies $V\left(\tilde{r}\right) > V\left(r\right)$, so contract r cannot be optimal. By the FOCs for investors, if either bank offers a contract such that $r_{1\tau} \geq 1$ and $r_{2\tau}^l + p\left(\theta_T\right)r_{2\tau}^h \geq 1$, with at least one inequality strict, investors will set $S_a = 0$, so they invest in at least one bank. Therefore, $d_\tau > 0$ in equilibrium, so any optimal contract offered by the banks will set intermediate payoffs so that $r_{1A}d_A + r_{1B}d_B = \tilde{c}$. If banks set $r_{1\tau}^* = \frac{\tilde{c}}{2}$ and $r_{2\tau}^{*h} = \left(1 - \lambda r_{1\tau}\right)$, all ICs are lax because $2 < \tilde{c} < 2\frac{p(\theta_T)R}{\lambda p(\theta_T)R + (1-\lambda)}$, and investors optimally set $d_\tau = 1$. Investors will run following bad news on bank τ iff $\phi < \frac{r_{1\tau}^*}{p(\theta_T)r_{2\tau}^{*h}}$, resulting in the expression for $\underline{\phi}$ in the statement of the theorem.

Uncertainty-Averse Investors: When investors are uncertainty averse, the withdrawal decisions become interrelated, so the optimal run behavior of investors depends on the specific contract offered by banks. Investors who remain in a bank following bad news will receive either (9) or (10), depending on the size of ϕ. Further, investor beliefs are now given by the worst-case scenario, as stated in Lemma 2. Given bad news is realized on bank τ', we can find the optimal withdrawal behavior of late investors, and thus the utility of investors, Υ_τ. As above, all left-hand and right-hand derivatives of Υ_A and Υ_B exist and are finite. Similar to the proof of Theorem 2, investors will refuse any contract that induces runs with probability 1, so it cannot be optimal for banks to offer such a contract. Therefore, it is optimal for the bank to offer a contract that induces investors to remain in both banks if there is no bad news. Thus, if there is no bad news, investors receive $\Upsilon_N\left(\overrightarrow{\theta}^a\right)$, where Υ_N is defined in (11) and $\overrightarrow{\theta}^a$ is from Lemma 2.

Thus, investor's expected payoff, given their optimal withdrawal strategy and the contracts from the bank, is $\Upsilon = (1-2\varepsilon)\Upsilon_N + \varepsilon\Upsilon_A + \varepsilon\Upsilon_B$. Thus, investors solve

$$\begin{aligned} V &= \max_{S_a,d_A,d_B} \Upsilon \\ s.t. &\quad S_a + d_A + d_B \leq 2 \end{aligned}$$

Let κ be the multiplier for the constraint, and let L be the Lagrangian function for investors. For signs $x \in \{-,+\}$,

$$\frac{\partial_x L}{\partial S_a} = \lambda u_x'\left(S_a + r_{1\tau}d_{1\tau} + r_{1\tau'}d_{\tau'}\right) + (1-\lambda) + \varepsilon\left(\frac{\partial_x\Upsilon_A}{\partial S_a} + \frac{\partial_x\Upsilon_B}{\partial S_a} - 2\frac{\partial_x\Upsilon_N}{\partial S_a}\right) - \kappa,$$

$$\begin{aligned} \frac{\partial_x\Upsilon}{\partial d_\tau} &= \lambda u_x'\left(S_a + r_{1\tau}d_{1\tau} + r_{1\tau'}d_{\tau'}\right)r_{1\tau} + (1-\lambda)\left(r_{2\tau}^l + p\left(\theta_\tau^a\right)r_{2\tau}^h\right) \\ &\quad + \varepsilon\left(\frac{\partial_x\Upsilon_A}{\partial d_\tau} + \frac{\partial_x\Upsilon_B}{\partial d_\tau} - 2\frac{\partial_x\Upsilon_N}{\partial d_\tau}\right) - \kappa \end{aligned}$$

Let $D^*(r)$ be the set of optimal investment policies for investors, given contracts $r = \{r_A, r_B\}$.

Now we have the optimal investment policy by investors, $D^*(r)$, consider the optimal contract offered by banks.

Suppose that (13) and (14) are lax (we will verify later). Bank τ solves

$$\begin{aligned} \max_{r_\tau} \quad & V \\ s.t. \quad & \lambda r_{1\tau} + (1-\lambda)\left(r^l_{2\tau} + \frac{r^h_{2\tau}}{R}\right) \leq 1 \\ & S_a + r_{1\tau} d_\tau + r_{1\tau'} d_{\tau'} \leq \min_{\vec{\theta} \in C} U_1\left(\vec{\theta}\right) \end{aligned}$$

where $U_1(\theta) = S_a + (r^l_{2\tau} + p(\theta_\tau) r^h_{2\tau}) d_\tau + (r^l_{2\tau'} + p(\theta_{\tau'}) r^h_{2\tau'}) d_{\tau'}$.

Safe Equilibrium: If the other bank sets $r^h_{2\tau'} = 0$, it will be optimal for bank τ to set $r^h_{2\tau} = 0$. Suppose to the contrary that $r^h_{2\tau} > 0$. By Lemma 2, $\theta_\tau = \theta_L$ for all $r^h_{2\tau} > 0$. Consider an alternative contract $\tilde{r}_\tau$ such that $\tilde{r}_{1\tau} = r_{1\tau}$, $\tilde{r}^l_{2\tau} = r^l_{2\tau} + \frac{r^h_{2\tau}}{R}$ and $\tilde{r}^h_{2\tau} = 0$. Pick $d \in D^*(r)$ and $\tilde{d} \in D^*(\tilde{r})$. By switching from r_τ to $\tilde{r}_\tau$, the bank provides investors with utility $V(\tilde{r}) = \Upsilon\left(\tilde{r}, \tilde{d}\right)$. By definition of the maximum, $\Upsilon\left(\tilde{r}, \tilde{d}\right) \geq \Upsilon(\tilde{r}, d)$. Also, $\Delta\Upsilon_N = (1-\lambda)(1-2\varepsilon)(1-p(\theta_L)R) R r^h_{2\tau} d_\tau$. If $d_\tau = 0$, $\Delta\Upsilon_\tau = 0$ for all $\tau \in \{A, B, N\}$, so $V(\tilde{r}) = V(r)$, and it is WLOG optimal to set $r^h_{2\tau} = 0$. If $d_\tau > 0$, $\Delta\Upsilon_N > 0$, and because ε is small, $\Upsilon(\tilde{r}, d) > \Upsilon(r, d)$, which implies $V(\tilde{r}) > V(r)$. Therefore, $r^h_{2\tau} = 0$: if one bank does not invest in the risky assets, the best response of the other bank is to not invest either. Because $\tilde{c} > 2$, banks provide as much insurance against the liquidity shock as possible, so the IC binds. Thus, $r_{1\tau} = r^l_{2\tau} = 1$ and $r^h_{2\tau} = 0$. Note that (13) simplifies to $r^l_{2\tau} \geq r_{1\tau}$, so it is satisfied. Similarly, (14) is also satisfied. Because $y_\tau(\eta, \theta, s) = 1 = z_\tau(\eta)$ for all η, θ, and s, there are no runs in the safe equilibrium.

Risky Equilibrium: Banks are only willing to set $r^h_{2\tau} > 0$ if $p(\theta^a_\tau) R \geq 1$, which requires interior beliefs. For investors to have interior beliefs, it must be that $r^h_{2\tau} d_\tau > 0$ for $\tau \in \{A, B\}$. Guess that $p(\theta^a_\tau) R > 1$ and $r_{1\tau} \geq 1$ (we will verify these later). This implies that $\frac{\partial_x L}{\partial S_a} < \frac{\partial_x L}{\partial d_\tau}$, so $S_a = 0$. Because the investor's budget constraint binds, $d_{\tau'} = 2 - d_\tau$. Substituting in, it can quickly be verified that the investor's problem is strictly concave in d_τ when beliefs are interior, so there exists a unique solution to the investors problem. Guess that all IC constraints are lax. Let κ_τ be the multiplier for the budget constraint of bank τ, and L_τ be the Lagrangian for bank τ. For $x \in \{-, +\}$, $\frac{\partial_x L_\tau}{\partial r_{1\tau}} = \frac{\partial_x \Upsilon_N}{\partial r_{1\tau}} + \varepsilon\left(\frac{\partial_x \Upsilon_A}{\partial r_{1\tau}} + \frac{\partial_x \Upsilon_B}{\partial r_{1\tau}} - 2\frac{\partial_x \Upsilon_N}{\partial r_{1\tau}}\right) - \lambda\kappa$, $\frac{\partial_x L_\tau}{\partial r^l_{2\tau}} = \frac{\partial_x \Upsilon_N}{\partial r^l_{2\tau}} + \varepsilon\left(\frac{\partial_x \Upsilon_A}{\partial r^l_{2\tau}} + \frac{\partial_x \Upsilon_B}{\partial r^l_{2\tau}} - 2\frac{\partial_x \Upsilon_N}{\partial r^l_{2\tau}}\right) - (1-\lambda)\kappa$, and $\frac{\partial_x L_\tau}{\partial r^h_{2\tau}} = \frac{\partial_x \Upsilon_N}{\partial r^h_{2\tau}} + \varepsilon\left(\frac{\partial_x \Upsilon_A}{\partial r^h_{2\tau}} + \frac{\partial_x \Upsilon_B}{\partial r^h_{2\tau}} - 2\frac{\partial_x \Upsilon_N}{\partial r^h_{2\tau}}\right) - \frac{1-\lambda}{R}\kappa$. Because $\frac{\partial_x \Upsilon_N}{\partial r^l_{2\tau}} = (1-\lambda) d_\tau$ and $\frac{\partial_x \Upsilon_N}{\partial r^h_{2\tau}} = (1-\lambda) p(\theta^a_\tau) d_\tau$, $\frac{\partial_x L_\tau}{\partial r^l_{2\tau}} < R\frac{\partial_x L_\tau}{\partial r^h_{2\tau}}$ for ε sufficiently small, so $r^l_{2\tau} = 0$. Also, $\frac{\partial_x \Upsilon_N}{\partial r_{1\tau}} = \lambda u'_x(c_1) d_\tau$, where $c_1 = r_{1\tau} d_\tau + r_{1\tau'} d_{\tau'}$.

Suppose to the contrary that $c_1 < \tilde{c}$, which implies $u'_x(c_1) = \psi > p(\theta_T) R$. By Lemma 2, the bank that provides investors with a higher exposure of risky assets, $r_\tau d_\tau \geq r_{\tau'} d_{\tau'}$, would have $\theta^a_\tau \leq \theta_T$. Thus, $\frac{1}{\lambda}\frac{\partial_x \Upsilon_N}{\partial r_{1\tau}} > \frac{R}{1-\lambda}\frac{\partial_x \Upsilon_N}{\partial r^h_{2\tau}}$, so for ε sufficiently small, $\frac{1}{\lambda}\frac{\partial_x L_\tau}{\partial r_{1\tau}} > \frac{R}{1-\lambda}\frac{\partial_x L_\tau}{\partial r^h_{2\tau}}$, so bank τ would have an incentive to decrease $r^h_{2\tau}$ and increase $r_{1\tau}$. Thus, the optimal contracts from banks must provide $c_1 \geq \tilde{c}$. Suppose to the contrary that $c_1 > \tilde{c}$, which implies $u'_x(c_1) = 1$. Because $p(\theta^a_\tau) R > 1$, $\frac{1}{\lambda}\frac{\partial_x \Upsilon_N}{\partial r_{1\tau}} < \frac{R}{1-\lambda}\frac{\partial_x \Upsilon_N}{\partial r^h_{2\tau}}$, so for sufficiently small ε, $\frac{1}{\lambda}\frac{\partial_x L_\tau}{\partial r_{1\tau}} < \frac{R}{1-\lambda}\frac{\partial_x L_\tau}{\partial r^h_{2\tau}}$, so either bank would have an incentive to decrease $r_{1\tau}$ and increase $r_{2\tau}$. Thus, the optimal contract from banks must provide $c_1 \leq \tilde{c}$, so $c_1 = \tilde{c}$ in equilibrium.

By setting $r_{1\tau} = r_{1\tau'} = \frac{1}{2}\tilde{c}$, $r^h_{2\tau} = r^h_{2\tau'}$, investors optimally select $d_\tau = 1$, so $\theta_\tau = \theta_T$, and the FOCs are satisfied. Because $2 < \tilde{c} < 2\frac{p(\theta_T)R}{\lambda p(\theta_T)R + (1-\lambda)}$, (12) is lax. Because $p(\theta_L) R < 1$, (13) and (14) are lax. Therefore, all IC constraint are lax. Because $r_{1\tau} > 1 > p(\theta_L) R$, if there is a shock bad enough to run one bank, investors will run both. The cutoff for runs follows from substitution into the expression from Lemma 3. ■